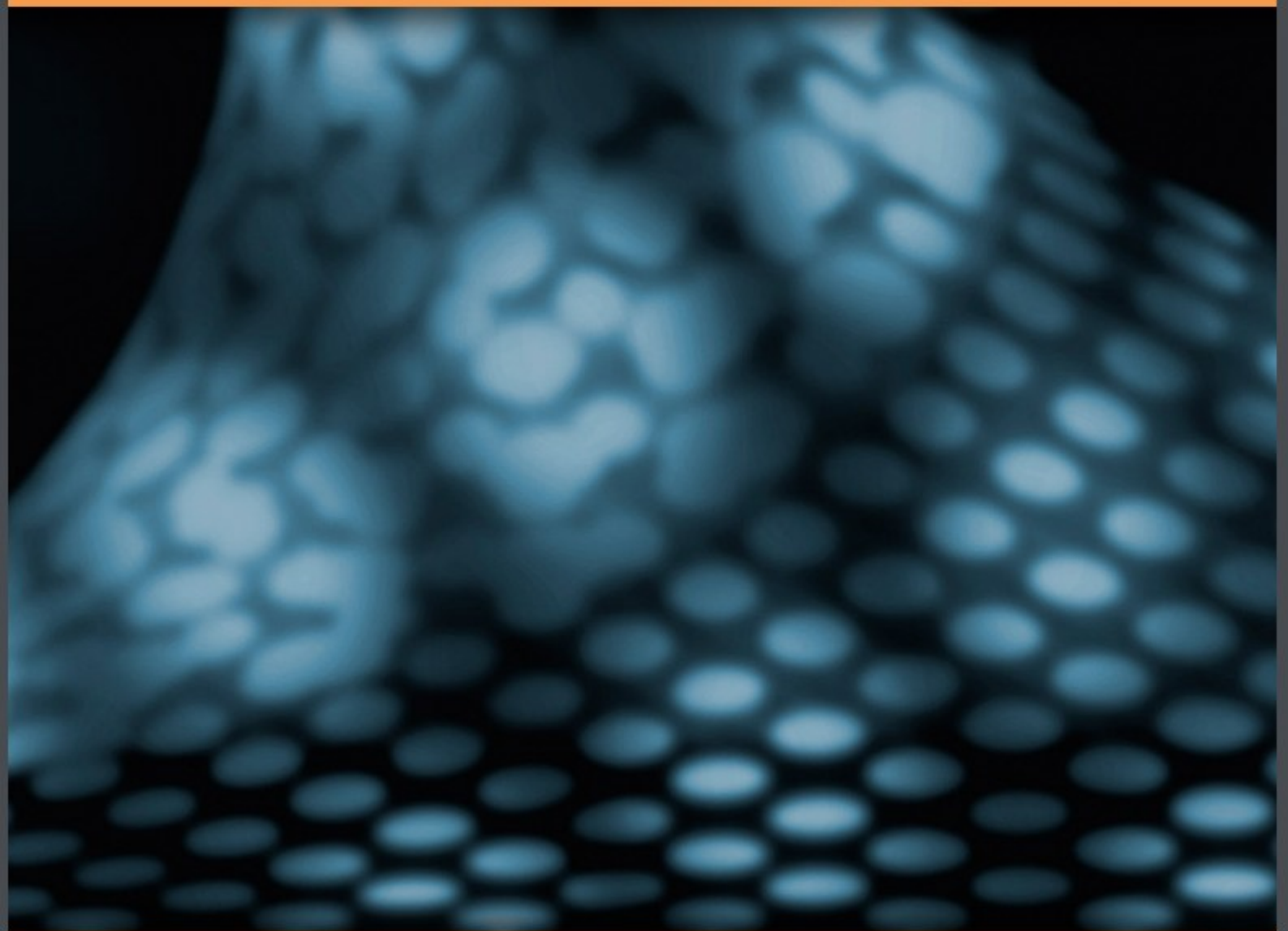


# Continuous Distributions

Probability Examples c-6

Leif Mejlbro



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# Probability Examples c-6

## Continuous Distributions

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Probability Examples c-6 – Continuous Distributions  
© 2009 Leif Mejlbro & Ventus Publishing ApS  
ISBN 978-87-7681-522-6

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## Introduction

This is the sixth book of examples from the *Theory of Probability*. This topic is not my favourite, however, thanks to my former colleague, Ole Jørsboe, I somehow managed to get an idea of what it is all about. The way I have treated the topic will often diverge from the more professional treatment. On the other hand, it will probably also be closer to the way of thinking which is more common among many readers, because I also had to start from scratch.

The prerequisites for the topics can e.g. be found in the *Ventus: Calculus 2* series, so I shall refer the reader to these books, concerning e.g. plane integrals.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro  
27th October 2009

# 1 Some theoretical background

## 1.1 The exponential distribution

A random variable  $X$  follows an *exponential distribution* with parameter  $a > 0$ , if its distribution function  $F(x)$  is given by

$$F(x) = \begin{cases} 1 - e^{-ax}, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

The corresponding *frequency*  $f(x)$  is given by

$$f(x) = \begin{cases} a e^{-ax}, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

We have for an exponentially distributed random variable  $X$  with parameter  $a > 0$ ,

$$E\{X\} = \frac{1}{a} \quad \text{and} \quad V\{X\} = \frac{1}{a^2}.$$

In general, if  $X$  is exponentially distributed, then

$$P\{X > s + t \mid X > s\} = P\{X > t\}, \quad \text{for } s, t > 0,$$

which is equivalent with the formula

$$P\{X > s + t\} = P\{X > s\} \cdot P\{X > t\}, \quad \text{for } s, t > 0.$$

We say that the exponential distribution is *forgetful*.

In practice, the exponential distribution often occurs as a distribution of *lifetimes*, which is in particular the case in queuing theory. In this case the forgetfulness is of paramount importance.

An exponentially distributed random variable  $X$  with parameter  $a > 0$  is a special *gamma distribution* (cf. the following), so one also writes,

$$X \in \Gamma\left(1, \frac{1}{a}\right)$$

for the exponential distribution.

Another type of generalized exponential distributions is the *Weibull distribution* with parameters  $a, b > 0$ . This is given by the distribution function

$$F(x) = \begin{cases} 1 - \exp(-ax^b), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

We note that we get the exponential distribution for  $b = 1$ . The *Weibull distribution* is used in connection with the *theory of reliability*.

## 1.2 The normal distribution

A random variable  $X$  is following a *normal distribution with mean 0 and variance 1*, and we write  $X \in N(0, 1)$ , if its frequency  $\varphi(x)$  is given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad \text{for } x \in \mathbb{R}.$$

Its distribution function is traditionally denoted by  $\Phi(x)$ . It is given by

$$\Phi(x) = \int_{-\infty}^x \varphi(t) dt, \quad \text{for } x \in \mathbb{R},$$

which cannot be expressed simpler by elementary functions.

We notice that if  $X \in N(0, 1)$ , then

$$E\{X\} = 0 \quad \text{and} \quad V\{X\} = 1,$$

and  $\Phi(-x) = 1 - \Phi(x)$ , and

$$1 - \Phi(x) \sim \frac{1}{x} \varphi(x) \quad \text{for } x \rightarrow +\infty,$$

or more precisely,

$$\left\{ \frac{1}{x} - \frac{1}{x^3} \right\} \varphi(x) < 1 - \Phi(x) < \frac{1}{x} \varphi(x).$$

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A random variable  $X$  is following a *normal distribution with mean  $\mu$  and variance  $\sigma^2$* , and we write  $X \in N(\mu, \sigma^2)$ , if its frequency  $f(x)$  is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad \text{for } x \in \mathbb{R}.$$

The normal distribution is also called the *Gaussian distribution*. It plays a central role in the theory of probability.

Let  $X_1$  and  $X_2$  be two independent  $N(0, 1)$  distributed random variables, and let  $a, b$  be two real constants. Then the linear combination is normally distributed with mean 0 and variance  $c^2 = a^2 + b^2$ , i.e.

$$aX_1 + bX_2 \in N(0, a^2 + b^2).$$

### 1.3 2-dimensional normal distributions

A 2-dimensional random variable  $(Z_1, Z_2)$  is *normally distributed with the parameters  $\mu_1, \mu_2 \in \mathbb{R}$ , and  $\sigma_1^2, \sigma_2^2 > 0$  and  $\varrho \in [0, 1[$* , if its frequency is given by

$$f(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}} \exp\left(-\frac{1}{2(1-\varrho^2)} \left\{ \left(\frac{z_1-\mu_1}{\sigma_1}\right)^2 - 2\varrho \frac{z_1-\mu_1}{\sigma_1} \cdot \frac{z_2-\mu_2}{\sigma_2} + \left(\frac{z_2-\mu_2}{\sigma_2}\right)^2 \right\}\right).$$

In this case we write  $(Z_1, Z_2) \in N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \varrho)$ .

If  $(Z_1, Z_2) \in N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \varrho)$ , then we have for the marginal random variables that

$$Z_1 \in N(\mu_1, \sigma_1^2) \quad \text{and} \quad Z_2 \in N(\mu_2, \sigma_2^2),$$

and

$$E\{Z_1\} = \mu_1, \quad V\{Z_1\} = \sigma_1^2, \quad \text{og} \quad E\{Z_2\} = \mu_2, \quad V\{Z_2\} = \sigma_2^2,$$

and concerning the correlation coefficient between them,

$$\varrho(Z_1, Z_2) = \varrho.$$

In general, if two random variables are independent, then they are also non-correlated, while the converse does not need to be true. However, if  $(Z_1, Z_2)$  is following a 2-dimensional normal distribution, where  $Z_1$  and  $Z_2$  are non-correlated, then  $Z_1$  and  $Z_2$  are in fact also independent, so we obtain a stronger result in this case.

Let  $(Z_1, Z_2)$  follow a 2-dimensional normal distribution, and let  $a, b, c$  and  $d$  be real constants, which satisfy the condition (of the determinant)  $ad - bc \neq 0$ . Then the 2-dimensional random variable

$$(U_1, U_2) := (aZ_1 + bZ_2, cZ_1 + dZ_2)$$

is again following a 2-dimensional normal distribution.

The theory above has a ring of geometry, and one may get a better description by using matrices for the more general *n-dimensional normal distribution*. We introduce for a general *n-dimensional* random variable

$$\mathbf{X} = (X_1, X_2, \dots, X_n)$$

the corresponding *mean vector*,

$$\boldsymbol{\mu} := (E\{X_1\}, E\{X_2\}, \dots, E\{X_n\}),$$

and its *covariance matrix*,

$$\mathbf{C} = \left\{ \begin{array}{cccc} V\{X_1\} & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & V\{X_2\} & \cdots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \cdots & V\{X_n\} \end{array} \right\},$$

provided that they exist.

Then the frequency of  $\mathbf{X} \in N(\boldsymbol{\mu}, \sigma_1^2, \sigma_2^2, \rho)$  is written

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^2 \sqrt{\det \mathbf{C}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}) \mathbf{C}^{-1}(\mathbf{x} - \boldsymbol{\mu})^\top\right), \quad \mathbf{x} \in \mathbb{R}^2,$$

where

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad \text{and} \quad \mathbf{C}^{-1} = \frac{1}{1 - \rho^2} \begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}.$$

This leads to the following generalization: An *n-dimensional* random variable  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  is a *non-singular normal distribution with mean vector*  $\boldsymbol{\mu}$  and a *positive definite covariance matrix*  $\mathbf{C}$ , if the corresponding frequency  $f(\mathbf{x})$  is given in the form

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det \mathbf{C}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}) \mathbf{C}^{-1}(\mathbf{x} - \boldsymbol{\mu})^\top\right), \quad \mathbf{x} \in \mathbb{R}^n.$$

### 1.4 Conditional normal distribution

Let  $(X, Y) \in N(\boldsymbol{\mu}, \sigma_1^2, \sigma_2^2, \rho)$  denote a 2-dimensional normal distribution. Then the conditional frequency  $f(x | y)$  for  $X$ , for given  $Y = y$ , is defined as

$$f(x | y) = \frac{1}{\sqrt{2\pi} \sigma_1 \sqrt{1 - \rho^2}} \exp\left(-\frac{1}{2\sigma_1^2(1 - \rho^2)} \left\{x - \mu_1 - \rho \frac{\sigma_1}{\sigma_2} (y - \mu_2)\right\}^2\right).$$

It follows that the conditional distribution of  $X$ , given  $Y = y$ , is a *normal distribution* with

$$\text{mean: } \mu_1 + \varrho \frac{\sigma_1}{\sigma_2} (y - \mu_2) \quad \text{and} \quad \text{variance: } \sigma_1^2 (1 - \varrho^2).$$

This is exploited, whenever one wants to estimate  $Y$  as a *linear function*  $Y = aX + b$  of  $X$ , where we shall find the constants  $a$  and  $b$ , such that  $V\{[Y - (aX + b)]^2\}$  becomes as small as possible. This is the case, when we choose

$$a = \varrho \frac{\sigma_2}{\sigma_1} \quad \text{and} \quad b = \mu_2 - \varrho \frac{\sigma_2}{\sigma_1} \mu_1.$$

The line

$$y = ax + b = \mu_2 + \varrho \frac{\sigma_2}{\sigma_1} (x - \mu_1)$$

is called the *regression line of  $Y$  with respect to  $X$* . Analogously, the line

$$x = \mu_1 + \varrho \frac{\sigma_1}{\sigma_2} (y - \mu_2)$$

is the *regression line of  $X$  with respect to  $Y$* . Summing up, they are of course written

$$\frac{x - \mu_1}{\sigma_1} = \frac{y - \mu_2}{\sigma_2},$$

which is easier to remember in practice.

## 1.5 Sums of independent normal distributed random variables

The main result is that if the normal distributed random variables  $X_1, X_2, \dots, X_n$  are mutually independent and normally distributed with  $X_i \in N(\mu_i, \sigma_i^2)$ , then the sum  $\sum X_i$  also normally distributed with

$$\sum_{i=1}^n X_i \in N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$

We express this result by saying that the *normal distribution is reproductive with respect to the parameters  $\mu$  and  $\sigma^2$* .

## 1.6 The Central Limit Theorem

The normal distribution is of paramount importance for the Theory of Probability, in particular because we have the following result, which shows that we can approximate a distribution of sums of independent identically distributed random variables by  $\Phi(x)$ .

We assume that a sequence of random variables  $\{X_i\}$  are all mutually independent and identically distributed of mean  $\mu$  and variance  $\sigma^2 > 0$ . If we put

$$Y_n = X_1 + X_2 + \dots + X_n = \sum_{i=1}^n X_i,$$

then

$$\frac{Y_n - n\mu}{\sigma\sqrt{n}}, \quad n \in \mathbb{N},$$

converges *in distribution* towards  $\Phi(x)$  for  $n \rightarrow +\infty$ , i.e.

$$\lim_{n \rightarrow +\infty} P \left\{ \frac{Y_n - n\mu}{\sigma\sqrt{n}} \leq x \right\} = \Phi(x) \quad \text{for every } x \in \mathbb{R}.$$

This means, roughly speaking, that

$Y_n$  is almost normally distributed  $N(n\mu, n\sigma^2)$  for large values of  $n$ .

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### 1.7 The Maxwell distribution

A random variable  $X$  is *Maxwell distributed*, if it has a frequency of the form

$$f(x) = \begin{cases} \sqrt{\frac{1}{\pi}} \frac{1}{\sigma^3} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right), & \text{for } x > 0, \\ 0, & \text{for } x \leq 0, \end{cases}$$

where the parameter  $\sigma > 0$ . For such a distribution,

$$E\{X\} = 2\sqrt{\frac{2}{\pi}}\sigma \quad \text{and} \quad V\{X\} = \left\{3 - \frac{8}{\pi}\right\}\sigma^2.$$

The *Maxwell distribution* occurs typically, when three given independent random variables  $X_1, X_2, X_3$ , are all  $N(0, \sigma^2)$  distributed. Then the random variable

$$X = \sqrt{X_1^2 + X_2^2 + X_3^2}$$

is *Maxwell distributed* of parameter  $\sigma > 0$ .

Conversely, if  $X_1, X_2$  and  $X_3$  are mutually independent identically distributed random variables of a *spherically symmetric* distribution of the 3-dimensional random variable  $(X_1, X_2, X_3)$ , i.e. the distribution depends only on the radius  $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$ , then  $X_1, X_2$  and  $X_3$  are all normally distributed random variables of the same type,  $X_i \in N(0, \sigma^2)$ .

### 1.8 The Gamma distribution

It is well-known that the *Gamma function* is given by

$$\Gamma(\mu) := \int_0^{+\infty} t^{\mu-1} e^{-t} dt, \quad \text{for } \mu > 0,$$

with the properties

$$\Gamma(\mu + 1) = \mu \Gamma(\mu) \quad \text{for } \mu > 0, \quad \text{in particular, } \Gamma(n + 1) = n! \quad \text{for } n \in \mathbb{N}_0.$$

Note also that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

In connection with the Gamma function we also mention the *Beta function*, which is defined by

$$B(\mu_1, \mu_2) = \int_0^1 t^{\mu_1-1} (1-t)^{\mu_2-1} dt = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1 + \mu_2)}, \quad \text{for } \mu_1, \mu_2 > 0.$$

A random variable  $X$  is *Gamma distributed* with *form parameter*  $\mu > 0$  and *scale parameter*  $\alpha > 0$ , if its frequency is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\alpha^\mu} x^{\mu-1} \exp\left(-\frac{x}{\alpha}\right), & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

We write  $X \in \Gamma(\mu, \alpha)$ . Note that

$$E\{X\} = \alpha\mu \quad \text{and} \quad V\{X\} = \alpha^2\mu.$$

It was mentioned previously that the *exponential distribution* is a special Gamma distribution,  $\Gamma\left(1, \frac{1}{a}\right)$ , of the frequency

$$f(x) = \begin{cases} a e^{-ax}, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

If  $X_1 \in \Gamma(\mu_1, \alpha)$  and  $X_2 \in \Gamma(\mu_2, \alpha)$  are independent Gamma distributed random variables of the same scale parameter, then the sum  $X_1 + X_2$  is also Gamma distributed,  $X_1 + X_2 \in \Gamma(\mu_1 + \mu_2, \alpha)$ . We say that the *Gamma distribution is reproductive in the form parameter for fixed scale parameter*.

If more generally,  $X_1, X_2, \dots, X_n$  are independent Gamma distributed random variables with the same scale parameter,  $X_i \in \Gamma(\mu_i, \alpha)$ , then their sum is also Gamma distributed,

$$\sum_{i=1}^n X_i \in \Gamma\left(\sum_{i=1}^n \mu_i, \alpha\right).$$

### 1.9 The $\chi^2$ distribution

Let  $X_1, X_2, \dots, X_n$  be identically distributed independent normal distributed random variables, thus  $X_i \in N(\mu, \sigma^2)$ . Then

$$Y_n = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu)^2 \in \Gamma\left(\frac{n}{2}, 2\right) := \chi^2(n),$$

where we call a distribution from  $\Gamma\left(\frac{n}{2}, 2\right) = \chi^2(n)$  a  $\chi^2$  *distribution* with  $n$  *degrees of freedom*. It follows immediately that if  $X_1, X_2, \dots, X_n$  are independent random variables, where  $X_i \in \chi^2(n_i)$ , then for the sum

$$\sum_{i=1}^n X_i \in \chi^2\left(\sum_{i=1}^n n_i\right).$$

If  $X \in \chi^2(n)$ , then

$$E\{X\} = n \quad \text{and} \quad V\{X\} = 2n.$$

Given a sequence  $\{X_n\}$  of random variables, where

$$X_n \in \chi^2(n), \quad \text{for every } n \in \mathbb{N}.$$

Then it follows from the *Central Limit Theorem* that

$$\lim_{n \rightarrow +\infty} P\left\{\frac{X_n - n}{\sqrt{2n}} \leq x\right\} = \Phi(x) \quad \text{for every } x \in \mathbb{R}.$$

In particular,  $X_n \in \chi^2(n)$  is approximately normal distributed  $N(n, 2n)$  for large  $n \in \mathbb{N}$ .

Notice that if the random variable  $X$  is  $\chi^2(3)$  distributed then the random variable  $Y = \sqrt{X}$  is a Maxwell distribution.

### 1.10 The $t$ distribution

A random variable  $X$  is *t-distributed*, or following a *Student distribution*, with  $n$  degrees of freedom, if its frequency is given by

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{1}{2}(n+1)}, \quad x \in \mathbb{R}.$$

We write  $X \in t(n)$ .

We obtain a *Student distribution* in the following way:

Let  $Y \in N(0, 1)$  and  $Z \in \chi^2(n)$  be independent random variables. If

$$X := \frac{Y}{\sqrt{\frac{Z}{n}}}, \quad n \in \mathbb{N},$$

then  $X \in t(n)$  is *Student distributed* with  $n$  degrees of freedom.

If  $n = 1$  we get the *Cauchy distribution* of the frequency

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

We notice that this distribution does not have a mean.

In general, the  $t$  distribution is symmetric with respect to 0, and if  $n > 1$ , then its mean is  $E\{X\} = 0$ .

When  $n$  is large, then the  $N(0, 1)$  distribution is a fair approximation of the  $t$  distribution. (In practice usually for  $n > 25$ ).

We mention that the  $t$  distribution is important because we often consider  $n$  independent identically normal distributed random variables  $X_1, X_2, \dots, X_n$ , where we know neither the mean nor the variance. Using  $X_1, X_2, \dots, X_n$  we want to estimate the mean  $\mu$  and the variance  $\sigma^2$ . We assume of course that  $n > 1$ .

As an *estimator* of the mean  $\mu$  we use the normal distributed random variable

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \in N\left(\mu, \frac{\sigma^2}{n}\right).$$

Since the variance decreases, when  $n$  increases, it follows immediately that the estimator becomes better, the larger  $n$ .

As an estimator of the variance  $\sigma^2$  we use the random variable

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2,$$

where it is only for technical reasons that we use the denominator  $n - 1$ . This trick assures that  $E\{S^2\} = \sigma^2$ .

It can be proved that

$$(n - 1) \frac{S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \in \chi^2(n - 1).$$

The “loss” of one degree of freedom is due to the fact that we have used one degree of freedom to estimate  $\mu$ .

It follows after some computations that the random variable

$$\frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \in t(n - 1)$$

is *Student distributed* with  $n - 1$  degrees of freedom.

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### 1.11 The $F$ distribution.

A random variable  $Z$  is  $F$  distributed, or *Fisher distributed* with the *degrees of freedom*  $n_1$  and  $n_2$ , if its frequency is given in the following way, using the Beta function,

$$f(z) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{z^{\frac{1}{2}n_1-1}}{\left(1 + \frac{n_1}{n_2}z\right)^{\frac{1}{2}(n_1+n_2)}}, \quad z > 0.$$

We write  $Z \in F(n_1, n_2)$ . When  $n_2 > 2$ , the mean is given by

$$E\{Z\} = \frac{n_2}{n_2 - 2},$$

while it does not exist for  $n_2 \leq 2$ .

The *Fisher distribution* occurs in the following casw: If  $X_1 \in \chi^2(n_1)$  and  $X_2 \in \chi^2(n_2)$  are two  $\chi^2$  distributed independent random variables, then

$$Z = \frac{\frac{X_1}{n_1}}{\frac{X_2}{n_2}} \in F(n_1, n_2)$$

is  $F$  distributed with the degrees of freedom  $n_1$  and  $n_2$ . We notice that the random variable  $Z$  is used to compare the *normed* random variables  $\frac{1}{n_1}X_1$  and  $\frac{1}{n_2}X_2$ .

### 1.12 Estimation of parameters

We shall shortly describe *Fisher's maximum method*.

Let  $X$  be a random variable with distribution function  $F(x; \alpha_1, \alpha_2, \dots, \alpha_k)$  and with  $k$  independent parameters  $\alpha_i$ ,  $i = 1, \dots, k$ . The task is from  $n$  independent observations  $X_1, \dots, X_n$  of the random variable  $X$  with the results of the observations  $x_1, \dots, x_n$  to *estimate* these unknown parameters.

**1. The discrete case.** We consider the function

$$h(\alpha_1, \dots, \alpha_k) := P\{X_1 = x_1 \wedge \dots \wedge X_n = x_n\} = P\{X_1 = x_1\} \cdots P\{X_n = x_n\} = \prod_{i=1}^n P\{X_i = x_i\}$$

in the parameters  $\alpha_1, \dots, \alpha_k$ .

Using methods known from e.g. *Ventus: Calculus 2* series we find the values of  $\alpha_1^*, \dots, \alpha_k^*$ , for which the function  $h(\alpha)$  has its *maximum*, i.e. the values of the parameters, for which

$$P\{X_1 = x_1 \wedge \dots \wedge X_n = x_n\}$$

is largest. These values  $\alpha_1^*, \dots, \alpha_k^*$  are called *estimates* of the parameters.

We notice that  $\alpha_i^*$  is dependent on the observations  $x_1, \dots, x_n$ ,

$$\alpha_i^* = g_i(x_1, \dots, x_n), \quad i = 1, \dots, n.$$

The corresponding *random variables*

$$g_i(X_1, \dots, X_n)$$

are called *estimators* of the parameters  $\alpha_1, \dots, \alpha_k$ . We emphasize that we distinguish between an *estimate*, which is a number, and an *estimator*, which is a random variable.

**2. The continuous case.** In this case the random variable  $X$  is given by the frequency

$$f(x) = f(x; \alpha_1, \dots, \alpha_k),$$

which depends on the  $k$  unknown parameters  $\alpha_1, \dots, \alpha_k$ . Let  $X_1, \dots, X_n$  denote  $n$  independent observations of  $X$  with the results of the observations  $x_1, \dots, x_n$ . Then like in the discrete case we introduce the function

$$h(\alpha_1, \dots, \alpha_k) = f(x_1) \cdots f(x_n) = \prod_{i=1}^n f(x_i),$$

where the results of the observations  $x_1, \dots, x_n$  are given, while the parameters  $\alpha_1, \dots, \alpha_k$  are the unknowns. We shall find the values  $\alpha_1^*, \dots, \alpha_k^*$  of  $\alpha_1, \dots, \alpha_k$ , for which the function  $h(\alpha)$  has its maximum. This is again a task, which has been described elsewhere, e.g. in the *Ventus: Calculus 2* series.

Using methods of this type we usually obtain *estimates* (i.e. numbers)

$$\alpha_i^* = g_i(x_1, \dots, x_n), \quad i = 1, \dots, k,$$

of the parameters  $\alpha_1, \dots, \alpha_k$ , with corresponding *estimators* in the same parameters,

$$g_i(X_1, \dots, X_n), \quad i = 1, \dots, k,$$

which are *random variables*.

**3. Estimators in general.** Let  $Y = g(X_1, \dots, X_n)$  be an estimator for some parameter  $\alpha$ . We say that the estimator is *central*, if

$$E\{Y\} = \alpha,$$

or roughly speaking that  $Y$  has the “right mean”.

That an estimator is central means that if we by some samples get some estimates (i.e. numbers) of  $\alpha$ , then these estimates will close in on  $\alpha$ .

If in particular  $X$  is normal distributed, then

$$\bar{X} := \frac{1}{n} \sum_{i=1}^n X_i,$$

is a *central estimator* of the mean, and

$$S^2 := \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

is a *central estimator* of the variance  $\sigma^2$ . We note that

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \quad \left( = \frac{n-1}{n} S^2 \right)$$

is *not* a central estimator of the variance  $\sigma^2$ .

In general,

$$\bar{X} := \frac{1}{n} \sum_{i=1}^n X_i$$

is *always* a central estimator of the mean  $\mu$ , *provided that the distribution has a mean!* Note, however, that one does not always get  $\bar{X}$  as an estimator by Fisher's maximum method.

A sequence of estimators  $Y_n = g_n(X_1, \dots, X_n)$  for a parameter  $\alpha$  is called *consistent*, if for every  $\varepsilon > 0$ ,

$$P\{|Y_n - \alpha| \geq \varepsilon\} \rightarrow 0 \quad \text{for } n \rightarrow +\infty.$$

It follows from the *Weak Law of Large Numbers* that if the distribution has a mean  $\mu$ , then the sequence of estimators

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$$

is a consistent sequence of estimators for the mean  $\mu$ .

Let

$$Y = g_1(X_1, \dots, X_n) \quad \text{and} \quad Z = g_2(X_1, \dots, X_n)$$

be two *central* estimators for the same parameter  $\alpha$ . Then  $Y$  is said to be more *efficient* than  $Z$ , if

$$V\{Y\} < V\{Z\} \quad \text{for every value of } \alpha.$$

If  $Y$  is more *efficient* than  $Z$ , then  $Y$  has a smaller variance, so roughly speaking,  $Y$  will have "more estimates" lying close to  $\alpha$  than  $Z$ . Therefore, one prefers  $Y$  as a (central) estimator in such a case.

## 2 The Exponential Distribution

**Example 2.1** Let  $Y, X_1, X_2$  be independent positive random variables, all with a continuous distribution, and let  $Y$  be exponentially distributed. It is well-known that for all  $s, t \in \mathbb{R}_+$ ,

$$(1) P\{Y > s + t\} = P\{Y > s\} \cdot P\{Y > t\},$$

(the property of forgetfulness).

1) Prove by using (1) that we have the following generalization

$$(2) P\{Y > X_1 + X_2\} = P\{Y > X_1\} \cdot P\{Y > X_2\}.$$

2) Is (2) also true, if  $X_1$  and  $X_2$  are mutually dependent, while they are both independent of  $Y$ ?

3) Let  $Y, X_1, X_2, \dots, X_n$  be independent positive variables, all with a continuous distribution, and let  $Y$  be exponentially distributed.

Prove that

$$P\{Y > X_1 + X_2 + \dots + X_n\} = \prod_{i=1}^n P\{Y > X_i\}.$$

1) Let  $f_1(x_1)$  and  $f_2(x_2)$  denote the frequencies of  $X_1$  and  $X_2$ , resp.. Since  $X_1$  and  $X_2$  are independent and positive, we conclude that the frequency of  $Z = X_1 + X_2$  for  $z > 0$  is given by

$$(3) g(z) = \int_0^z f_1(x)f_2(z-x) dx \quad (\text{and } g(z) = 0 \text{ for } z \leq 0).$$

Let  $a$  be the parameter for  $Y$ . The frequency  $h(t)$  of  $Y - Z$  is for  $t \geq 0$  (note the the frequency is of no interest for  $t < 0$ ) given by

$$\begin{aligned} h(t) &= a \int_0^\infty e^{-ay}g(y-t) dy = a \cdot e^{-at} \int_0^\infty e^{-az}g(z) dz \\ &= a e^{-at} \int_0^\infty e^{-az} \left\{ \int_0^z f_1(x)f_2(z-x) dx \right\} dz \\ &= a e^{-at} \int_0^\infty e^{-ax} f_1(x) \left\{ \int_x^\infty e^{-a(z-x)} f_2(z-x) dz \right\} dx \\ &= a e^{-at} \cdot \int_0^\infty e^{-ax} f_1(x) dx \cdot \int_0^\infty e^{-ay} f_2(y) dy, \end{aligned}$$

so

$$P\{Y > X_1 + X_2\} = P\{Y - Z > 0\} = \int_0^\infty h(t) dt = \int_0^\infty e^{-ax} f_1(x) dx \cdot \int_0^\infty e^{-ay} f_2(y) dy.$$

Analogously, the frequency of  $Y - X_i$  is given by

$$h_i(t) = a \int_0^\infty e^{-ay} f_i(y-t) dy = a \int_t^\infty e^{-ay} f_i(y-t) dy = a e^{-at} \int_0^\infty e^{-ax} f_i(x) dx,$$

hence

$$P\{Y > X_i\} = P\{Y - X_i > 0\} = \int_0^\infty h_i(t) dt = \int_0^\infty e^{-ax} f_i(x) dx, \quad i = 1, 2,$$

and it follows that

$$P\{Y > X_1 + X_2\} = P\{Y > X_1\} \cdot P\{Y > X_2\}.$$

- 2) Formula (2) has been proved under the assumption of (3), which again presupposes that  $X_1$  and  $X_2$  are independent. We may therefore expect that the answer is “no”.

An explicit example, which *shows* that the answer in general is “no” is given by  $X_2 = 1 - X_1$ , where  $X_1$  is uniformly distributed over  $[0, 1]$ . In this case,

$$P\{Y > X_1 + X_2\} = P\{Y > 1\} = \int_1^\infty a e^{-ay} dy = e^{-a}.$$

It follows that the frequency  $h_1(t)$  of  $Y - X_1$  is given by

$$h_1(t) = a e^{-at} \int_0^1 e^{-ax} dx = e^{-at} (1 - e^{-a}) \quad \text{for } t > -1,$$

hence

$$P\{Y > X_1\} = \int_0^\infty f_1(t) dt = \frac{1}{a} (1 - e^{-a}).$$

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Since

$$\{Y > X_2 = 1 - X_1\} = \{Y + X_1 > 1\},$$

we first find the frequency of  $Y + X_1$ , i.e. for  $t > 0$ ,

$$k(t) = \int_0^\infty a e^{-ay} f_1(t - y) dy = \int_{\max\{0, t-1\}}^t a e^{-ay} dy = \exp(-a \max\{0, t - 1\}) - e^{-at},$$

thus

$$\begin{aligned} P\{T > X_2\} &= P\{Y + X_1 > 1\} = \int_1^\infty k(t) dt = \int_1^\infty \{e^{-at} e a - e^{-at}\} dt \\ &= \frac{1}{a} (e^a - 1) \cdot e^{-a} = \frac{1}{a} \cdot (1 - 1^{-a}) = P\{Y > X_2\}, \end{aligned}$$

which of course also can be seen directly by an argument of symmetry. We finally get

$$P\{Y > X_1\} \cdot P\{Y > X_2\} = \left(\frac{1 - e^{-a}}{a}\right)^2 \neq e^{-a} = P\{Y > X_1 + X_2\},$$

and it follows from this example that the claim does not hold in general.

3) If we put  $Z_j = X_1 + \dots + X_j$ , then we get by recursion and (2),

$$\begin{aligned} P\{Y > X_1 + X_2 + \dots + X_n\} &= P\{Y > Z_n\} = P\{Y > Z_{n-1} + X_n\} \\ &= P\{Y > Z_{n-1}\} \cdot P\{Y > X_n\} = \dots = \prod_{i=1}^n P\{Y > X_i\}. \end{aligned}$$

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**Example 2.2** Assume that  $X_1$  and  $X_2$  are independent random variables of the frequencies

$$f_{X_1}(x_1) = \begin{cases} 4e^{-4x_1}, & x_1 > 0, \\ 0, & x_1 \leq 0, \end{cases} \quad f_{X_2}(x_2) = \begin{cases} e^{x_2}, & x_2 < 0, \\ 0, & x_2 \geq 0. \end{cases}$$

1. Find the means  $E\{X_1\}$  and  $E\{X_2\}$ .
2. Find the variances  $V\{X_1\}$  and  $V\{X_2\}$ .

Let the two-dimensional random variable  $(Y_1, Y_2) = \tau(X_1, X_2)$  be given by

$$Y_1 = 4X_1 - X_2, \quad Y_2 = 4X_1 + X_2.$$

3. Prove that  $\tau$  maps  $\mathbb{R}_+ \times \mathbb{R}_-$  bijectively onto

$$D' = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0, -y_1 < y_2 < y_1\}.$$

4. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .
5. Find the marginal frequencies of  $Y_1$  and of  $Y_2$ .
6. Prove that  $Y_1$  and  $Y_2$  are non-correlated.
7. Check if  $Y_1$  and  $Y_2$  are independent.

1. and 2. Since  $X_1$  is exponentially distributed,  $a = 4$ , and  $-X_2$  is exponentially distributed with  $a = 1$ , we get

$$E\{X_1\} = \frac{1}{4}, \quad V\{X_1\} = \frac{1}{16},$$

and

$$E\{X_2\} = -1, \quad V\{X_2\} = 1.$$

3. It follows from  $y_1 = 4x_1 - x_2$  and  $y_2 = 4x_1 + x_2$  that

$$x_1 = \frac{1}{8}(y_1 + y_2) \quad \text{and} \quad x_2 = \frac{1}{2}(y_2 - y_1).$$

Since  $y_1 = 4x_1 - x_2 > 0$  and  $8x_1 = y_1 + y_2 > 0$ ,  $2x_2 = y_2 - y_1 < 0$ , we get

$$D' = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0 \wedge -y_1 < y_2 < y_1\}.$$

4. The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{16} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -\frac{1}{8}.$$

Since the simultaneous frequency of  $(X_1, X_2)$  is

$$g(x_1, x_2) = \begin{cases} 4 \exp(-4x_1 + x_2) & \text{for } x_1 > 0 \text{ og } x_2 < 0, \\ 0 & \text{ellers,} \end{cases}$$

the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$  is given by

$$k(y_1, y_2) = \begin{cases} \frac{1}{2} \exp(-y_1) & \text{i } D', \\ 0 & \text{otherwise.} \end{cases}$$

5. The marginal frequency of  $Y_1$  for  $y_1 > 0$  is given by

$$\int_{-y_1}^{y_1} k(y_1, y_2) dy_2 = y_1 \exp(-y_1),$$

thus

$$h_{Y_1}(y_1) = \begin{cases} y_1 \exp(-y_1) & \text{for } y_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal frequency of  $Y_2$  is given by

$$h_{Y_2}(y_2) = \int_{|y_2|}^{\infty} \frac{1}{2} \exp(-y_1) dy_1 = \frac{1}{2} \exp(-|y_2|), \quad y_2 \in \mathbb{R}.$$

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6. Since  $X_1$  and  $X_2$  are independent, and  $\text{Cov}(\cdot, \cdot)$  is bilinear, we get

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(4X_1 - X_2, 4X_1 + X_2) = 16V\{X_1\} - V\{X_2\} = 16 \cdot \frac{1}{16} - 1 = 0,$$

hence  $Y_1$  and  $Y_2$  are non-correlated.

7. Since  $D'$  is not an axes parallel domain,  $Y_1$  and  $Y_2$  cannot be independent.

ALTERNATIVELY it follows immediately that

$$h(y_1) \cdot h_{Y_2}(y_2) \neq k(y_1, y_2).$$

**Example 2.3** Assume that  $X_1, X_2, X_3, \dots$  are independent random variables, such that  $X_k$  for every  $k \in \mathbb{N}$  has the frequency

$$f_k(x) = \begin{cases} k e^{-kx}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Furthermore, let

$$Y_n = \sum_{k=1}^n X_k \quad \text{and} \quad Z_n = Y_n - \ln n, \quad n \in \mathbb{N}.$$

1) Find mean and variance of  $X_k$ , and mean and variance of  $Y_n$ .

2) Find the frequency of  $Y_2$ .

3) Prove for every  $n \in \mathbb{N}$  that the frequency  $g_n(y)$  of  $Y_n$  is given by

$$g_n(y) = \begin{cases} n e^{-y} (1 - e^{-y})^{n-1} = n e^{-ny} (e^y - 1)^{n-1}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

HINT: Use e.g. induction; the essential step is to prove that if the formula is true for some given value of  $n_0 \in \mathbb{N}$ , then it also holds for the following value  $n_0 + 1$ .

4) Find the distribution function  $G_n(y)$  for  $Y_n$ .

5) Find the distribution function  $H_n(z)$  for  $Z_n$ .

6) Prove that the sequence  $(Z_n)_{n=1}^\infty$  converges in distribution towards a random variable  $Z$ , and find the distribution function and the frequency of  $Z$ .

1) Every  $X_k$  is exponentially distributed with  $a = k$ , so

$$E\{X_k\} = \frac{1}{k} \quad \text{og} \quad V\{X_k\} = \frac{1}{k^2}.$$

Since the  $X_k$  are mutually independent, we get

$$E\{Y_n\} = \sum_{k=1}^n E\{X_k\} = \sum_{k=1}^n \frac{1}{k},$$

and

$$V\{Y_n\} = \sum_{k=1}^n V\{X_k\} = \sum_{k=1}^n \frac{1}{k^2}.$$

2) Since  $Y_2 = X_1 + X_2$  only has positive values, we have  $g_2(y) = 0$  for  $y \leq 0$ . If  $y > 0$ , then

$$\begin{aligned} g_2(y) &= \int_0^\infty f_1(x) f_2(y-x) dx = \int_0^y e^{-x} \cdot 2e^{-2(y-x)} dx = 2e^{-2y} \int_0^y e^x dx = 2e^{-2y} (e^y - 1) \\ &= 2e^{-y} (1 - e^{-y}). \end{aligned}$$

3) It is obvious that the formula is true for  $n = 1$  and  $n = 2$ , cf. **2**. Then assume that the formula holds for some  $n \in \mathbb{N}$ . Since  $Y_{n+1}$  only has positive values, we have  $g_{n+1}(y) = 0$  for  $y \leq 0$ . If  $y > 0$ , then

$$\begin{aligned} g_{n+1}(y) &= \int_0^y g_n(x) \cdot f_{n+1}(y-x) dx = \int_0^y n e^{-nx} (e^x - 1)^{n-1} \cdot (n+1) e^{-(n+1)(y-x)} dx \\ &= n(n+1) e^{-(n+1)y} \int_0^y (e^x - 1)^{n-1} \cdot e^x dx, \quad t = e^x \\ &= (n+1) e^{-(n+1)y} \int_1^{e^y} n(t-1)^{n-1} dt = (n+1) e^{-(n+1)y} (e^y - 1)^n, \end{aligned}$$

which is the formula under consideration with  $n$  replaced by  $n + 1$ , and the formula follows by induction.

4) The distribution function of  $Y_n$  is  $G_n(y) = 0$  for  $y \leq 0$ . If  $y > 0$ , then

$$G_n(y) = \int_0^y g_n(t) dt = \int_0^y n e^{-t} (1 - e^{-t})^{n-1} dt = [(1 - e^{-t})^n]_0^y = (1 - e^{-y})^n,$$

hence

$$G_n(y) = \begin{cases} (1 - e^{-y})^n & \text{for } y > 0, \\ 1 & \text{for } y \leq 0. \end{cases}$$

5) By a rearrangement,

$$H_n(z) = P\{Z_n \leq z\} = P\{Y_n - \ln n \leq z\} = P\{Y_n \leq z + \ln n\}.$$

If  $z \leq -\ln n$ , we get  $H_n(z) = 0$ . If  $z > -\ln n$ , then instead

$$H_n(z) = G_n(z + \ln n) = \{1 - \exp(-z - \ln n)\}^n = \left(1 - \frac{1}{n} e^{-z}\right)^n,$$

so

$$H_n(z) = \begin{cases} \left(1 - \frac{1}{n} e^{-z}\right)^n & \text{for } z > -\ln n, \\ 0 & \text{for } z \leq -\ln n. \end{cases}$$

6) To any  $z \in \mathbb{R}$  there exists an  $n_0 \in \mathbb{N}$ , such that  $z > -\ln n$  for every  $n \geq n_0$ . Then for such  $n \geq n_0$ ,

$$H_n(z) = \left\{1 - \frac{e^{-z}}{n}\right\}^n \rightarrow \exp(-e^{-z}), \quad \text{for } n \rightarrow \infty,$$

hence  $(Z_n)$  converges in distribution towards a random variable  $Z$ , the distribution function of which is

$$H(z) = \exp(-e^{-z}), \quad z \in \mathbb{R}.$$

The frequency is obtained by a differentiation,

$$h(z) = e^{-z} \cdot \exp(-e^{-z}) = \exp(-z - e^{-z}), \quad z \in \mathbb{R}.$$

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**Example 2.4** An instrument  $A$  contains two components of lifetimes  $X_1$  and  $X_2$ , which are assumed to be independent random variables, both of the frequency

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $\alpha$  is a positive constant. The instrument  $A$  can only operate if both components work, so the lifetime  $X$  for  $A$  is  $X = \min\{X_1, X_2\}$ .

Another instrument  $B$  also contains two components, the lifetimes of which,  $Y_1$  and  $Y_2$ , are independent random variables, both of the frequency

$$g(y) = \begin{cases} 4\alpha e^{-4\alpha y}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

Instrument  $B$  first uses one component, and when it breaks down, one immediately changes to the other component, so the lifetime  $Y$  of  $B$  is  $Y = Y_1 + Y_2$ .

- 1) Find frequency, mean and variance of  $X$ .
- 2) Find frequency, mean and variance of  $Y$ .
- 3) Find the frequency and the distribution function of the random variable  $U = \frac{X}{Y}$ .
- 4) Check if  $U$  has a mean, and if this is the case find it.
- 5) Find  $P\{Y > X\}$ .

It is immediately seen that  $X_1$  and  $X_2$  are exponentially distributed with parameter  $a = \alpha$  and the distribution function

$$F_{X_i} = \begin{cases} 1 - e^{-\alpha x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Analogously,  $Y_1$  and  $Y_2$  are exponentially distributed with  $a = 4\alpha$ ; however, in this case we do not need to indicate the distribution function.

- 1) The distribution function of  $X = \min\{X_1, X_2\}$  is

$$F_X(x) = 1 - (1 - F_{X_1}(x)) \cdot (1 - F_{X_2}(x)) = \begin{cases} 1 - e^{-2\alpha x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

hence  $X$  is exponentially distributed with  $a = 2\alpha$ , so

$$f_X(x) = \begin{cases} 2\alpha e^{-2\alpha x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

and

$$E\{X\} = \frac{1}{2\alpha} \quad \text{and} \quad V\{X\} = \frac{1}{4\alpha^2}.$$

2) Since  $Y$  only has positive values,  $g_Y(y) = 0$  for  $y \leq 0$ . If  $y > 0$ , then

$$g_Y(y) = \int_0^y g(t) g(y-t) dt = 16\alpha^2 \int_0^y e^{-4\alpha t} \cdot e^{-4\alpha(y-t)} dt = 16\alpha^2 y e^{-4\alpha y},$$

so  $Y \in \Gamma\left(2, \frac{1}{4\alpha}\right)$  is Gamma distributed, and

$$E\{Y\} = 2 \cdot \frac{1}{4\alpha} = \frac{1}{2\alpha} \quad \text{og} \quad V\{Y\} = \frac{1}{16\alpha^2} \cdot 2 = \frac{1}{8\alpha^2}.$$

At this stage we cannot assume that the Gamma distribution is known, so we have ALTERNATIVELY by direct computations,

$$E\{Y\} = 16\alpha^2 \int_0^\infty y^2 e^{-4\alpha y} dy = \frac{1}{4\alpha} \int_0^\infty t^2 e^{-t} dt = \frac{2}{4\alpha} = \frac{1}{2\alpha},$$

and

$$E\{Y^2\} = 16\alpha^2 \int_0^\infty y^3 e^{-4\alpha y} dy = \frac{1}{16\alpha^2} \int_0^\infty t^3 e^{-t} dt = \frac{6}{16\alpha^2} = \frac{3}{8\alpha^2},$$

thus

$$V\{Y\} = E\{Y^2\} - (E\{Y\})^2 = \frac{3}{8\alpha^2} - \frac{1}{4\alpha^2} = \frac{1}{8\alpha^2}.$$

3) Since  $U = \frac{X}{Y}$  only has positive values, the frequency is  $h(u) = 0$  for  $u \leq 0$ . Since  $X$  and  $Y$  are stochastically independent, we get  $u > 0$ ,

$$\begin{aligned} h(u) &= \int_0^\infty f_X(uy) \cdot g_Y(y) \cdot |y| dy = \int_0^\infty 2\alpha e^{-2\alpha uy} \cdot 16\alpha^2 y e^{-4\alpha y} \cdot y dy \\ &= 32\alpha^3 \int_0^\infty y^2 \exp(-2\alpha(u+2)y) dy = \frac{32\alpha^3}{\{2\alpha(u+2)\}^3} \int_0^\infty t^2 e^{-t} dt \\ &= \frac{32\alpha^3}{8\alpha^3(u+2)^3} \cdot 2! = \frac{8}{(u+2)^3}, \end{aligned}$$

hence

$$h(u) = \begin{cases} \frac{8}{(u+2)^3} & \text{for } u > 0, \\ 0 & \text{for } u \leq 0. \end{cases}$$

The distribution function is then obtained by an integration,

$$H(u) = \begin{cases} 1 - \frac{4}{(u+2)^2} & \text{for } u > 0, \\ 0 & \text{for } u \leq 0. \end{cases}$$

4) It follows by an explicit computation that

$$\begin{aligned} \int_{-\infty}^{\infty} |u| h(u) du &= \int_{-\infty}^{\infty} u h(u) du = \int_0^{\infty} u h(u) du = 8 \int_0^{\infty} \frac{u+2-2}{(u+2)^3} du \\ &= 8 \int_0^{\infty} \left\{ \frac{1}{(u+2)^2} - \frac{1}{(u+2)^3} \right\} du = 8 \left[ -\frac{1}{u+2} + \frac{1}{(u+2)^2} \right]_0^{\infty} = 8 \left( \frac{1}{2} - \frac{1}{4} \right) = 2 = E\{U\}, \end{aligned}$$

hence the mean *exists* and it is equal to  $E\{U\} = 2$ .

ALTERNATIVELY, it follows that

$$E \left\{ \frac{1}{Y} \right\} = \int_0^{\infty} \frac{1}{y} f_Y(y) dy = \int_0^{\infty} 16\alpha^2 e^{-4\alpha y} dy = \frac{16\alpha^2}{4\alpha} = 4\alpha,$$

from which

$$E\{U\} = E\{X\} \cdot E \left\{ \frac{1}{Y} \right\} = 2.$$

5) From  $U = \frac{X}{Y}$  follows that

$$P\{Y > X\} = P\{U < 1\} = H(1) = 1 - \frac{4}{(1+2)^2} = \frac{5}{9}.$$

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### 3 The Normal Distribution

**Example 3.1** Given a random variable  $X$ , which is normally distributed of mean 2.12. Given that

$$P\{X \geq 3\} = 0.324,$$

find the variance of  $X$ .

If follows from

$$0.324 = P\{X \geq 3\} = P\{X - \mu \geq 0.88\} = 1 - P\left\{\frac{X - \mu}{\sigma} < \frac{0.88}{\sigma}\right\} = 1 - \Phi\left(\frac{0.88}{\sigma}\right),$$

that

$$\Phi\left(\frac{0.88}{\sigma}\right) = 1 - 0.324 = 0.676 \approx \Phi(0.4567),$$

thus

$$\frac{0.88}{\sigma} = 0.4567,$$

and hence

$$V\{X\} = \sigma^2 = \left(\frac{0.88}{0.4567}\right)^2 = 3.713.$$

**Example 3.2** Given a normally distributed random variable  $X$ , for which

$$P\{X \leq 3\} = 0.9087 \quad \text{and} \quad P\{X \leq 2\} = 0.6030.$$

Find mean and variance of  $X$ .

First rearrange the given data in the following way

$$0.9087 = P\{X \leq 3\} = P\left\{\frac{X - \mu}{\sigma} \leq \frac{3 - \mu}{\sigma}\right\} = \Phi\left(\frac{3 - \mu}{\sigma}\right)$$

and

$$0.6030 = P\{X \leq 2\} = P\left\{\frac{X - \mu}{\sigma} \leq \frac{2 - \mu}{\sigma}\right\} = \Phi\left(\frac{2 - \mu}{\sigma}\right).$$

Using a table we get the inverse of  $\Phi$ ,

$$\frac{3 - \mu}{\sigma} = \Phi^{-1}(0.9087) = 1.333, \quad \text{thus } 3 - \mu = 1.333\sigma,$$

and

$$\frac{2 - \mu}{\sigma} = \Phi^{-1}(0.6030) = 0.261, \quad \text{thus } 2 - \mu = 0.261\sigma.$$

Hence,  $1.071 \sigma = 1$ , so  $\sigma = 0.934$ , and

$$V\{X\} = \sigma^2 = 0.872,$$

and

$$E\{X\} = \mu = 3 - 1.333 \sigma = 1.7555.$$

**Example 3.3** As usual,  $\varphi(x)$  and  $\Phi(x)$  denote the frequency and the distribution function, resp. in an  $N(0, 1)$ -distribution. Obviously, we have the inequality for every  $x > 0$ ,

$$\int_x^\infty \varphi(y) \left(1 - \frac{3}{y^4}\right) dy < \int_x^\infty \varphi(y) dy < \int_x^\infty \varphi(y) \left(1 + \frac{1}{y^2}\right) dy.$$

1) Apply this inequality to prove that for  $x > 0$  we have the inequality

$$\frac{1}{x} \left(1 - \frac{1}{x^2}\right) \varphi(x) < 1 - \Phi(x) < \frac{1}{x} \varphi(x),$$

and prove that

$$\lim_{x \rightarrow \infty} \frac{1 - \Phi(x)}{\varphi(x)} = 0.$$

2) Prove that

$$\frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) = -1 + x \cdot \frac{1 - \Phi(x)}{\varphi(x)}.$$

3) Prove that

$$\lim_{x \rightarrow \infty} \frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) = 0.$$

1) Clearly,

$$\int_x^\infty \varphi(y) dy = 1 - \Phi(x).$$

From  $x > 0$  we immediately get the right hand side of the inequality, because  $1 \leq \frac{y}{x}$  for  $y \geq x$ ,

$$\int_x^\infty \varphi(y) dy \leq \frac{1}{x} \int_x^\infty y \varphi(y) dy = \frac{1}{x} [-\varphi(y)]_x^\infty = \frac{\varphi(x)}{x},$$

where  $\varphi'(y) = -y \varphi(y)$ .

Furthermore, by a partial integration,

$$\begin{aligned} \int_x^\infty \varphi(y) \left(1 - \frac{3}{y^4}\right) dy &= \left[ \varphi(y) \left(y + \frac{1}{y \cdot 3}\right) \right]_x^\infty - \int_x^\infty \varphi'(y) \left(y + \frac{1}{y^3}\right) dy \\ &= \int_x^\infty \varphi(y) \left(y^2 + \frac{1}{y^2}\right) dy - \varphi(x) \cdot x - \varphi(x) \cdot \frac{1}{x^3} \\ &= \frac{1}{x} \left(1 - \frac{1}{x^2}\right) \varphi(x) + \int_x^\infty \varphi(y) \left(y^2 + \frac{1}{y^2}\right) dy - \varphi(x) \left(x + \frac{1}{x}\right). \end{aligned}$$



Now,

$$\begin{aligned} \frac{d}{dx} \left\{ \varphi(x) \left( x + \frac{1}{x} \right) \right\} &= \varphi'(x) \left( x + \frac{1}{x} \right) + \varphi(x) - \frac{1}{x^2} \varphi(x) \\ &= -x^2 \varphi(x) - \varphi(x) + \varphi(x) - \frac{1}{x^2} \varphi(x) = -\varphi(x) \left( x^2 + \frac{1}{x^2} \right), \end{aligned}$$

so

$$\begin{aligned} \int_x^\infty \varphi(y) \left( y^2 + \frac{1}{y^2} \right) dy - \varphi(x) \cdot \left( x + \frac{1}{x} \right) \\ = \int_x^\infty \varphi(y) \left( y^2 + \frac{1}{y^2} \right) dy - \int_x^\infty \varphi(y) \cdot \left( y^2 + \frac{1}{y^2} \right) dy = 0, \end{aligned}$$

and thus

$$1 - \Phi(x) > \int_x^\infty \varphi(y) \left( 1 - \frac{3}{y^4} \right) dy = \frac{1}{x} \left( 1 - \frac{1}{x^2} \right) \varphi(x) + 0 = \frac{1}{x} \left( 1 - \frac{1}{x^2} \right) \varphi(x),$$

and we have proved the inequalities.

Now,  $\varphi(x) > 0$ , so it follows from the former result that

$$\frac{1}{x} \left( 1 - \frac{1}{x^2} \right) < \frac{1 - \Phi(x)}{\varphi(x)} < \frac{1}{x}.$$

Both limits tend towards 0 for  $x \rightarrow \infty$ , hence

$$\lim_{x \rightarrow \infty} \frac{1 - \Phi(x)}{\varphi(x)} = 0.$$

2) It follows from  $\Phi'(x) = \varphi(x)$ , and  $\varphi'(x) = -x \varphi(x)$  that

$$\begin{aligned} \frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) &= \frac{-\Phi'(x)}{\varphi} - \frac{1 - \Phi(x)}{\varphi(x)^2} \cdot \varphi'(x) = -\frac{\varphi(x)}{\varphi(x)} + \frac{1 - \Phi(x)}{\varphi(x)^2} \cdot x \varphi(x) \\ &= -1 + x \cdot \frac{1 - \Phi(x)}{\varphi(x)}. \end{aligned}$$

3) If we again use the inequalities proved in (1), we get

$$-1 + \frac{x}{\varphi(x)} \cdot \frac{1}{x} \left( 1 - \frac{1}{x^2} \right) \varphi(x) < \frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) < -1 + \frac{x}{\varphi(x)} \cdot \frac{1}{x} \varphi(x),$$

hence by a reduction,

$$-\frac{1}{x^2} < \frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) = 0.$$

Both limits tend towards 0 for  $x \rightarrow \infty$ , so

$$\lim_{x \rightarrow \infty} \frac{d}{dx} \left( \frac{1 - \Phi(x)}{\varphi(x)} \right) = 0.$$

**Example 3.4** Let  $X \in N(0, 1)$ . Compute for every  $n \in \mathbb{N}$  the moments

$$E\{X^n\} \quad \text{and} \quad E\{|X|^n\}.$$

If  $n = 2m + 1$  is an odd number, then

$$\begin{aligned} E\{|X|^{2m+1}\} &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^{2m+1} \exp\left(-\frac{1}{2}x^2\right) dx = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^{2m+1} \exp\left(-\frac{1}{2}x^2\right) dx \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \left(2 \cdot \frac{x^2}{2}\right)^m \exp\left(-\frac{1}{2}x^2\right) d\left(\frac{x^2}{2}\right) = \frac{2^{m+1}}{\sqrt{2\pi}} \int_0^\infty y^m e^{-y} dy = \frac{2^{m+1}m!}{\sqrt{2\pi}}. \end{aligned}$$

It follows in particular that all moments exist.

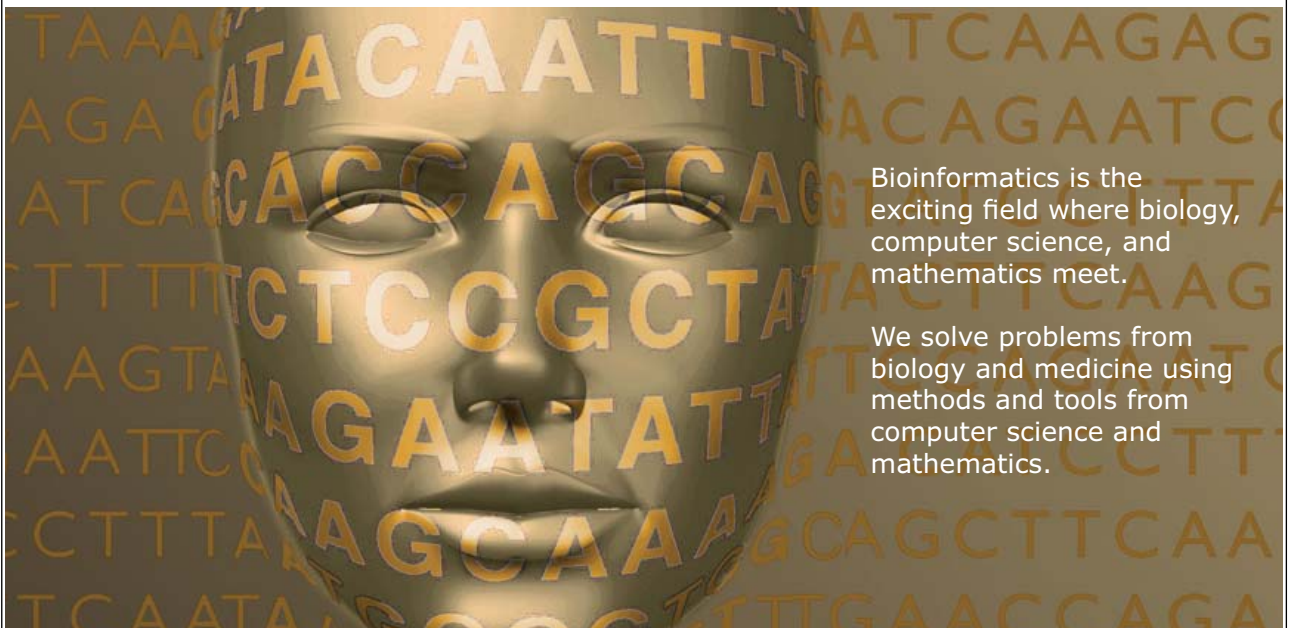
If  $n = 2m + 1$  is odd, then the integrand is odd, and it follows by the symmetry that

$$E\{X^{2m+1}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^{2m+1} \exp\left(-\frac{1}{2}x^2\right) dx = 0.$$



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If  $n = 2m$  is even, then

$$\begin{aligned} E\{X^{2m}\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m} \exp\left(-\frac{1}{2}x^2\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{x^{2m+1}}{2m+1} \exp\left(-\frac{1}{2}x^2\right) \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x^{2m+1}}{2m+1} x \exp\left(-\frac{1}{2}x^2\right) dx \\ &= \frac{1}{2m+1} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{2m+2} \exp\left(-\frac{1}{2}x^2\right) dx = \frac{1}{2m+1} E\{X^{2m+2}\}, \end{aligned}$$

thus

$$E\{X^{2m+2}\} = (2m+1) E\{X^{2m}\},$$

and hence by a change of variable,

$$E\{X^{2m}\} = (2m-1) E\{X^{2m-2}\}.$$

Then by recursion,

$$\begin{aligned} E\{X^{2m}\} &= E\{|X|^{2m}\} = (2m-1) E\{X^{2m-2}\} = (2m-1)(2m-3)\cdots 3 \cdot 1 \\ &= \frac{2m(2m-1)(2m-2)\cdots 3 \cdot 2 \cdot 1}{(2 \cdot m) \cdot (2(m-1)) \cdots (2 \cdot 1)} = \frac{(2m)!}{2^m \cdot m!}. \end{aligned}$$

**Example 3.5** Let  $X_1$  and  $X_2$  be independent random variables,  $X_i \in N(0, \sigma^2)$ ,  $i = 1, 2$ . Find the distribution function and the frequency of the random variable

$$Y = \sqrt{X_1^2 + X_2^2}.$$

Find also the mean and the variance of  $Y$ .

The distribution of  $Y$  is called the Rayleigh distribution.

If  $y \leq 0$ , then  $P\{Y \leq y\} = 0$ . If instead  $y > 0$ , then we apply that  $X_1$  and  $X_2$  are independent and identically distributed, and use polar coordinates,

$$\begin{aligned} P\{Y \leq y\} &= P\{X_1^2 + X_2^2 \leq y^2\} = \frac{1}{2\pi\sigma^2} \int_{\{x_1^2+x_2^2 \leq y^2\}} \exp\left(-\frac{1}{2\sigma^2}(x_1^2+x_2^2)\right) dx_1 dx_2 \\ &= \frac{1}{2\pi\sigma^2} \cdot 2\pi \int_0^y r \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right). \end{aligned}$$

Here we have used that the simultaneous frequency of  $(X_1, X_2)$  is  $\varphi(x_1) \cdot \varphi(x_2)$ .

Summing up,

$$F_Y(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right) & \text{for } y > 0, \end{cases}$$

and

$$f_Y(y) = \begin{cases} 0 & \text{for } y < 0, \\ \frac{y}{\sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right) & \text{for } y \geq 0. \end{cases}$$

The mean is

$$E\{Y\} = \int_0^\infty \frac{y^2}{\sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = \sigma \int_0^\infty z^2 \exp\left(-\frac{z^2}{2}\right) dz = \sigma \sqrt{\frac{\pi}{2}}.$$

Since

$$E\{Y^2\} = E\{X_1^2\} + E\{X_2^2\} = 2\sigma^2,$$

we get the variance

$$V\{Y\} = E\{Y^2\} - (E\{Y\})^2 = 2\sigma^2 - \sigma^2 \cdot \frac{\pi}{2} = \left(2 - \frac{\pi}{2}\right) \sigma^2.$$

**Example 3.6** Let  $X_1$  and  $X_2$  be independent and identically distributed random variables of the frequency

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{x}{2}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

- 1) Find the frequency of  $Y = X_1 + X_2$ .
- 2) Prove that if  $Z$  is normally distributed of mean 0 and variance 1, then  $Z^2$  has the frequency  $f(x)$ .
- 3) Find mean and variance of  $Y$ .

1) If  $s > 0$ , then the frequency of  $Y = X_1 + X_2$  is

$$\begin{aligned} h(s) &= \int_0^s \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{x}{2}\right) \cdot \frac{1}{\sqrt{2\pi(s-x)}} \exp\left(-\frac{1}{2}(s-x)\right) dx \\ &= \frac{1}{2\pi} \exp\left(-\frac{s}{2}\right) \int_0^s \frac{dx}{\sqrt{x(s-x)}} = \frac{1}{2\pi} \exp\left(-\frac{s}{2}\right) \int_0^s \frac{\frac{1}{s} dx}{\sqrt{\frac{x}{s} \left(1 - \frac{x}{s}\right)}} \\ &= \frac{1}{2} \exp\left(-\frac{s}{2}\right) \cdot \frac{1}{\pi} \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = C \cdot \frac{1}{2} \exp\left(-\frac{1}{2}s\right) = \frac{1}{2} \exp\left(-\frac{1}{2}s\right), \end{aligned}$$

because  $\int_0^\infty h(s) ds = 1$ , so  $C = 1$ . Hence

$$h(s) = \begin{cases} \frac{1}{2} \exp\left(-\frac{s}{2}\right) & \text{for } s > 0, \\ 0 & \text{for } s \leq 0, \end{cases}$$

and it follows that  $Y = X_1 + X_2$  is exponentially distributed.

2) Clearly,  $F_{Z^2}(x) = 0$  for  $x \leq 0$ . If  $x > 0$ , then

$$F_{Z^2}(x) = P\{Z^2 \leq x\} = P\{-\sqrt{x} \leq Z \leq \sqrt{x}\} = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}).$$

When  $z > 0$ , the frequency is obtained by a differentiation,

$$f_{Z^2} = \varphi(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - \varphi(-\sqrt{x}) \cdot \left(-\frac{1}{2\sqrt{x}}\right) = \frac{1}{\sqrt{x}} \varphi(\sqrt{x}) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{1}{2}x\right),$$

and it follows from  $f_{Z^2}(x) = 0$  for  $x \leq 0$ , that  $f_{Z^2}(x) = f(x)$ .

3) Since  $Y$  is exponentially distributed with  $a = \frac{1}{2}$ , we get

$$E\{Y\} = \frac{1}{a} = 2 \quad \text{and} \quad V\{Y\} = \frac{1}{a^2} = 4.$$

**Remark 3.1** Since  $X_1$  and  $X_2$  are independent and identically distributed, we have

$$E\{Y\} = E\{X_1\} + E\{X_2\} \quad \text{og} \quad V\{Y\} = V\{X_1\} + V\{X_2\},$$

hence

$$E\{X_1\} = E\{X_2\} = 1 \quad \text{og} \quad V\{X_1\} = V\{X_2\} = 2. \quad \diamond$$

**Example 3.7** Let  $(X_1, X_2)$  be a two-dimensional random variable of the frequency

$$h(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right).$$

Let the two-dimensional random variable  $(Y_1, Y_2)$  be given by

$$X_1 = \sqrt{-2 \ln Y_1} \cdot \cos(2\pi Y_2), \quad X_2 = \sqrt{-2 \ln Y_1} \cdot \sin(2\pi Y_2),$$

where  $0 < Y_1 < 1$ ,  $0 < Y_2 < 1$ .

Find the frequency of  $(Y_1, Y_2)$ .

Are  $Y_1$  and  $Y_2$  independent?

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{1}{2} \frac{1}{\sqrt{-2 \ln y_1}} \left\{-\frac{2}{y_1}\right\} \cos(2\pi y_2) & -2\pi \sqrt{-2 \ln y_1} \cdot \sin(2\pi y_2) \\ \frac{1}{2} \frac{1}{\sqrt{-2 \ln y_1}} \left\{-\frac{2}{y_1}\right\} \sin(2\pi y_2) & 2\pi \sqrt{-2 \ln y_1} \cdot \cos(2\pi y_2) \end{vmatrix} = -\frac{2\pi}{y_1}.$$

We get from  $x_1^2 + x_2^2 = -2 \ln y_1$  that the frequency of  $(Y_1, Y_2)$  is

$$k(y_1, y_2) = \frac{1}{2\pi} \exp(\ln y_1) \cdot \left|-\frac{2\pi}{y_1}\right| = 1 \quad \text{for } (y_1, y_2) \in ]0, 1[ \times ]0, 1[.$$

It follows immediately that the marginal frequencies are

$$K_{Y_i}(y_i) = 1 \quad \text{for } y_i \in ]0, 1[, \quad i = 1, 2,$$

and  $k_{Y_i}(y_i) = 0$  otherwise. Hence

$$k(y_1, y_2) = k_{Y_1}(y_1) \cdot k_{Y_2}(y_2),$$

so  $Y_1$  and  $Y_2$  are independent.

**Example 3.8** A random variable  $U \in N(0, 1)$  is normally distributed of mean 0 and variance 1.

1. Prove that  $E\{U^4\} = 3$ .

A two-dimensional random variable  $(X, Y)$  is following a two-dimensional normal distribution with

$$E\{X\} = E\{Y\} = 0, \quad V\{X\} = V\{Y\} = \sigma^2 > 0, \quad \varrho(X, Y) = \varrho, \quad (|\varrho| < 1),$$

i.e.  $(X, Y) \in N(0, 0, \sigma^2, \sigma^2, \varrho)$ .

2. Compute the real constant  $a$ , such that the random variables  $X - aY$  and  $Y$  are non-correlated.

3. Explain why the random variables  $X - aY$  and  $Y$  are stochastically independent for the value of  $a$  found in 2.

4. Compute  $E\{X^2Y^2\}$  by using that  $X = (X - aY) + aY$ .

1) Since  $\varphi'(u) = -u\varphi(u)$ , we get by partial integration,

$$\begin{aligned} E\{U^4\} &= \int_{-\infty}^{\infty} u^4 \varphi(u) du = - \int_{-\infty}^{\infty} u^3 \varphi'(u) du \\ &= [-u^3 \varphi(u)]_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} u^2 \varphi(u) du = 0 - 3 \int_{-\infty}^{\infty} u \cdot \varphi'(u) du \\ &= [-3u \varphi(u)]_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} 1 \cdot \varphi(u) du = 0 + 3 = 3. \end{aligned}$$

2) Then by a direct computation,

$$\text{Cov}(X - aY, Y) = \text{Cov}(X, Y) - aV\{Y\} = \varrho \sqrt{V\{X\}V\{Y\}} - aV\{Y\} = (\varrho - a)\sigma^2.$$

This expression is equal to 0 for  $a = \varrho$ .

3) From  $E\{X\} = E\{Y\}$  follows that

$$\begin{aligned} V\{X - \varrho Y\} &= E\{(X - \varrho Y)^2\} = E\{X^2\} - 2\varrho E\{XY\} + E\{Y^2\} \\ &= V\{X\} - 2\varrho \text{Cov}(X, Y) + V\{Y\} = \sigma^2 + \sigma^2 - 2\varrho^2\sigma^2 \\ &= 2(1 - \varrho^2)\sigma^2, \end{aligned}$$

so the covariance matrix of  $(X - \varrho Y, Y)$  is given by

$$\mathbf{C} = \begin{pmatrix} 2(1 - \varrho^2)\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix},$$

and the frequency is

$$f(x, y) = \frac{1}{2\pi\sqrt{2(1-\rho^2)}\sigma^2} \exp\left(-\frac{1}{2}\left\{\left(\frac{x}{\sqrt{2(1-\rho^2)}\sigma}\right)^2 + \left(\frac{y}{\rho}\right)^2\right\}\right).$$

The structure of the frequency shows that  $X - \rho Y$  and  $Y$  are independent.

4) By using the given trick,

$$\begin{aligned} E\{X^2Y^2\} &= E\{(X - \rho Y) + \rho Y\}^2 Y^2 \\ &= E\{(X - \rho Y)^2 Y^2 + 2\rho(X - \rho Y)Y^3 + \rho^2 Y^4\} \\ &= E\{(X - \rho Y)^2\} E\{Y^2\} + 2\rho E\{X - \rho Y\} E\{Y^3\} + \rho^2 E\{Y^4\} \\ &= \{\sigma^2 + \rho^2\sigma^2 - 2\rho \cdot \rho\sigma^2\} \sigma^2 + 0 + \rho^2 \cdot 3\sigma^4 \\ &= \sigma^4(1 - \rho^2) + 3\rho^2\sigma^4 = \sigma^4(1 + 2\rho^2). \end{aligned}$$

Here we have applied that  $X - \rho Y$  and  $Y$  are independent and that

$$E\{Y^3\} = 0 \quad \text{and} \quad E\{Y^4\} = 3\sigma^4.$$

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**Example 3.9** Let  $X_1, X_2, \dots, X_n$  be mutually independent  $N(0,1)$ -distributed random variables, and let  $S_k = \sum_{i=1}^k X_i$ ,  $k < n$ .

1) Find the distribution of  $S_n$ .

2) Find the simultaneous distribution of  $(S_m, S_n)$ , when  $m < n$ .

1) From  $S_n \in N(0, n)$  follows that the distribution function is

$$F_n(s) = \Phi\left(\frac{s}{\sqrt{n}}\right).$$

2) Analogously we get  $S_m \in N(0, m)$ . The covariance matrix is

$$\mathbf{C} = \begin{pmatrix} V\{S_m\} & \text{Cov}(S_m, S_n) \\ \text{Cov}(S_m, S_n) & V\{S_n\} \end{pmatrix},$$

where  $V\{S_m\} = m$  and  $V\{S_n\} = n$ , and

$$\text{Cov}(S_m, S_n) = \text{Cov}\left(S_m, S_m + \sum_{i=m+1}^n X_i\right) = V\{S_m\} = m,$$

thus

$$\mathbf{C} = \begin{pmatrix} m & m \\ m & n \end{pmatrix}, \quad \det \mathbf{C} = m(n - m),$$

and hence

$$\mathbf{C}^{-1} = \frac{1}{m(n - m)} \begin{pmatrix} n & -m \\ -m & m \end{pmatrix},$$

so

$$\begin{aligned} (x, y) \mathbf{C}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{m(n - m)} (nx^2 - 2mxy + my^2) \\ &= \frac{1}{1 - \frac{m}{n}} \left\{ \left(\frac{x}{\sqrt{m}}\right)^2 - 2 \cdot \sqrt{\frac{m}{n}} \cdot \frac{x}{\sqrt{m}} \cdot \frac{y}{\sqrt{n}} + \left(\frac{y}{\sqrt{n}}\right)^2 \right\}. \end{aligned}$$

This result corresponds to

$$\sigma_1 = \sqrt{m}, \quad \sigma_2 = \sqrt{n} \quad \text{og} \quad \rho = \sqrt{\frac{m}{n}},$$

so the distribution is

$$(S_m, S_n) \in N\left(0, 0, m, n, \sqrt{\frac{m}{n}}\right),$$

and the frequency is

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi\sqrt{m(n - m)}} \exp\left(-\frac{nx^2 - 2mxy + my^2}{2m(n - m)}\right) \\ &= \frac{1}{2\pi\sqrt{m} \cdot \sqrt{n} \cdot \sqrt{1 - \frac{m}{n}}} \cdot \exp\left(-\frac{1}{2} \left\{ \left(\frac{x}{\sqrt{m}}\right)^2 - 2\sqrt{\frac{m}{n}} \cdot \frac{x}{\sqrt{m}} \cdot \frac{y}{\sqrt{n}} + \left(\frac{y}{\sqrt{n}}\right)^2 \right\}\right). \end{aligned}$$



**Example 3.10** Let  $(X_1, X_2)$  follow the two-dimensional normal distribution

$$(X_1, X_2) \in N(0, 0, \sigma_1^2, \sigma_2^2, \varrho),$$

and let the random variable  $Y$  be given by  $Y = X_1/X_2$ .

Prove that  $Y$  is Cauchy distributed of median  $\varrho \frac{\sigma_1}{\sigma_2}$ .

Find in particular the frequency in the case  $\sigma_1 = \sigma_2$ ,  $\varrho = 0$ .

Since  $(X_1, X_2) \in N(0, 0, \sigma_1^2, \sigma_2^2, \varrho)$ , we immediately get the frequency,

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}} \cdot \exp \left[ -\frac{1}{2(1-\varrho^2)} \left\{ \left( \frac{x_1}{\sigma_1} \right)^2 - 2\varrho \frac{x_1}{\sigma_1} \frac{x_2}{\sigma_2} + \left( \frac{x_2}{\sigma_2} \right)^2 \right\} \right].$$

Then by a known formula, the frequency of  $Y = X_1/X_2$  is given by

$$\begin{aligned} g(y) &= \int_{-\infty}^{\infty} f(yx, x) \cdot |x| dx \\ &= \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}} \cdot \int_0^{\infty} \exp \left[ -\frac{1}{2(1-\varrho^2)} \left\{ \frac{y^2}{\sigma_1^2} x^2 - 2\frac{\varrho y}{\sigma_1\sigma_2} x^2 + \frac{1}{\sigma_2^2} x^2 \right\} \right] |x| dx \\ &= \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}} \cdot \int_0^{\infty} \exp \left[ -\frac{1}{1-\varrho^2} \left\{ \left( \frac{y}{\sigma_1} - \frac{\varrho}{\sigma_2} \right)^2 + \left( \frac{\sqrt{1-\varrho^2}}{\sigma_2} \right)^2 \right\} \frac{x^2}{2} \right] x dx \\ &= \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}} \cdot \frac{1-\varrho^2}{\left( \frac{y}{\sigma_1} - \frac{\varrho}{\sigma_2} \right)^2 + \left( \frac{\sqrt{1-\varrho^2}}{\sigma_2} \right)^2} \\ &= \frac{1}{\pi} \cdot \frac{\sigma_1}{\sigma_2} \sqrt{1-\varrho^2} \cdot \frac{1}{\left( y - \frac{\varrho\sigma_1}{\sigma_2} \right)^2 + \left( \frac{\sigma_1}{\sigma_2} \sqrt{1-\varrho^2} \right)^2}, \quad y \in \mathbb{R}, \end{aligned}$$

which is the frequency of a Cauchy distribution of median  $\varrho \frac{\sigma_1}{\sigma_2}$ .

If  $\varrho = 0$  and  $\sigma_1 = \sigma_2 = \sigma$ , then

$$g(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}, \quad y \in \mathbb{R}.$$

**Example 3.11** Let  $X_1$  and  $X_2$  be independent  $N(0, 1)$ -distributed random variables, i.e.  $(X_1, X_2)$  has the two-dimensional frequency

$$h(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Let the random variables  $Y_1$  and  $Y_2$  be given by

$$Y_1 = \frac{X_1}{X_2}, \quad Y_2 = \sqrt{X_1^2 + X_2^2}.$$

- 1) Find frequency and distribution function of  $Y_1$ .
- 2) Find frequency and distribution function of  $Y_2$ .
- 3) Compute for  $y_1 \in \mathbb{R}$  and  $y_2 > 0$  the probability

$$P\{Y_1 \leq y_1, Y_2 \leq y_2\}.$$

HINT: Express the probability as a plane integral of  $h$  over a suitable set, and then compute the integral by using polar coordinates.

- 4) Find the simultaneous frequency of  $(Y_1, Y_2)$ .

- 1) By a standard formula the frequency of  $X/Y$  is given by

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(z^2 + 1)x^2\right) \cdot |x| dx = \frac{1}{\pi} \int_0^{\infty} \exp\left(-\frac{1}{2}(z^2 + 1)\frac{x^2}{2}\right) \cdot x dx \\ &= \frac{1}{\pi} \cdot \frac{1}{z^2 + 1}, \end{aligned}$$

thus we have a Cauchy-distribution, and the distribution function is

$$F(z) = \frac{1}{\pi} [\text{Arctan } t]_{-\infty}^z = \frac{1}{\pi} \text{Arctan } z + \frac{1}{2}.$$

- 2) The distribution function of  $Y_2$  is 0 for  $r \leq 0$ . If  $r > 0$ , then

$$\begin{aligned} G(r) &= P\{Y_2 \leq r\} = \int_{\{\sqrt{x_1^2 + x_2^2} \leq r\}} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right) dx_1 dx_2 \\ &= \frac{1}{2\pi} \cdot 2\pi \int_0^r \exp\left(-\frac{1}{2}\varrho^2\right) \varrho d\varrho = \left[-\exp\left(-\frac{1}{2}\varrho^2\right)\right]_0^r = 1 - \exp\left(-\frac{1}{2}r^2\right), \end{aligned}$$

so the distribution function is

$$G(r) = \begin{cases} 0 & \text{for } r \leq 0, \\ 1 - \exp\left(-\frac{1}{2}r^2\right) & \text{for } r > 0. \end{cases}$$

The corresponding frequency is

$$g(r) = \begin{cases} 0 & \text{for } r \leq 0, \\ r \exp\left(-\frac{1}{2}r^2\right) & \text{for } r > 0. \end{cases}$$

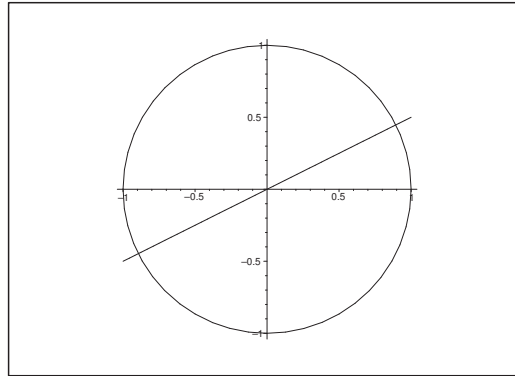



Figure 1: When  $y_1 = 2$ , the domain of integration is the union of two circular sections of the complementary angle  $\varphi_0 = \text{Arccot } y_1$ .

- 3) It follows by a geometrical analysis that  $P\{Y_1 \leq y_1, Y_2 \leq y_2\}$  is the integral of  $h(x_1, x_2)$  over the union of two circular sections of the same angle. The upper circular section has a part of the negative  $x_1$ -axis as part of its boundary, and the complementary angle is

$$\varphi_0 = \text{Arccot } y_1.$$

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We get by using polar coordinates

$$\begin{aligned} P\{Y_1 \leq y_1, Y_2 \leq y_2\} &= 2 \int_{\varphi_0}^{\pi} \left\{ \int_0^{y_2} \frac{1}{2\pi} \exp\left(-\frac{1}{2} \varrho^2\right) \varrho d\varrho \right\} d\varphi \\ &= \left\{ 1 - \exp\left(-\frac{1}{2} y_2^2\right) \right\} \cdot \left\{ 1 - \frac{1}{\pi} \operatorname{Arccot} y_1 \right\}. \end{aligned}$$

4) The simultaneous frequency of  $(Y_1, Y_2)$  is

$$k(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \frac{1}{\pi} \cdot \frac{1}{1+y_1^2} \cdot y_2 \exp\left(-\frac{1}{2} y_2^2\right).$$

It follows that  $Y_1$  and  $Y_2$  are stochastically independent.

**Example 3.12** Give an example of a two-dimensional random variable  $(X, y)$  which has the following three properties:

- 1)  $X$  is normally distributed,  $X \in N(0, 1)$ ,
- 2)  $Y$  is normally distributed,  $Y \in N(0, 1)$ ,
- 3)  $(X, Y)$  does not follow a two-dimensional normal distribution.

HINT: Try to find a frequency  $f(x, y)$ , which is 0, if  $xy < 0$ .

If we put

$$f(x, y) = \begin{cases} \frac{1}{\pi} \exp\left(-\frac{1}{2}(x^2 + y^2)\right) & \text{for } xy \geq 0, \\ 0 & \text{for } xy < 0, \end{cases}$$

then all three conditions are satisfied.

**Example 3.13** Let  $X$  and  $Y$  be independent random variables of the same frequency  $f$ , which is positive and continuously differentiable, and assume that

$$(4) \quad f(x) f(y) = g(x^2 + y^2) \quad \text{for every } (x, y).$$

(The meaning of (4) is that the distribution of  $(X, Y)$  is rotational invariant).

Prove that

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

hence,  $X$  and  $Y$  are normally distributed.

HINT: Differentiate (4) with respect to  $x$  and derive that

$$\frac{f'(x)}{f(x)} = \text{constant} \cdot x.$$

When (4) is differentiated with respect to  $x$  and  $y$ , resp., we get

$$f'(x)f(y) = 2x g'(x^2 + y^2) \quad \text{and} \quad f(x)f'(y) = 2y g'(x^2 + y^2).$$

If  $x \neq 0$  and  $y \neq 0$ , we get

$$g'(x^2 + y^2) = \frac{1}{2x} f'(x)f(y) = \frac{1}{2y} f(x)f'(y).$$

Since  $f(x) \cdot f(y) \neq 0$ , it follows by separating the variables that

$$\frac{1}{x} \frac{f'(x)}{f(x)} = \frac{1}{y} \frac{f'(y)}{f(y)}.$$

The left hand side only depends on  $x$ , and the right hand side only depends on  $y$ , and since they are equal, they must be equal to a constant  $c$ , thus

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x) = c \cdot x,$$

and whence by integration,

$$f(x) = k \cdot \exp\left(c \frac{1}{2} x^2\right) = k \cdot \exp\left(-\frac{1}{2} \left\{\frac{x}{\sigma}\right\}^2\right) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \left\{\frac{x}{\sigma}\right\}^2\right).$$

In fact,

- 1) since  $\int_{-\infty}^{\infty} f(x) dx < \infty$ , we see that  $c = -\frac{1}{\sigma^2}$  must be negative, and
- 2) since  $\int_{-\infty}^{\infty} f(x) dx = 1$ , we have  $k = \frac{1}{\sqrt{2\pi} \sigma}$ , and
- 3)  $f(0) = \frac{1}{\sqrt{2\pi} \sigma}$  by an continuous extension.

## 4 The Central Limit Theorem

**Example 4.1** Prove that

$$e^{-n} \sum_{k=0}^n \frac{n^k}{k!} \rightarrow \frac{1}{2} \quad \text{for } n \rightarrow \infty.$$

HINT: Apply the Central Limit Theorem on a sequence of independent Poisson distributed random variables.

Let  $X_n$ ,  $n \in \mathbb{N}$  be independent Poisson distributed random variables with  $\lambda = 1$ , i.e.

$$P\{X_n = k\} = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{1}{k!} \cdot e^{-1}, \quad k \in \mathbb{N}_0.$$

The Poisson distribution is reproductive, so  $Y_n = X_1 + \cdots + X_n$  is also Poisson distributed, and  $Y_n \in P(1 + \cdots + 1) = P(n)$ . This means that the distribution function of  $Y_n$  is

$$P\{Y_n = k\} = \frac{n^k}{k!} e^{-n}, \quad k \in \mathbb{N}_0,$$

hence

$$E\{Y_n\} = \lambda_n = n \quad \text{and} \quad s_n^2 = V\{Y_n\} = n.$$

Then by the Central Limit Theorem,

$$\lim_{n \rightarrow \infty} P\left\{\frac{Y_n - E\{Y_n\}}{s_n} \leq x\right\} = \Phi(x) \quad \text{for every } x \in \mathbb{R}.$$

This means that

$$\begin{aligned} \lim_{n \rightarrow \infty} P\left\{\frac{Y_n - n}{\sqrt{n}} \leq x\right\} &= \lim_{n \rightarrow \infty} P\{Y_n \leq n + x\sqrt{n}\} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n+[x\sqrt{n}]} P\{Y_n = k\} \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n+[x\sqrt{n}]} P\{Y_n = k\} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n+[x\sqrt{n}]} \frac{n^k}{k!} e^{-n} = \Phi(x). \end{aligned}$$

If we choose  $x = 0$ , then

$$\lim_{n \rightarrow \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!} = \Phi(0) = \frac{1}{2}.$$

**Example 4.2** A coin is thrown 10,000 gänge, where we get the result head in 4979 of the throws. Using this information one wants to judge if the coin may be considered as “honest” (i.e. the probability is  $\frac{1}{2}$  for heads). One may follow the following procedure:

- 1) Assuming that the coin is honest, apply the Central Limit Theorem to find the probability that the number of heads,  $X$ , lies between 4979 and 5021 (both numbers included).
- 2) Accepting the hypothesis that the coin is honest if the probability above is  $\leq 0.95$ , check if the coin can be considered as honest.
- 3) Repeat the test with another coin, in which case we get  $5000 - N$  heads among the 10,000 throws. Find the largest number  $N$ , for which the hypothesis that this coin is honest will be accepted by the method described above.

- 1) When we assume that the coin is “honest”, then the number  $X$  of heads is binomially distributed,  $X \in B\left(10,000, \frac{1}{2}\right)$  with

$$E\{X\} = 5000 \quad \text{and} \quad V\{X\} = 10,000 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2500 = 50^2.$$

By the *Central Limit Theorem*  $X$  is approximated by

$$Y \in N(5000, 50^2),$$

thus

$$\begin{aligned} P\{4979 \leq X \leq 5021\} &\approx P\{4978.5 \leq Y \leq 5021.5\} = P\{-21.5 \leq Y - 5000 \leq 21.5\} \\ &= P\left\{-0.43 \leq \frac{Y - 5000}{50} \leq 0.43\right\} = 2\Phi(0.43) - 1 = 0.3328. \end{aligned}$$

- 2) Since  $0.3328 < 0.95$ , we shall *accept the hypothesis that the coin is honest*.
- 3) We find in the same way as in (1) that

$$\begin{aligned} P\{5000 - N \leq X \leq 5000 + N\} &\approx P\left\{5000 - N - \frac{1}{2} \leq Y \leq 5000 + N + \frac{1}{2}\right\} \\ &= P\left\{\left|\frac{Y - 5000}{50}\right| \leq \frac{N + \frac{1}{2}}{50}\right\} = 2\Phi\left(\frac{N + \frac{1}{2}}{50}\right) - 1. \end{aligned}$$

We shall find the largest integer  $N$ , for which

$$2\Phi\left(\frac{N + \frac{1}{2}}{50}\right) - 1 \leq 0.95,$$

thus

$$\Phi\left(\frac{N + \frac{1}{2}}{50}\right) \leq 0.975.$$

By using tables we get

$$\frac{N + \frac{1}{2}}{50} \leq 1.96, \quad \text{i.e.} \quad N + \frac{1}{2} \leq 98.$$

the largest integer  $N$ , for which the hypothesis is accepted, is 97.

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**Example 4.3** Given the random variables  $X_1, X_2, \dots, X_{900}$ , which are mutually independent of the distribution

$$P\{X_i = 1\} = \frac{1}{10}, \quad P\{X_i = 0\} = \frac{9}{10}, \quad i = 1, 2, \dots, 900.$$

Put

$$X = \sum_{i=1}^{900} X_i.$$

- 1) Find the distribution of  $X$ .
- 2) Find mean and variance of  $X$ .
- 3) Apply the Central Limit Theorem to find the largest integer  $M$ , for which

$$P\{90 - M \leq X \leq 90 + M\} \leq 0.95.$$

- 4) In a given table of “random numbers” there is a page of 900 numbers, of which 70 are zeros, which apparently gives a too small number of zeros, if the numbers should have been chosen “randomly” (because we would have expected a number close to 90).

Explain why it is reasonable to reject the hypothesis of randomness if the number 70 does not belong to the interval  $[90 - M, 90 + M]$ , where  $M$  is given by (3).

Shall one in the present case reject the hypothesis of randomness?

- 1) Clearly,

$$X = \sum_{i=1}^{900} X_i \in B\left(900, \frac{1}{10}\right).$$

- 2) Since the distribution is known, it follows immediately that

$$E\{X\} = 90, \quad V\{X\} = 81 = 9^2.$$

- 3) Choosing  $Y \in N(90, 9^2)$ , cf. (2), it follows from the Central Limit Theorem that

$$P\{90 - M \leq X \leq 90 + M\}$$

is approximately equal to

$$P\left\{90 - M - \frac{1}{2} \leq Y \leq 90 + M + \frac{1}{2}\right\} = P\left\{\left|\frac{Y - 90}{9}\right| \leq \frac{M + \frac{1}{2}}{9}\right\} = 2\Phi\left(\frac{M + \frac{1}{2}}{9}\right) - 1.$$

This is again smaller than or equal to 0.95, if

$$\Phi\left(\frac{M + \frac{1}{2}}{9}\right) \leq 0.975, \quad \text{i.e.} \quad \frac{M + \frac{1}{2}}{9} \leq 1.96,$$

hence  $M \leq 17$ .

Thus the largest integer is  $M = 17$ .

- 4) If we have randomness, then the probability of being *outside* the interval  $[90 - M, 90 + M]$  smaller than 0.05.

Since one does not believe in an event of so small probability, the hypothesis is rejected.

We have in the case under consideration that

$$[90 - M, 90 + M] = [73, 107].$$

Since 70 does not lie in this interval, we *reject* the hypothesis of randomness.

**Example 4.4** A schoolteacher wants to demonstrate to his pupils that one by throwing a coin many times will obtain heads in approximately half of the throws. They agree to throw the coin 100 times and consider the result as acceptable if the number of heads,  $X$ , satisfies the inequalities  $48 \leq X \leq 52$ . Considering results obtained by an application of the Central Limit Theorem one shall

- 1) Compute  $P\{48 \leq X \leq 52\}$ .
- 2) Find the smallest integer  $M$ , such that

$$P\{50 - M \leq X \leq 50 + M\} \geq 0.95.$$

- 3) They then decide to throw the coin  $50N$  times and consider the result as acceptable if the number of heads,  $Y$ , satisfies

$$24N \leq Y \leq 26N.$$

Find by using the Central Limit Theorem the smallest integer for which the probability of the event above is  $\geq 0.95$ .

- 1) Since  $X \in B\left(100, \frac{1}{2}\right)$ , we see that  $X$  can be approximated by  $N(50, 25)$ , so

$$\begin{aligned} P\{48 \leq X \leq 52\} &= F\left(52 \frac{1}{2}\right) - F\left(47 \frac{1}{2}\right) \approx \Phi\left(\frac{5}{5}\right) - \Phi\left(-\frac{5}{5}\right) = 2\Phi\left(\frac{1}{2}\right) - 1 \\ &= 2 \cdot 0.6915 - 1 = 0.3830. \end{aligned}$$

**Remark 4.1** We get by using a pocket calculator

$$\begin{aligned} P\{48 \leq X \leq 52\} &= \left\{ \binom{100}{48} + \binom{100}{49} + \binom{100}{50} + \binom{100}{51} + \binom{100}{52} \right\} \frac{1}{2^{100}} \\ &= \left\{ 2 \binom{100}{48} + 2 \binom{100}{49} + \binom{100}{50} \right\} \cdot \frac{1}{2^{100}} = 0,38270. \end{aligned}$$

so we see that the approximation above is indeed a good one.  $\diamond$

2) Since  $\Phi(1.96) = 0.975$ , and

$$P\{50 - M \leq X \leq 50 + M\} \approx \Phi\left(\frac{M + \frac{1}{2}}{5}\right) - \Phi\left(-\frac{M + \frac{1}{2}}{5}\right) = 2\Phi\left(\frac{2M + 1}{10}\right) - 1 \geq 0.95$$

for

$$\frac{2M + 1}{10} \geq 1.96, \quad \text{i.e.} \quad M \geq 9.3,$$

we conclude that  $M = 10$  is the smallest integer for which

$$P\{50 - M \leq X \leq 50 + M\} \geq 0.95.$$

3) Since  $Y \in B\left(50N, \frac{1}{2}\right)$  can be approximated by an  $N\left(25N, 25 \cdot \frac{N}{2}\right)$ -distribution with  $s_N = 5\sqrt{\frac{N}{2}}$ , it follows that

$$P\{24N \leq Y \leq 26N\} = F\left(26N + \frac{1}{2}\right) - F\left(24N - \frac{1}{2}\right) \approx 2\Phi\left(\frac{2N + 1}{5\sqrt{2N}}\right) - 1.$$

We get in the same way as in **(2)** the condition

$$\frac{2N + 1}{5\sqrt{2N}} \geq 1.96,$$

which can also be written as an inequality containing a polynomial of second order in  $\sqrt{2N}$ ,

$$\left(\sqrt{2N}\right)^2 - 9.8\sqrt{2N} + 1 \geq 0.$$

The roots of the equation  $z^2 - 9.8z + 1 = 0$  are  $z = 4.9 \pm \sqrt{4.9^2 - 1}$ . Now,

$$4.9 - \sqrt{4.9^2 - 1} = 0.10313 < 1,$$

so only  $+$  can be used. Hence,

$$2N \geq z^2 = 4.9^2 + 4.9^2 - 1 + 2 \cdot 4.9\sqrt{4.9^2 - 1} = 2 \cdot 4.9^2 - 1 + 9.8\sqrt{4.9^2 - 1},$$

from which

$$N \geq 4.9^2 - 0.5 + 4.9\sqrt{4.9^2 - 1} = 47.015.$$

The smallest integer, which fulfils this condition is  $N = 48$ .

**Example 4.5** Let  $X_1, X_2, \dots$  be independent identically distributed random variables of mean  $\mu$  and positive variance  $\sigma^2$ , and let

$$Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N}.$$

Find for every  $x > 0$  by using the Central Limit Theorem

$$\lim_{n \rightarrow \infty} P \left\{ \left| \frac{Y_n - n\mu}{\sqrt{n}} \right| \leq x \right\} \quad \text{and} \quad \lim_{n \rightarrow \infty} P \left\{ \left( \frac{Y_n - n\mu}{\sqrt{n}} \right)^2 \leq x \right\}.$$

The results shall be expressed in the distribution function  $\Phi(x)$ .

It follows from

$$E \{Y_n\} = \sum_{i=1}^n E \{X_i\} = n\mu$$

and

$$V \{Y_n\} = \sum_{i=1}^n V \{X_i\} = n\sigma^2, \quad \text{i.e. } s_n = \sigma\sqrt{n},$$

by the Central Limit Theorem that

$$\lim_{n \rightarrow \infty} P \left\{ \left| \frac{Y_n - n\mu}{\sqrt{n}} \right| \leq x \right\} = \lim_{n \rightarrow \infty} P \left\{ -\frac{x}{\sigma} \leq \frac{Y_n - n\mu}{\sqrt{n}} \leq \frac{x}{\sigma} \right\} = \begin{cases} 2\Phi\left(\frac{x}{\sigma}\right) - 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

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Now,

$$\lim_{n \rightarrow \infty} P \left\{ \left( \frac{Y_n - n\mu}{\sqrt{n}} \right)^2 \leq x \right\} = \lim_{n \rightarrow \infty} P \left\{ \left| \frac{Y_n - n\mu}{\sqrt{n}} \right| \leq \sqrt{x} \right\},$$

so it follows from the above that

$$\lim_{n \rightarrow \infty} P \left\{ \left( \frac{Y_n - n\mu}{\sqrt{n}} \right)^2 \leq x \right\} = \begin{cases} 2\Phi\left(\frac{\sqrt{x}}{\sigma}\right) - 1, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

**Example 4.6** Given a roulette, where the possible event of each game is either red, black or green of the probabilities  $r$ ,  $b$  and  $g$ , where

$$r > 0, \quad b > 0, \quad g > 0 \quad \text{and} \quad r + b + g = 1.$$

When we continually play on this roulette we assume that the games are independent.

Let  $n \in \mathbb{N}$  be a fixed number, and let  $X_n$ ,  $Y_n$  and  $Z_n$  denote the number of games among the  $n$  games which results in either red, black or green.

- 1) Find the distribution functions of  $X_n$ ,  $Y_n$  and  $Z_n$ .
- 2) Find the variances  $V\{X_n\}$ ,  $V\{Y_n\}$  and  $V\{Z_n\}$ .
- 3) Find, e.g. by using that  $X_n + Y_n + Z_n = n$ , the variance  $V\{X_n + Y_n\}$ .
- 4) Compute the correlation  $\rho(X_n, Y_n)$ .
- 5) Allowing  $n$  to vary, we shall prove that the sequence  $\left(\frac{1}{n} X_n\right)$  for  $n \rightarrow \infty$  converges in probability towards the constant  $r$ , and that the sequence  $\left(\frac{1}{n} Y_n\right)$  converges towards the constant  $b$ .
- 6) Prove for any  $\varepsilon \in \mathbb{R}_+$  and any  $a > \frac{1}{2}$ ,

$$P\{|X_n - nr| < \varepsilon n^a \text{ and } |Y_n - nb| < \varepsilon n^a\} \rightarrow 1 \quad \text{for } n \rightarrow \infty.$$

- 1) Clearly,  $X_n \in B(n, r)$ ,  $Y_n \in B(n, b)$  and  $Z_n \in B(n, g)$  are all binomially distributed with

$$P\{X_n = k\} = \binom{n}{k} r^k (1-r)^{n-k}, \quad k = 0, 1, 2, \dots, n,$$

$$P\{Y_n = k\} = \binom{n}{k} b^k (1-b)^{n-k}, \quad k = 0, 1, 2, \dots, n,$$

$$P\{Z_n = k\} = \binom{n}{k} g^k (1-g)^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

2) The variance of  $U \in B(n, p)$  is  $V\{U\} = np(1 - p)$ , hence

$$V\{X_n\} = nr(1 - r), \quad V\{Y_n\} = nb(1 - b), \quad V\{Z_n\} = ng(1 - g).$$

3) It follows from  $X_n + Y_n + Z_n = n$  that

$$\begin{aligned} V\{X_n + Y_n\} &= V\{n - Z_n\} = V\{n\} + V\{Z_n\} = 0 + ng(1 - g) \\ &= ng(1 - g) = n(1 - r - b)(r + b). \end{aligned}$$

4) From

$$\begin{aligned} X_n Y_n &= \frac{1}{2} \left\{ (X_n + Y_n)^2 - X_n^2 - Y_n^2 \right\} = \frac{1}{2} \left\{ (n - Z_n)^2 - X_n^2 - Y_n^2 \right\} \\ &= \frac{1}{2} \left\{ Z_n^2 - 2n Z_n + n^2 - X_n^2 - Y_n^2 \right\}, \end{aligned}$$

follows that

$$\begin{aligned} \text{Cov}(X_n, Y_n) &= E\{X_n Y_n\} - E\{X_n\} E\{Y_n\} \\ &= \frac{1}{2} E\{Z_n^2\} - n E\{X_n\} + \frac{n^2}{2} - \frac{1}{2} E\{X_n^2\} - \frac{1}{2} E\{Y_n^2\} - E\{X_n\} E\{Y_n\} \\ &= \frac{1}{2} \left[ E\{Z_n^2\} - (E\{Z_n\})^2 \right] + \frac{1}{2} \left[ (E\{Z_n\})^2 - 2n E\{Z_n\} + n^2 \right] \\ &\quad - \frac{1}{2} \left[ E\{X_n^2\} - (E\{X_n\})^2 \right] - \frac{1}{2} (E\{X_n\})^2 \\ &\quad - \frac{1}{2} \left[ E\{Y_n^2\} - (E\{Y_n\})^2 \right] - \frac{1}{2} (E\{Y_n\})^2 - E\{X_n\} E\{Y_n\} \\ &= \frac{1}{2} V\{Z_n\} + \frac{1}{2} (n - E\{Z_n\})^2 - \frac{1}{2} V\{X_n\} - \frac{1}{2} (E\{X_n\})^2 \\ &\quad - \frac{1}{2} V\{Y_n\} - \frac{1}{2} (E\{Y_n\})^2 - E\{X_n\} E\{Y_n\} \\ &= \frac{1}{2} V\{Z_n\} - \frac{1}{2} V\{X_n\} - \frac{1}{2} V\{Y_n\} \\ &\quad + \frac{1}{2} (n - E\{Z_n\})^2 - \frac{1}{2} (E\{X_n\} + E\{Y_n\})^2 \\ &= \frac{1}{2} V\{Z_n\} - \frac{1}{2} V\{X_n\} - \frac{1}{2} V\{Y_n\} \\ &= \frac{n}{2} \{g(1 - g) - r(1 - r) - b(1 - b)\} \\ &= \frac{n}{2} \{r + b - r^2 - b^2 - 2rb - r + r^2 - b + b^2\} \\ &= -nr b. \end{aligned}$$

Since

$$\sqrt{V\{X_n\} V\{Y_n\}} = n\sqrt{rb(1 - r)(1 - b)},$$

the correlation is

$$\rho = \frac{-nr b}{n\sqrt{rb(1 - r)(1 - b)}} = -\sqrt{\frac{r \cdot b}{(1 - r)(1 - b)}}.$$

ALTERNATIVELY it is immediately seen that

$$\begin{aligned} \text{Cov}(X_n, Y_n) &= \frac{1}{2} (V\{X_n + Y_n\} - V\{X_n\} - V\{Y_n\}) \\ &= \frac{n}{2} \{gr + gb - rb - rg - br - bg\} = -nr b. \end{aligned}$$

5) It suffices to prove that  $\left(\frac{1}{n} X_n\right) \xrightarrow{D} r$  (in distribution), because the proof for  $\left(\frac{1}{n} Y_n\right) \xrightarrow{D} b$  is analogous.

Since

$$E\left\{\frac{1}{n} X_n\right\} = r \quad \text{and} \quad V\left\{\frac{1}{n} X_n\right\} = \frac{r(1-r)}{n} = s_n,$$

it follows from the Central Limit Theorem that

$$\lim_{n \rightarrow \infty} P\left\{\frac{\frac{1}{n} X_n - r}{\sqrt{\frac{r(1-r)}{n}}} \leq y\right\} = \lim_{n \rightarrow \infty} P\left\{X_n - nr \leq \sqrt{n} \cdot \sqrt{r(1-r)} \cdot y\right\}.$$

Now,

$$\begin{aligned} F_n(x) &= P\left\{\frac{1}{n} X_n \leq x\right\} = P\{X_n - nr \leq nx - nr\} \\ &= P\left\{X_n - nr \leq \sqrt{n} \cdot \sqrt{r(1-r)} \cdot \frac{\sqrt{n} \cdot (x - r)}{\sqrt{r(1-r)}}\right\}. \end{aligned}$$

hence

$$\lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \Phi\left(\frac{\sqrt{n}}{\sqrt{r(1-r)}}(x - r)\right) = \begin{cases} 0 & \text{for } x < r, \\ 1 & \text{for } x > r, \end{cases}$$

and the claim follows.

ALTERNATIVELY one may use Chebyshev's inequality. We have

$$\begin{aligned} P\left\{\left|\frac{1}{n} X_n - r\right| > \varepsilon\right\} &= P\{|X_n - nr| > n\varepsilon\} \leq \frac{nr(b+g)}{n^2\varepsilon^2} \\ &= \frac{1}{n} r(b+g) \cdot \frac{1}{\varepsilon^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty, \end{aligned}$$

proving that  $\left(\frac{1}{n} X_n\right)$  converges in probability towards  $r$ .

It follows similarly that  $\left(\frac{1}{n} Y_n\right)$  converges in probability towards  $b$ .

6) If  $a > \frac{1}{2}$  and  $\varepsilon > 0$ , then

$$P\{|X_n - nr| \leq \varepsilon n^a\} = P\left\{|X_n - nr| \leq \sqrt{n} \cdot \sqrt{r(1-r)} \cdot \frac{\varepsilon}{\sqrt{r(1-r)}} \cdot n^{a-\frac{1}{2}}\right\}$$

$$\approx \Phi\left(\frac{\varepsilon}{\sqrt{r(1-r)}} \cdot n^{a-\frac{1}{2}}\right) \rightarrow 1 \quad \text{for } n \rightarrow \infty.$$

Analogously,

$$P\{|Y_n - nb| < \varepsilon n^a\} \approx \Phi\left(\frac{\varepsilon}{\sqrt{b(1-b)}} n^{a-\frac{1}{2}}\right) \rightarrow 1 \quad \text{for } n \rightarrow \infty.$$

Hence, for every  $\delta > 0$  there is an  $n_0$ , such that

$$P\{|X_n - nr| < \varepsilon n^a\} \geq 1 - \frac{\delta}{2}$$

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and

$$P\{|Y_n - nb| < \varepsilon n^a\} \geq 1 - \frac{\delta}{2}$$

for  $n \geq n_0$ . Then

$$\begin{aligned} P\{|X_n - nr| < \varepsilon n^a \text{ and } |Y_n - nb| < \varepsilon n^a\} \\ &\geq P\{|X_n - nr| < \varepsilon n^a\} - (1 - P\{|Y_n - nb| < \varepsilon n^a\}) \\ &\geq 1 - \frac{\delta}{2} - \frac{\delta}{2} = 1 - \delta, \quad \text{for } n \geq n_0. \end{aligned}$$

This holds for every  $\delta > 0$ , so by taking the limit,

$$\lim_{n \rightarrow \infty} P\{|X_n - nr| < \varepsilon n^a \text{ and } |Y_n - nb| < \varepsilon n^a\} = 1.$$

ALTERNATIVELY we may also here apply Chebyshev's inequality. The complementary event has the probability

$$\begin{aligned} P\{\{|X_n - nr| \geq \varepsilon n^a\} \cup \{|Y_n - nb| \geq \varepsilon n^a\}\} &\leq P\{|X_n - nr| \geq \varepsilon n^a\} + P\{|Y_n - nb| \geq \varepsilon n^a\} \\ &\leq \frac{V\{X_n\}}{\varepsilon^2 n^{2a}} + \frac{V\{Y_n\}}{\varepsilon^2 n^{2a}} = \frac{nr(b+g)}{\varepsilon^2 n^{2a}} + \frac{nb(r+g)}{\varepsilon^2 n^{2a}} \leq \frac{1}{\varepsilon^2} n^{1-2a} \rightarrow 0 \quad \text{for } n \rightarrow \infty, \end{aligned}$$

(because  $a > \frac{1}{2}$ ). Hence, we find the probability of the event,

$$P\{|X_n - nr| < \varepsilon n^a \wedge |Y_n - nb| < \varepsilon n^a\} \rightarrow 1 \quad \text{for } n \rightarrow \infty.$$

**Example 4.7** In an experiment an event  $S$  (success) occurs with the probability  $p$ , where  $0 < p < 1$ , and the complementary event  $F$  (failure) occurs with the probability  $q = 1 - p$ . The experiment is repeated under identical conditions and such that the events are mutually independent. We define for every  $r \in \mathbb{N}$  the random variable  $Y_r$  by

$Y_r = k$ , if we have had precisely  $k$  failures before the  $r$ -th success.

1) Prove that  $Y_1$  has its distribution given by

$$P\{Y_1 = k\} = pq^k, \quad k \in \mathbb{N}_0.$$

2) Prove that  $Y_r$  has its distribution given by

$$P\{Y_r = k\} = \binom{k+r-1}{k} p^r q^k, \quad k \in \mathbb{N}_0.$$

3) Explain why  $Y_r$  can be written in the form

$$Y_r = \sum_{i=1}^r X_i,$$

where the  $X_i$  are independent, identically distributed random variables.

4) Prove by applying Chebyshev's inequality that for every  $\varepsilon > 0$  and every  $a > \frac{1}{2}$ ,

$$\lim_{r \rightarrow \infty} P\left\{\left|Y_r - r \frac{p}{q}\right| \leq \varepsilon r^a\right\} = 1.$$

5) Prove by applying the Central Limit Theorem that for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} P\left\{\left|Y_r - r \frac{q}{p}\right| \leq \varepsilon \sqrt{r}\right\} = f(\varepsilon),$$

where  $0 < f(\varepsilon) < 1$ .

1) The event  $\{Y_1 = k\}$  means that we have had precisely  $k$  failures before the first success, i.e. in totally  $k + 1$  experiments, so

$$P\{Y_1 = k\} = \binom{k+1-1}{k} p^1 q^k = pq^k, \quad k \in \mathbb{N}_0.$$

2) The event  $\{Y_r = k\}$  means that we have had  $k$  failures and  $r - 1$  successes in the first  $k + r - 1$  experiments, at that the  $(k + r)$ -th experiment is a success. The occurrences of the  $k$  failures can be chosen in  $\binom{k+r-1}{k}$  ways, so

$$P\{Y_r = k\} = \binom{k+r-1}{k} p^r q^k, \quad k \in \mathbb{N}_0.$$

This describes a negative binomial distribution,  $Y_r \in NB(r, p)$ .

3) Since  $Y_1 = X_1$  has the distribution

$$P\{X_1 = k\} = P\{Y_1 = k\} = pq^k, \quad k \in \mathbb{N}_0,$$

we see that

$$P\{X_1 = k - 1\} = P\{X_1 + 1 = k\} = pq^{k-1}, \quad k \in \mathbb{N},$$

is geometrically distributed. Then  $Y_r + r$  is Pascal distributed,

$$Y_r + r = \sum_{i=1}^r (X_i + 1) \in \text{Pas}(r, p),$$

provided that all  $X_i + 1$  are independent and identically distributed, thus

$$Y_r = \sum_{i=1}^r X_i, \quad X_i \in NB(1, p), \text{ i.e. } P\{X_i = k\} = pq^k, \quad k \in \mathbb{N}_0,$$

and the negative binomial distribution is reproductive in the form parameter.

4) By Chebyshev's inequality,

$$P\{|Y - \mu| \geq b\} \leq \frac{\sigma^2}{b^2}, \quad \text{where } \mu = E\{Y\} \text{ and } \sigma^2 = V\{Y\}.$$

It follows from (3) and a formula that

$$\mu_r = E\{Y_r\} = E\{Y_r + r\} - r = \frac{r}{p} - r = r \frac{1-p}{p} = r \cdot \frac{q}{p},$$

and

$$\sigma_r^2 = V\{Y_r\} = V\{Y_1 + r\} = \frac{rq}{p^2}.$$

Then by insertion into Chebyshev's inequality,

$$(5) \quad P\left\{\left|Y_r - r \frac{q}{p}\right| \geq b\right\} \leq \frac{rq}{b^2 p^2}.$$

Choosing  $b = \varepsilon r^a$ , where  $a > \frac{1}{2}$ , we get by (5),

$$0 \leq P\left\{\left|Y_r - r \frac{q}{p}\right| > \varepsilon r^a\right\} \leq \frac{q}{p^2 \varepsilon} \cdot \frac{1}{r^{2a-1}} \rightarrow 0 \quad \text{for } r \rightarrow \infty,$$

thus

$$\lim_{r \rightarrow \infty} P\left\{\left|Y_r - r \frac{q}{p}\right| \leq \varepsilon r^a\right\} = 1 - \lim_{r \rightarrow \infty} P\left\{\left|Y_r - r \frac{q}{p}\right| > \varepsilon r^a\right\} = 1.$$

5) By the Central Limit Theorem,

$$\begin{aligned} \lim_{r \rightarrow \infty} P\left\{\left|Y_r - r \frac{q}{p}\right| \leq \varepsilon \sqrt{r}\right\} &= \lim_{r \rightarrow \infty} P\left\{\left|\frac{Y_r - r \frac{q}{p}}{\sigma_1 \sqrt{r}}\right| \leq \frac{\varepsilon \sqrt{r}}{\sigma_1 \sqrt{r}}\right\} = \lim_{r \rightarrow \infty} P\left\{\left|\frac{Y_r - r \frac{q}{p}}{\sigma_1 \sqrt{r}}\right| \leq \frac{\varepsilon p}{\sqrt{q}}\right\} \\ &= 2\Phi\left(\varepsilon \frac{p}{\sqrt{q}}\right) - 1 = f(\varepsilon). \end{aligned}$$

Here we have used that

$$Y_r = \sum_{i=1}^r X_i \quad \text{and} \quad E\{X_i\} = \frac{q}{p} \quad \text{and} \quad \sigma_1^2 = \frac{qp}{p^2}.$$

If  $Z$  is  $N(0, 1)$ -distributed, then clearly,

$$f(\varepsilon) = 2\Phi\left(\varepsilon \frac{p}{\sqrt{q}}\right) - 1 = P\left\{|Z| \leq \varepsilon \frac{p}{\sqrt{q}}\right\} \in ]0, 1[.$$

**Example 4.8** Let  $X_1, X_2, \dots$  be independent random variables, all rectangularly distributed over the interval  $]0, 1[$ .

Introduce for every  $n \in \mathbb{N}$  the random variables

$$Y_n = \sum_{i=1}^n X_i, \quad Z_n = \sum_{i=1}^n X_i^2.$$

- 1) Compute the means  $E\{Y_n\}$  and  $E\{Z_n\}$ .
- 2) Compute the variances  $V\{Y_n\}$  and  $V\{Z_n\}$ .
- 3) Compute for  $j = 1, 2, \dots, n$ , the covariance  $\text{Cov}(Y_n, X_j^2)$ .
- 4) Compute the correlation  $\rho(Y_n, Z_n)$ , and prove that it does not depend on  $n$ .
- 5) Let  $n$  vary. Prove for every  $\varepsilon \in \mathbb{R}_+$  and every  $a > \frac{1}{2}$  that

$$P\left\{\left|Y_n - \frac{n}{2}\right| < \varepsilon n^a\right\} \rightarrow 1 \quad \text{for } n \rightarrow \infty,$$

and

$$P\left\{\left|Z_n - \frac{n}{3}\right| < \varepsilon n^a\right\} \rightarrow 1 \quad n \rightarrow \infty.$$

- 6) Find

$$\lim_{n \rightarrow \infty} P\left\{\left|Z_n - \frac{n}{3}\right| < \frac{\sqrt{n}}{4}\right\},$$

(3 decimals).

- 1) Obviously,  $E\{Y_n\} = \frac{n}{2}$ . Furthermore,

$$E\{Z_n\} = \sum_{i=1}^n E\{X_i^2\} = \sum_{i=1}^n \int_0^1 x_i^2 dx_i = \frac{n}{3}.$$

2) Clearly,  $V\{Y_n\} = \frac{n}{12}$ . Since

$$E\{(X_i^2)^2\} = \int_0^1 x_i^4 dx_i = \frac{1}{5} \quad \text{and} \quad E\{X_i^2\} = \frac{1}{3},$$

we get

$$V\{X_i^2\} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45},$$

so

$$V\{Z_n\} = \frac{4n}{45}.$$

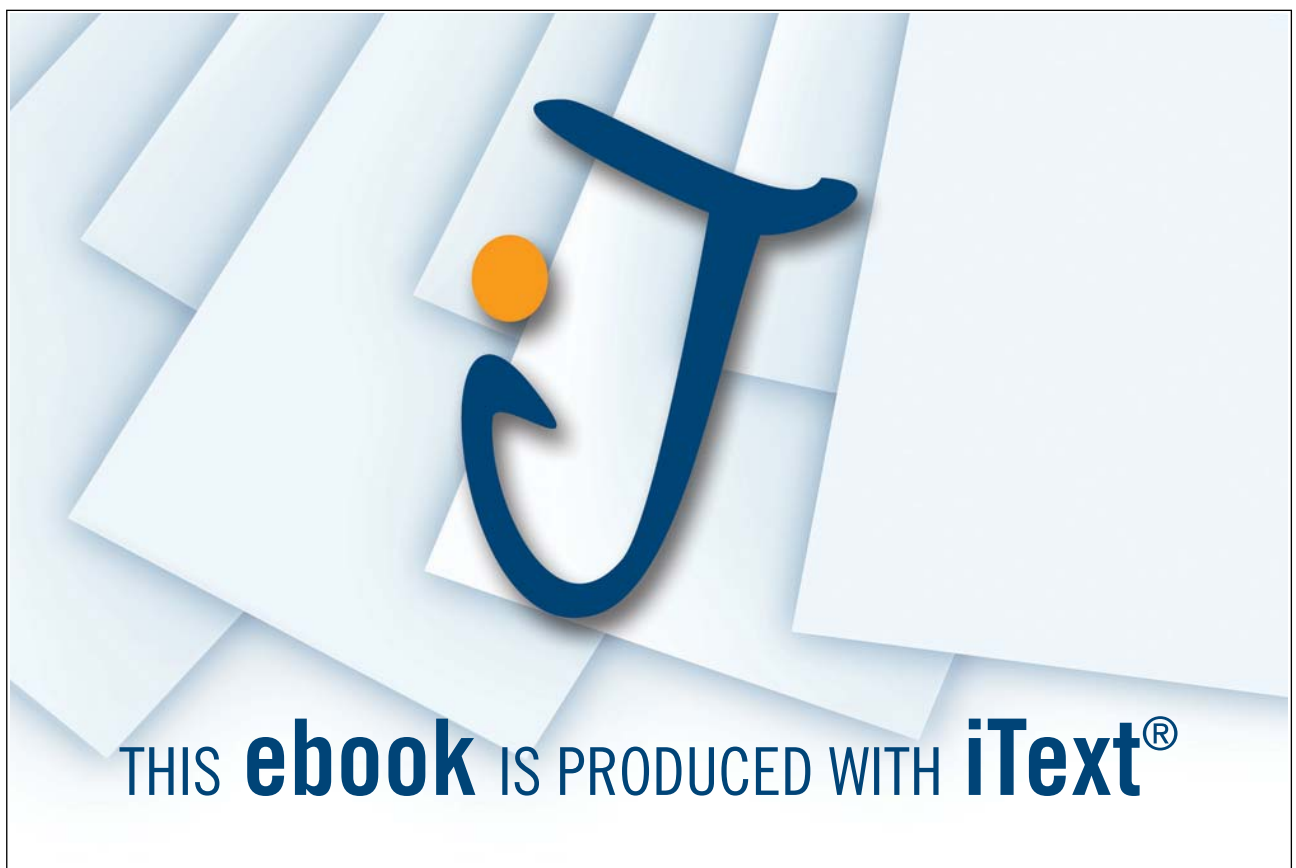
3) A direct computation gives

$$\text{Cov}(X_n, X_j^2) = \text{Cov}(X_j, X_j^2) = E\{X_j^3\} - E\{X_j\} E\{X_j^2\} = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}.$$

4) By (3),

$$\text{Cov}(Y_n, Z_n) = \sum_{j=1}^n \text{Cov}(Y_n, X_j^2) = \frac{n}{12},$$

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thus

$$\varrho(Y_n, Z_n) = \frac{n}{12} \cdot \frac{1}{\sqrt{\frac{n}{12}} \cdot \sqrt{\frac{4n}{45}}} = \sqrt{\frac{45}{48}} = \sqrt{\frac{15}{16}} = \frac{1}{4} \sqrt{15},$$

independently of  $n$ .

5) By Chebyshev's inequality,

$$P \left\{ \left| Y_n - \frac{n}{2} \right| \geq \varepsilon n^a \right\} \leq \frac{\frac{n}{12}}{\varepsilon^2 n^{2a}} = \frac{1}{12 \varepsilon^2 n^{2a-1}} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

so

$$\lim_{n \rightarrow \infty} P \left\{ \left| Y_n - \frac{n}{2} \right| < \varepsilon n^a \right\} = 1 - 0 = 1.$$

It follows in the same way that

$$P \left\{ \left| Z_n - \frac{n}{3} \right| \geq \varepsilon n^a \right\} \leq \frac{\frac{4n}{45}}{\varepsilon^2 n^{2a}} = \frac{4}{45 \varepsilon^2} \cdot \frac{1}{n^{2a-1}} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

so

$$\lim_{n \rightarrow \infty} P \left\{ \left| Z_n - \frac{n}{3} \right| < \varepsilon n^a \right\} = 1 - 0 = 1.$$

6) By the Central Limit Theorem,

$$\begin{aligned} P \left\{ \left| Z_n - \frac{n}{3} \right| < \frac{\sqrt{n}}{4} \right\} &= P \left\{ \frac{\left| Z_n - \frac{n}{3} \right|}{\sqrt{\frac{4n}{45}}} \leq \frac{\frac{\sqrt{n}}{4}}{\sqrt{\frac{4n}{45}}} \right\} \rightarrow 2 \Phi \left( \frac{3}{8} \sqrt{5} \right) - 1 \\ &= 2 \Phi(0.83853) - 1 \approx 2 \cdot 0.799 - 1 = 0.598. \end{aligned}$$

**Example 4.9** Let  $U_1$  and  $U_2$  be two mutually independent random variables of the means  $\mu_1$  and  $\mu_2$ , resp., and the variances  $\sigma_1^2$  and  $\sigma_2^2$ , resp., and let  $U = U_1 \cdot U_2$ .

1. Prove that the variance of  $U$  is  $\sigma_1^2 \sigma_2^2 + \mu_1^2 \sigma_2^2 + \mu_2^2 \sigma_1^2$ .

A rectangle has the edges  $X_1$  and  $X_2$ , where  $X_1$  and  $X_2$  are independent, identically distributed random variables of frequency

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $Y = X_1 \cdot X_2$  denote the area of the rectangle.

2. Find, by e.g. using the result of 1., the mean and variance of  $Y$ .

Let  $Y_1, Y_2, \dots$  be a sequence of mutually independent random variables, all following the same distribution as  $Y$ , and let

$$Z_n = \sum_{i=1}^n Y_i, \quad n \in \mathbb{N}.$$

3. Find, by means of Chebyshev's inequality a positive constant  $a$ , such that

$$P \left\{ \left| Z_n - \frac{4n}{9} \right| < \frac{\sqrt{n}}{4} \right\} \geq a \quad \text{for every } n \in \mathbb{N}.$$

4. Find by means of the Central Limit Theorem,

$$\lim_{n \rightarrow \infty} P \left\{ \left| Z_n - \frac{4n}{9} \right| < \frac{\sqrt{n}}{4} \right\}, \quad (3 \text{ decimals}).$$

1) It follows from

$$E \{U^2\} = E \{U_1^2\} \cdot E \{U_2^2\} = (\sigma_1^2 + \mu_1^2) \cdot (\sigma_2^2 + \mu_2^2),$$

that

$$V\{U\} = E \{U^2\} - (E\{U\})^2 = \sigma_1^2 \sigma_2^2 + \mu_1^2 \sigma_2^2 + \mu_2^2 \sigma_1^2 + \mu_1^2 \mu_2^2 - (\mu_1 \mu_2)^2 = \sigma_1^2 \sigma_2^2 + \mu_1^2 \sigma_2^2 + \mu_2^2 \sigma_1^2.$$

2) In this case we have (at least) two variants.

a) First compute

$$E\{X_1\} = \int_0^1 2x^2 dx = \frac{2}{3} \quad \text{and} \quad E\{X_1^2\} = \int_0^1 2x^3 dx = \frac{1}{2},$$

thus

$$V\{X_1\} = E\{X_1^2\} - (E\{X_1\})^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

We obtain the same results for  $X_2$ .

Then

$$E\{Y\} = E\{X_1\} \cdot E\{X_2\} = \frac{4}{9},$$

and by **1.**,

$$V\{Y\} = \frac{1}{18} \cdot \frac{1}{18} + 2 \cdot \frac{4}{9} \cdot \frac{1}{18} = \frac{17}{324}.$$

b) ALTERNATIVE SOLUTION. First find the frequency of  $Y = X_1 \cdot X_2$ . If  $0 < y < 1$ , then

$$f_Y(y) = \int_{x=y}^1 f_{X_1}(x) f_{X_2}\left(\frac{y}{x}\right) \frac{1}{x} dx = \int_{x=y}^1 2x \cdot 2 \frac{y}{x} \cdot \frac{1}{x} dx = 4y \int_{x=y}^1 \frac{1}{x} dx = -4y \ln y.$$

Hence

$$E\{Y\} = -4 \int_0^1 y^2 \ln y dy = -4 \left[ \frac{y^3}{3} \ln y - \int \frac{y^2}{3} dy \right]_0^1 = \frac{4}{9},$$

$$E\{Y^2\} = -4 \int_0^1 y^3 \ln y dy = -4 \left[ \frac{y^4}{4} \ln y - \int \frac{y^3}{4} dy \right]_0^1 = \frac{1}{4},$$

so

$$V\{Y\} = \frac{1}{4} - \left(\frac{4}{9}\right)^2 = \frac{1}{4} - \frac{16}{81} = \frac{17}{214}.$$

3) Put

$$Z_n = \sum_{i=1}^n Y_i.$$

Then

$$E\{Z_n\} = n \cdot \frac{4}{9} \quad \text{and} \quad V\{Z_n\} = n \cdot \frac{17}{324}.$$

It follows from Chebyshev's inequality that

$$P\left\{\left|Z_n - \frac{4n}{9}\right| \geq \frac{\sqrt{n}}{4}\right\} \leq n \cdot \frac{17}{324} \cdot \frac{16}{n} = \frac{68}{81},$$

hence

$$P\left\{\left|Z_n - \frac{4n}{9}\right| < \frac{\sqrt{n}}{4}\right\} \geq 1 - \frac{68}{81} = \frac{13}{81} \quad (\approx 0,160).$$



4) Obviously, the assumptions of the Central Limit Theorem are satisfied, so

$$P \left\{ \left| Z_n - \frac{4n}{9} \right| < \frac{\sqrt{n}}{4} \right\} = P \left\{ -\frac{18}{4\sqrt{17}} < \frac{Z_n - 4 \cdot \frac{n}{9}}{\sqrt{n} \cdot \frac{\sqrt{17}}{18}} < \frac{18}{4\sqrt{17}} \right\} \rightarrow 2\Phi \left( \frac{18}{4\sqrt{17}} \right) - 1$$

$$= 2\Phi(1.0914) - 1 = 2 \cdot 0.8624 - 1 = 0.7248.$$

Hence, with 3 decimals,

$$\lim_{n \rightarrow \infty} P \left\{ \left| Z_n - \frac{4n}{9} \right| < \frac{\sqrt{n}}{4} \right\} = 0.725.$$

**Example 4.10** Let  $X_1$  and  $X_2$  be independent random variable of the frequencies

$$f_{X_1}(x_1) = e^{-2|x_1|}, \quad x_1 \in \mathbb{R}, \quad f_{X_2}(x_2) = \begin{cases} e^{-x_2}, & x_2 > 0, \\ 0, & x_2 \leq 0. \end{cases}$$

1. Find the means  $E\{X_1\}$  and  $E\{X_2\}$ .
2. Find the variances  $V\{X_1\}$  and  $V\{X_2\}$ .
3. Prove that the random variables  $X_1 + X_2$  and  $2X_1 - X_2$  are non-correlated.
4. Compute the frequency of the random variable  $X_1 + X_2$ .

Let  $U_1, U_2, \dots, U_{200}$  be mutually independent random variables, all following the same distribution as  $X_1$ , and let

$$V = \sum_{i=1}^{200} U_i.$$

5. Find by using the Central Limit Theorem an approximate expression of

$$P\{-20 \leq V \leq 20\}.$$

- 1) The integral  $\int_{-\infty}^{\infty} |x_1| e^{-2|x_1|} dx_1 < \infty$  is convergent, and the integrand  $x_1 e^{-2|x_1|}$  is odd. Thus

$$E\{X_1\} = \int_{-\infty}^{\infty} x_1 e^{-2|x_1|} dx_1 = 0.$$

Furthermore, by using a  $\Gamma$ -integral,

$$E\{X_2\} = \int_0^{\infty} x_2 e^{-x_2} dx_2 = 1.$$

2) From  $E\{X_1\} = 0$  follows that

$$V\{X_1\} = E\{X_1^2\} = \int_{-\infty}^{\infty} x_1^2 e^{-2|x_1|} dx_1 = 2 \int_0^{\infty} x_1^2 e^{-2x_1} dx_1 = \frac{1}{4} \cdot 2! = \frac{1}{2}.$$

Since

$$E\{X_2^2\} = \int_0^{\infty} x_2^2 e^{-x_2} dx_2 = 2,$$

we get

$$V\{X_2\} = 2 - 1 = 1.$$

3) Now,  $X_1$  and  $X_2$  are independent, and  $\text{Cov}(\cdot, \cdot)$  is bilinear, hence

$$\text{Cov}(X_1 + X_2, 2X_1 - X_2) = 2\text{Cov}(X_1, X_2) - \text{Cov}(X_2, X_2) = 2V\{X_1\} - V\{X_2\} = 0.$$

This shows that  $X_1 + X_2$  and  $2X_1 - X_2$  are non-correlated (though they are *not* independent.)

4) Since  $X_1$  and  $X_2$  are independent, the frequency of  $Y = X_1 + X_2$  is given by the convolution integral

$$g(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y - x) dx.$$

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The integrand is positive, if  $y > x$ . We split the investigation according to whether  $y \leq 0$  or  $y > 0$ .

a) If  $y \leq 0$ , then

$$g(y) = \int_{-\infty}^y e^{2x} e^{-(y-x)} dx = e^{-y} \int_{-\infty}^y e^{3x} dx = \frac{1}{3} e^{2y}.$$

b) If instead  $y > 0$ , then

$$\begin{aligned} g(y) &= \int_{-\infty}^0 e^{2x} e^{-(y-x)} dx + \int_0^y e^{-2x} e^{-(y-x)} dx = e^{-y} \int_{-\infty}^0 e^{3x} dx + e^{-y} \int_0^y e^{-x} dx \\ &= \frac{1}{3} e^{-y} + (1 - e^{-y}) e^{-y} = \frac{4}{3} e^{-y} - e^{-2y}. \end{aligned}$$

Summing up we get

$$g(y) = \begin{cases} \frac{1}{3} e^{2y}, & \text{for } y \leq 0, \\ \frac{4}{3} e^{-y} - e^{-2y}, & \text{for } y > 0. \end{cases}$$

5) It follows from the Central Limit Theorem that  $V$  is almost normally distributed with mean 0 and variance  $100 = 10^2$ . When we put  $W \in N(0, 10^2)$ , the wanted probability is almost equal to

$$P\{-20 \leq W \leq 20\} = P\left\{-2 \leq \frac{W}{10} \leq 2\right\} = 2\Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 0.9544.$$

**Example 4.11** An instrument  $A$  contains two components, the lifetimes of which,  $X_1$  and  $X_2$ , are independent random variables, both of frequency

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $\alpha$  is a positive constant. The instrument  $A$  does only work, when at least one of the two components operates, i.e. the lifetime  $X$  of  $A$  is  $X = \max\{X_1, X_2\}$ .

Another instrument  $B$  also contains two components, the lifetimes of which,  $Y_1$  and  $Y_2$ , are independent random variables, both of frequency

$$g(y) = \begin{cases} \frac{\alpha}{3} \exp\left(-\frac{\alpha}{3}y\right), & y > 0, \\ 0, & y \leq 0. \end{cases}$$

The instrument  $B$  is only working, if both components are working, i.e. the lifetime  $Y$  of  $B$  is  $Y = \min\{Y_1, Y_2\}$ .

- 1) Find the frequency of  $X$ .
- 2) Find mean and variance of  $X$ .
- 3) Find frequency, mean and variance of  $Y$ .
- 4) Find the simultaneous frequency of  $(X, Y)$ , and then find  $P\{Y > X\}$ .
- 5) Consider 180 instruments of the same type as  $A$ . The random variable  $Z$  denotes the sum of the lifetimes of the 180 instrument. Find by using the Central Limit Theorem an approximate expression of

$$P\left\{\frac{240}{\alpha} < Z < \frac{300}{\alpha}\right\}.$$

- 1) According to a known formula,  $X = \max\{X_1, X_2\}$  has the distribution function

$$F_X(x) = \begin{cases} F_{X_1}(x) \cdot F_{X_2}(x) = (1 - e^{-\alpha x})^2 & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

so the frequency is

$$f(x) = \begin{cases} 2\alpha e^{-\alpha x} (1 - e^{-\alpha x}) = 2\alpha e^{-\alpha x} - 2\alpha e^{-2\alpha x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

- 2) The mean is

$$E\{X\} = \int_0^{\infty} 2\alpha x e^{-\alpha x} dx - \int_0^{\infty} 2\alpha x e^{-2\alpha x} dx = \frac{2}{\alpha} \cdot 1! - \frac{1}{2\alpha} \cdot 1! = \frac{3}{2\alpha}.$$

Since

$$\begin{aligned} E\{X^2\} &= \int_0^\infty 2\alpha x^2 e^{-\alpha x} dx - \int_0^\infty 2\alpha x^2 e^{-2\alpha x} dx \\ &= \frac{2}{\alpha^2} \cdot 2! - \frac{2}{(2\alpha)^2} \cdot 2! = \frac{1}{\alpha^2} \left(2 - \frac{1}{4}\right) = \frac{14}{4\alpha^2}, \end{aligned}$$

we get the variance

$$V\{X\} = \frac{14}{4\alpha^2} - \frac{14}{4\alpha^2} - \frac{9}{4\alpha^2} = \frac{5}{4\alpha^2}.$$

3) It follows by a formula that  $Y = \min\{Y_1, Y_2\}$  has the distribution function

$$\begin{aligned} F_Y(y) &= 1 - (1 - F_{Y_1}(y))(1 - F_{Y_2}(y)) = 1 - (1 - F_{Y_1}(y))^2 \\ &= \begin{cases} 1 - \exp\left(-\frac{2}{3}\alpha y\right), & \text{for } y > 0, \\ 0, & \text{for } y \leq 0. \end{cases} \end{aligned}$$

The frequency is obtained by a differentiation,

$$f_Y(y) = \begin{cases} \frac{2\alpha}{3} \exp\left(-\frac{2\alpha}{3}y\right) & \text{for } y > 0, \\ 0 & \text{for } y \leq 0, \end{cases}$$

proving that  $Y$  is exponentially distributed with

$$E\{Y\} = \frac{3}{2\alpha} \quad \text{and} \quad V\{Y\} = \frac{9}{4\alpha^2}.$$

4) Since  $X$  and  $Y$  are stochastically independent, the frequency of  $(X, Y)$  is given by

$$h(x, y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{4\alpha^2}{3} e^{-\alpha x} (1 - e^{-\alpha x}) \cdot \exp\left(-\frac{2\alpha}{3}y\right) & \text{for } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} P\{Y > X\} &= \int_0^\infty \left\{ \int_0^y f_X(x) \cdot f_Y(y) dx \right\} dy = \int_0^\infty [F_X(x)]_0^y f_Y(y) dy \\ &= \int_0^\infty (1 - e^{-\alpha y})^2 \cdot \frac{2\alpha}{3} \cdot \exp\left(-\frac{2\alpha}{3}y\right) dy \\ &= \frac{2\alpha}{3} \int_0^\infty \left\{ \exp\left(-\frac{2\alpha}{3}y\right) - 2 \exp\left(-\frac{5\alpha}{3}y\right) + \exp\left(-\frac{8\alpha}{3}y\right) \right\} dy \\ &= \frac{2\alpha}{3} \left\{ \frac{3}{2\alpha} - 2 \cdot \frac{3}{5\alpha} + \frac{3}{8\alpha} \right\} = \frac{2}{3} \left\{ \frac{3}{2} - \frac{6}{5} + \frac{3}{8} \right\} = 1 - \frac{4}{5} + \frac{1}{4} \\ &= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}. \end{aligned}$$

5) Since

$$\mu = E\{Z\} = 180 E\{X\} = \frac{270}{\alpha},$$

and

$$\sigma^2 = V\{Z\} = 180 V\{X\} = \frac{225}{\alpha^2} = \left(\frac{15}{\alpha}\right)^2,$$

it follows by the Central Limit theorem that

$$\begin{aligned} P\left\{\frac{240}{\alpha} < Z < \frac{300}{\alpha}\right\} &= P\left\{-\frac{30}{\alpha} < Z - \mu < \frac{30}{\alpha}\right\} = P\left\{-2 < \frac{Z - \mu}{\sigma} < 2\right\} \\ &\approx 2\Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 0.9544. \end{aligned}$$

**Example 4.12** Let  $X_1, X_2, X_3, \dots$  be mutually independent random variables with their distribution given by

$$P\{X_i = 1\} = P\{X_i = -1\} = \frac{1}{2}, \quad i \in \mathbb{N},$$

and let

$$Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N}.$$

- 1) Prove that if  $U_i$  is Bernoulli distributed with the probability parameter  $\frac{1}{2}$ ,  $U_i \in B\left(1, \frac{1}{2}\right)$ , then  $2U_i - 1$  has the same distribution as  $X_i$ .
- 2) Find by e.g. applying the result of **1.**, the distribution of the random variable  $Y_5$ .
- 3) Find the characteristic function of  $X_i$ , and then find the characteristic function of  $Y_n$ .
- 4) Prove that the sequence  $\left(\frac{1}{\sqrt{n}} Y_n\right)$  converges in distribution towards a random variable  $Y$ , and find the distribution of  $Y$ .  
HINT: One may either use the Central Limit Theorem, or the result of **3.**
- 5) Find an approximate expression of  $P\{-30 \leq Y_{400} \leq 30\}$ .

- 1) When  $U_i \in B\left(1, \frac{1}{2}\right)$ , then

$$P\{U_i = 0\} = P\{2U_i - 1 = -1\} = \frac{1}{2} = P\{U_i = 1\} = P\{2U_i - 1 = 1\},$$

hence  $2U_i - 1$  has the same distribution as  $X_i$ .

2) It follows that

$$Y_5 = \sum_{i=1}^5 X_i = \sum_{i=1}^5 (2U_i - 1) = 2 \sum_{i=1}^5 U_i - 5 = 2U - 5,$$

where

$$U = \sum_{i=1}^5 U_i \in B\left(5, \frac{1}{2}\right)$$

is binomially distributed, hence

$$P\{U = k\} = \binom{5}{k} \frac{1}{2^5}, \quad k = 0, 1, 2, 3, 4, 5.$$

Then

$$\begin{aligned} P\{Y_5 = -5\} &= P\{U = 0\} = \frac{1}{32}, \\ P\{Y_5 = -3\} &= P\{U = 1\} = \frac{5}{32}, \\ P\{Y_5 = -1\} &= P\{U = 2\} = \frac{5}{16}, \\ P\{Y_5 = 1\} &= P\{U = 3\} = \frac{5}{16}, \\ P\{Y_5 = 3\} &= P\{U = 4\} = \frac{5}{32}, \\ P\{Y_5 = 5\} &= P\{U = 5\} = \frac{1}{32}. \end{aligned}$$


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3) The characteristic function of  $X_i$  is

$$k_{X_i}(\omega) = E \{ e^{i\omega X_i} \} = \frac{1}{2} e^{i\omega} + \frac{1}{2} e^{-i\omega} = \cos \omega.$$

The characteristic function of  $Y_n = \sum_{i=1}^n X_i$  is then

$$k_{Y_n} = \prod_{i=1}^n k_{X_i}(\omega) = \cos^n \omega.$$

4) FIRST VARIANT. The characteristic function of  $\frac{1}{\sqrt{n}} Y_n = Z_n$  is

$$k_{Z_n}(\omega) = \cos^n \left( \frac{\omega}{\sqrt{n}} \right) = \left\{ 1 - \frac{\omega^2}{2} \cdot \frac{1}{n} + \frac{1}{n} \varepsilon \left( \frac{1}{n} \right) \right\}^n \rightarrow \exp \left( -\frac{\omega^2}{2} \right) \quad \text{for } n \rightarrow \infty.$$

The limit function exists and is continuous at 0, hence the sequence  $\left( \frac{1}{\sqrt{n}} Y_n \right)$  of random variables converges in distribution towards a random variable  $Y$  with the characteristic function

$$k_Y(\omega) = \exp \left( -\frac{\omega^2}{2} \right),$$

and we conclude that  $Y \in N(0, 1)$  is normally distributed.

SECOND VARIANT. Now

$$E \{ Z_n \} = \frac{1}{\sqrt{n}} \sum_{i=1}^n E \{ X_i \} = 0,$$

and

$$V \{ Z_n \} = \frac{1}{n} \cdot n \cdot V \{ X_1 \} = V \{ 2U_1 - 1 \} = 4V \{ U_1 \} = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1,$$

it follows by the Central Limit Theorem that

$$F_{Z_n}(z) = P \{ Z_n \leq z \} = P \left\{ \frac{Z_n - E \{ Z_n \}}{\sqrt{V \{ Z_n \}}} \leq z \right\} \rightarrow \Phi(z) \quad \text{for } n \rightarrow \infty,$$

proving that  $Z_n \rightarrow Y$  in distribution, where  $Y \in N(0, 1)$  is normally distributed.

5) Since  $Y_{400} = \frac{1}{20} Z_{400}$ , it follows from the SECOND VARIANT in 4. above that

$$P \{ -30 < Y_{400} < 30 \} = P \left\{ -\frac{3}{2} < Z_{400} < \frac{3}{2} \right\} \approx 2\Phi \left( \frac{3}{2} \right) - 1 \approx 2 \cdot 0.9332 - 1 = 0.8664.$$

Here, it is however easier to apply the continuity correction, so we get the interval  $] -30.5, 30.5[$ . In this case we have the approximation

$$\begin{aligned} P \{ -30.5 < Y_{400} < 30.5 \} &= P \left\{ -\frac{305}{200} < Z_{400} < \frac{305}{200} \right\} \\ &\approx 2\Phi(1.525) - 1 = 2 \cdot 0.9364 - 1 = 0.8728. \end{aligned}$$



**Example 4.13** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} 1 - \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

- 1) Prove that  $E\{X\} = E\{X^2\}$ .
- 2) Find the variance  $V\{X\}$ .
- 3) Let  $X_1, X_2, \dots, X_{450}$  be mutually independent random variables, all distributed like  $X$  above, and let  $Y = \sum_{i=1}^{450} X_i$ . Find by using the Central Limit Theorem a number  $x$ , such that

$$P\{300 - x < Y < 300 + x\}$$

is (almost) 0.95.

- 1) By some simple computations,

$$E\{X\} = \int_0^2 x f(x) dx = \int_0^2 \left(x - \frac{1}{2}x\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = 2 - \frac{4}{3} = \frac{2}{3},$$

and

$$E\{X^2\} = \int_0^2 x^2 f(x) dx = \int_0^2 \left(x^2 - \frac{1}{2}x^3\right) dx = \left[\frac{1}{3}x^3 - \frac{1}{8}x^4\right]_0^2 = \frac{8}{3} - 2 = \frac{2}{3},$$

hence

$$E\{X\} = E\{X^2\} = \frac{2}{3}.$$

- 2) Another simple computation gives

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}.$$

- 3) If  $Y = \sum_{i=1}^{450} X_i$ , then

$$E\{Y\} = 450 \cdot \frac{2}{3} = 300 \quad \text{and} \quad V\{Y\} = 450 \cdot \frac{2}{9} = 100 = 10^2.$$

If  $Z \in N(300, 10^2)$ , it follows by the Central Limit Theorem that

$$\begin{aligned} P\{300 - x < Y < 300 + x\} &\approx P\{300 - x < Z < 300 + x\} \\ &= P\left\{-\frac{x}{10} < \frac{Z - 300}{10} < \frac{x}{10}\right\} = 2\Phi\left(\frac{x}{10}\right) - 1. \end{aligned}$$

From

$$2\Phi\left(\frac{x}{10}\right) - 1 = 0,95, \quad \text{dvs.} \quad \Phi\left(\frac{x}{10}\right) = 0,975,$$

follows that

$$\frac{x}{10} = 1.96, \quad \text{thus} \quad x = 19.6.$$

**Example 4.14** Let  $X_1, X_2, X_3, \dots$  be mutually independent random variables, all of the distribution given by

$$P\{X_i = 2\} = \frac{1}{3}, \quad P\{X_i = -1\} = \frac{2}{3}, \quad i \in \mathbb{N},$$

and let

$$Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N}.$$

- 1) Find mean and variance of  $X_i$ .
- 2) Explain why  $Y_n$  only takes values in the interval  $[-n, 2n]$ .
- 3) Find by means of Chebyshev's inequality a positive constant  $a$ , such that

$$P\{|Y_n| < 3\sqrt{n}\} \geq a \quad \text{for every } n \in \mathbb{N}.$$

- 4) Find by means of the Central Limit Theorem

$$\lim_{n \rightarrow \infty} P\{|Y_n| < 3\sqrt{n}\} \quad (3 \text{ decimals}).$$

- 5) Find the distribution of  $Y_4$ .

The simplest method is to introduce  $Z_i \in B\left(1, \frac{1}{3}\right)$ , which is Bernoulli distributed, and then note that  $X_i = 3Z_i - 1$ .

- 1) The mean is

$$E\{X_i\} = 3E\{Z_i\} - 1 = 3 \cdot \frac{1}{3} - 1 = 0.$$

ALTERNATIVELY,

$$E\{X_i\} = 2 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3} = 0.$$

- 2) The variance is

$$V\{X_i\} = 9V\{Z_i\} = 9 \cdot \frac{1}{3} \cdot \frac{2}{3} = 2.$$

ALTERNATIVELY we get from  $E\{X_i\} = 0$  that

$$V\{X_i\} = E\{X_i^2\} = 4 \cdot \frac{1}{3} + \frac{2}{3} = 2.$$

- 3) Since  $-1 \leq X_i \leq 2$  for all  $i$ , we get

$$-n \leq \sum_{i=1}^n X_i = Y_n \leq 2n.$$

4) Since  $E\{Y_n\} = 0$  and  $\sigma_n^2 = V\{Y_n\} = 2n$ , it follows by Chebyshev's inequality that

$$P\{|Y_n - 0| \geq 3\sqrt{n}\} \leq \frac{\sigma_n^2}{9n} = \frac{2}{9},$$

hence

$$P\{|Y_n| < 3\sqrt{n}\} \geq 1 - \frac{2}{9} = \frac{7}{9} = a.$$

5) According to the Central Limit Theorem,

$$\lim_{n \rightarrow \infty} P\left\{\frac{Y_n - 0}{\sqrt{2n}} \leq x\right\} = \lim_{n \rightarrow \infty} P\{Y_n \leq x \cdot \sqrt{2n}\} = \Phi(x).$$

If we put  $x = \frac{3}{\sqrt{2}}$ , then

$$\begin{aligned} P\{|Y_n| \leq 3\sqrt{n}\} &= P\left\{Y_n \leq \frac{3}{\sqrt{2}} \cdot \sqrt{2n}\right\} - P\left\{Y_n < -\frac{3}{\sqrt{2}} \cdot \sqrt{2n}\right\} \\ &\rightarrow 2\Phi\left(\frac{3}{\sqrt{2}}\right) - 1 \approx 2 \cdot \Phi(2, 121) - 1 \\ &\approx 2 \cdot 0.983 - 1 = 0.966. \end{aligned}$$

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6) Since

$$Y_4 = 3 \sum_{i=1}^4 Z_i - 4,$$

where

$$\sum_{i=1}^4 Z_i \in B\left(4, \frac{1}{3}\right)$$

is binomially distributed, we get

$$P\{Y_4 = 3k - 4\} = \binom{4}{k} \cdot \left(\frac{1}{3}\right)^k \cdot \left(\frac{2}{3}\right)^{4-k}, \quad k = 0, 1, 2, 3, 4,$$

hence

$$P\{Y_4 = -4\} = \frac{16}{81},$$

$$P\{Y_4 = -1\} = \frac{32}{81},$$

$$P\{Y_4 = 2\} = \frac{24}{81} = \frac{8}{27},$$

$$P\{Y_4 = 5\} = \frac{8}{81},$$

$$P\{Y_4 = 8\} = \frac{1}{81}$$

**Example 4.15** Let  $X_1, X_2, X_3, \dots$  be mutually independent random variables, all of the distribution given by

$$P\{X_i = 1\} = \frac{3}{4}, \quad P\{X_i = -3\} = \frac{1}{4}, \quad i \in \mathbb{N},$$

and let

$$Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N}.$$

- 1) Find mean and variance of  $X_i$ .
- 2) Find by means of the Central Limit Theorem,

$$\lim_{n \rightarrow \infty} P\{-2\sqrt{n} < Y_n < 2\sqrt{n}\} \quad (3 \text{ decimals}).$$

- 3) Explain why  $X_i$  has the same distribution as  $4Z_i - 3$ , where  $Z_i$  is Bernoulli distributed,

$$P\{Z_i = 1\} = \frac{3}{4}, \quad P\{Z_i = 0\} = \frac{1}{4},$$

and find by means of this for every  $n \in \mathbb{N}$  the probabilities

$$P\{Y_n = 4k - 3n\}, \quad k = 0, 1, 2, \dots, n.$$

- 1) The mean is

$$E\{X_i\} = 1 \cdot \frac{3}{4} - 3 \cdot \frac{1}{4} = 0.$$

Hence the variance

$$V\{X_i\} = E\{X_i^2\} - (E\{X_i\})^2 = 1 \cdot \frac{3}{4} + 9 \cdot \frac{1}{4} - 0^2 = \frac{3+9}{4} = 3.$$

2) The mean is

$$E\{Y_n\} = \sum_{i=1}^n E\{X_i\} = 0,$$

and the variance is  $V\{Y_n\} = 3n$ .

It follows from the Central Limit Theorem that

$$\begin{aligned} \lim_{n \rightarrow \infty} P\{-2\sqrt{n} < Y_n < 2\sqrt{n}\} &= \lim_{n \rightarrow \infty} P\left\{\left|\frac{Y_n - 0}{\sqrt{3n}}\right| < \frac{2}{\sqrt{3}}\right\} \\ &= 2\Phi\left(\frac{2}{\sqrt{3}}\right) - 1 = 2\Phi(1.1547) - 1 \\ &\approx 2 \cdot 0.876 - 1 = 0.752. \end{aligned}$$

3) This follows from 4., so we just indicate that

$$\begin{aligned} P\{Y_3 = -9\} &= \frac{1}{4^3} &&= \frac{1}{64}, \\ P\{Y_3 = -5\} &= \binom{3}{1} \cdot \frac{3}{4} \cdot \frac{1}{4^2} = \frac{9}{4^3} &&= \frac{9}{64}, \\ P\{Y_3 = -1\} &= \binom{3}{2} \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} = \frac{27}{4^3} &&= \frac{27}{64}, \\ P\{Y_3 = 3\} &= \left(\frac{3}{4}\right)^3 = \frac{27}{4^3} &&= \frac{27}{64}. \end{aligned}$$

4) It is obvious that if  $Z_i$  is Bernoulli distributed,

$$P\{Z_i = 1\} = \frac{3}{4}, \quad P\{Z_i = 0\} = \frac{1}{4},$$

then

$$P\{X_i = 1\} = P\{4Z_i - 3 = 1\} = P\{Z_i = 1\} = \frac{3}{4},$$

and

$$P\{X_i = -3\} = P\{4Z_i - 3 = -3\} = P\{Z_i = 0\} = \frac{1}{4},$$

hence  $X_i$  and  $4Z_i - 3$  have same distribution.

If we allow  $X_i = 4Z_i - 3$ , then

$$Y_n = \sum_{i=1}^n X_i = 4 \sum_{i=1}^n Z_i - 3n,$$

where

$$\sum_{i=1}^n Z_i \in B\left(n, \frac{3}{4}\right)$$

is binomially distributed.

Then

$$\begin{aligned} P\{Y_n = 4k - 3n\} &= P\left\{4 \sum_{i=1}^n Z_i - 3n = 4k - 3n\right\} = P\left\{\sum_{i=1}^n Z_i = k\right\} \\ &= \binom{n}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{n-k} \quad \text{for } k = 0, 1, \dots, n. \end{aligned}$$

Putting  $n = 3$  we get question **3.**, cf. the above.

## 5 The Maxwell distribution

**Example 5.1** 1) Prove that the function

$$f(x) = \begin{cases} \frac{4}{\sqrt{\pi}} x^2 \exp(-x^2), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

is the frequency of a random variable  $X$ .

2) Find mean and variance of the random variable  $X$ .

3) Find the frequency of the random variable  $Y = \frac{1}{2} X^2$ .

4) Find the mean of the random variable  $Y$ .

1) Obviously,  $f(x) \geq 0$  everywhere. Since

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 \exp(-x^2) dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} x \cdot \exp(-x^2) \cdot 2x dx \\ &= \frac{2}{\sqrt{\pi}} [x(-\exp(-x^2))]_0^{\infty} + \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-x^2) dx = 1, \end{aligned}$$

it follows that  $f(x)$  is the frequency of a random variable  $X$ .

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2) The mean is

$$E\{X\} = \frac{4}{\sqrt{\pi}} \int_0^{\infty} x \cdot x^2 \exp(-x^2) dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} y e^{-y} dy = \frac{1}{\sqrt{\pi}} \cdot 1! = \frac{2}{\sqrt{\pi}}.$$

Furthermore,

$$\begin{aligned} E\{X^2\} &= \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^4 \exp(-x^2) dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} x^3 \exp(-x^2) \cdot 2x dx \\ &= \frac{2}{\sqrt{\pi}} [-x^3 \exp(-x^2)]_0^{\infty} + \frac{6}{\sqrt{\pi}} \int_0^{\infty} x^2 \exp(-x^2) dx \\ &= \frac{3}{2} \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 \exp(-x^2) dx = \frac{3}{2} \cdot 1 = \frac{3}{2}, \end{aligned}$$

hence

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{3}{2} - \frac{4}{\pi}.$$

3) Since  $y = \frac{1}{2}x^2$  is a bijective map  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$  with the inverse  $x = \sqrt{2y}$ , the frequency of  $Y$  for  $y > 0$  is given by

$$g(y) = f(\sqrt{2y}) \cdot \frac{1}{2} \cdot \frac{2}{\sqrt{2y}} = \frac{4}{\sqrt{\pi}} \cdot 2y \cdot e^{-2y} \cdot \frac{1}{\sqrt{2y}} = \frac{4}{\sqrt{\pi}} \sqrt{2y} \cdot e^{-2y},$$

hence

$$g(y) = \begin{cases} \frac{4}{\sqrt{\pi}} \sqrt{2y} \cdot e^{-2y} & \text{for } y > 0, \\ 0 & \text{for } y \leq 0. \end{cases}$$

4) Then by 2.

$$E\{Y\} = E\left\{\frac{1}{2}X^2\right\} = \frac{1}{2}E\{X^2\} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

**Example 5.2** Define a random variable  $Z$  by  $Z = \frac{1}{2}mY^2$ , where  $Y$  is Maxwell distributed. Find the frequency of  $Z$ , and find mean and variance of  $Z$ . Which distribution has  $Z$ ?

The random variable  $Z = \frac{1}{2}mY^2$  represent the kinetic energy. Clearly,  $Z$  has only positive values. The frequency of  $Y$  is

$$f_Y(y) = \begin{cases} \sqrt{\frac{1}{\pi}} \cdot \frac{1}{\sigma^3} y^2 \exp\left(-\frac{y^2}{2\sigma^2}\right) & \text{for } y > 0, \\ 0 & \text{for } y \leq 0, \end{cases}$$

hence  $Y \geq 0$ . Therefore,  $Y = \sqrt{\frac{2}{m}} \cdot \sqrt{Z}$ , so when  $z > 0$ , the distribution function of  $Z$  is given by

$$F_Z(z) = P\{Z \leq z\} = P\left\{Y \leq \sqrt{\frac{2z}{m}}\right\} = F_Y\left(\sqrt{\frac{2z}{m}}\right).$$

By differentiation we get the frequency of  $Z$  for  $z > 0$ ,

$$\begin{aligned} f_Z(z) &= f_Y\left(\sqrt{\frac{2z}{m}}\right) \cdot \sqrt{\frac{2}{m}} \cdot \frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{2m}} \cdot \frac{1}{\sqrt{z}} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sigma^3} \cdot \frac{2z}{m} \exp\left(-\frac{1}{2} \cdot \frac{2z}{\sigma^2 m}\right) \\ &= \frac{1}{\sqrt{\pi}} \cdot \frac{2}{m\sigma^3} \cdot \sqrt{z} \cdot \exp\left(-\frac{1}{2} \frac{2z}{\sigma^2 m}\right) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \left(\frac{2z}{m\sigma^2}\right)^{\frac{3}{2}-1} \cdot \frac{1}{2^{3/2}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{2z}{m\sigma^2}\right) \frac{2}{\sigma^2 m}. \end{aligned}$$

Since  $f_Z(z) = 0$  for  $z \leq 0$ , it follows that we have a  $\chi^2$ -distribution in the new random variable,  $X = \frac{2Z}{m\sigma^2}$ , or more precisely,  $X \in \chi^2\left(\frac{3}{2}\right)$ .

Notice that  $\frac{dx}{dz} = \frac{2}{\sigma^2 m}$ .

The mean is found by using the change of variable  $t = \frac{y}{\sigma}$ ,

$$\begin{aligned} E\{Z\} &= \frac{m}{2} E\{Y^2\} = \frac{m}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sigma^3} y^2 y^2 \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \\ &= \frac{m\sigma^2}{2} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty t^4 \exp\left(-\frac{t^2}{2}\right) dt = \frac{m\sigma^2}{2} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty t^3 \exp\left(-\frac{t^2}{2}\right) t dt \\ &= \frac{m\sigma^2}{2} \sqrt{\frac{2}{\pi}} \left\{ \left[-t^3 \exp\left(-\frac{t^2}{2}\right)\right]_0^\infty + 3 \int_0^\infty t \cdot \exp\left(-\frac{t^2}{2}\right) dt \right\} = \frac{3}{2} m\sigma^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} E\{Z^2\} &= \frac{m^2}{4} E\{Y^4\} = \frac{m^2}{4} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sigma^3} y^4 y^2 \exp\left(-\frac{y^2}{2\sigma^2}\right) dy, \quad t = \frac{y}{\sigma} \\ &= \frac{m^2\sigma^4}{4} \sqrt{\frac{2}{\pi}} \int_0^\infty t^6 \exp\left(-\frac{t^2}{2}\right) dt = \frac{m^2\sigma^4}{4} \sqrt{\frac{2}{\pi}} \int_0^\infty t^5 \cdot \exp\left(-\frac{t^2}{2}\right) t dt \\ &= \frac{m^2\sigma^4}{4} \sqrt{\frac{2}{\pi}} \cdot 5 \int_0^\infty t^3 \cdot \exp\left(-\frac{t^2}{2}\right) t dt = \frac{m^2\sigma^4}{4} \sqrt{\frac{2}{\pi}} \cdot 5 \cdot 3 \int_0^\infty t \cdot \exp\left(-\frac{t^2}{2}\right) t dt \\ &= \frac{m^2\sigma^4}{4} \cdot 5 \cdot 3 \cdot 1 = \frac{15}{4} (m\sigma^2)^2, \end{aligned}$$

hence

$$V\{Z\} = E\{Z^2\} - (E\{Z\})^2 = \left(\frac{15}{4} - \frac{9}{4}\right) m^2\sigma^4 = \frac{3}{2} m^2\sigma^4.$$

## 6 The Gamma distribution

**Example 6.1** Let  $X_1$  be a non-negative random variable of frequency  $f_1(x)$ , mean  $\mu_1$  and variance  $\sigma_1^2$ , and let the function  $f_2$  be defined by

$$f_2(x) = \begin{cases} \frac{1}{\mu_1} x f_1(x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

1. Prove that  $f_2(x)$  is the frequency of a random variable  $X_2$ .
2. Compute the mean  $\mu_2$  of  $X_2$  (expressed by  $\mu_1$  and  $\sigma_1^2$ ).

Let  $X_1$  be exponentially distributed with parameter  $\alpha$  and frequency  $f_1(x)$ , and let the functions  $f_n(x)$ ,  $n \in \mathbb{N}$ , be defined recursively by

$$f_n(x) = \begin{cases} \frac{1}{\mu_{n-1}} x f_{n-1}(x), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

(where  $\mu_{n-1}$  is the mean corresponding to the frequency  $f_{n-1}(x)$ ).

3. Prove that  $f_n$  is the frequency of a gamma distribution of form parameter  $n$  and scale parameter  $\frac{1}{a}$ .

- 1) Since  $f_1(x) \geq 0$  and  $\mu_1 > 0$ , we have  $f_2(x) \geq 0$  and

$$\int_{-\infty}^{\infty} f_2(x) dx = \frac{1}{\mu_1} \int_0^{\infty} x f_1(x) dx = \frac{1}{\mu_1} E\{X_1\} = \frac{1}{\mu_1} \cdot \mu_1 = 1,$$

thus  $f_2$  is the frequency of a random variable  $X_2$ .

- 2) Then by a straightforward computation,

$$E\{X_2\} = \int_0^{\infty} x f_2(x) dx = \frac{1}{\mu_1} \int_0^{\infty} x^2 f_1(x) dx = \frac{1}{\mu_1} E\{X^2\} = \frac{1}{\mu_1} (V\{X\} + \mu_1^2) = \frac{\sigma_1^2}{\mu_1} + \mu_1.$$

- 3) If

$$f_1(x) = \begin{cases} a e^{-ax} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

then

$$f_2(x) = \begin{cases} \frac{a}{\mu_1} x e^{-ax} & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

This is a constant times the frequency of a  $\Gamma\left(2, \frac{1}{a}\right)$ -distribution. Since  $f_2(x)$  itself is a frequency, the constant is 1.

Then assume that

$$f_n(x) = \begin{cases} \frac{a^n}{\Gamma(n)} x^{n-1} \exp(-ax) & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

It follows by this construction that

$$f_{n+1}(x) = \begin{cases} \frac{1}{\mu_n} \cdot \frac{a^n}{\Gamma(n)} x^n \exp(-ax) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

and we conclude that

$$\frac{1}{\mu_n} \cdot \frac{a^n}{\Gamma(n)} = \frac{a^{n+1}}{\Gamma(n+1)},$$

and that  $X_{n+1} \in \Gamma\left(n+1, \frac{1}{a}\right)$ , and the claim follows by induction.

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**Remark 6.1** It follows from the above that

$$\mu_n = a \frac{\Gamma(n+1)}{\Gamma(n)} = na. \quad \diamond$$

**Example 6.2** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} \frac{1}{B(u,v)} x^{u-1}(1-x)^{v-1}, & 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $u$  and  $v$  denote positive constants, and where  $B(u, v)$  denotes the Beta function.

Find mean and variance of the random variable  $X$ .

The distribution of  $X$  is called the Beta distribution of form parameters  $u$  and  $v$  (this is written  $X \in Be(u, v)$ ). If in particular  $u = v = \frac{1}{2}$ , then we get the Arcussinus distribution, and if  $u = v = 1$ , then we get the uniform distribution over  $]0, 1[$ .

The Beta function is defined by

$$B(u, v) = \int_0^1 z^{u-1}(1-z)^{v-1} dz = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad u, v \in \mathbb{R}_+.$$

The mean is

$$E\{X\} = \frac{1}{B(u, v)} \int_0^1 x^u(1-x)^{v-1} dx = \frac{B(u+1, v)}{B(u, v)} = \frac{\Gamma(u+1)\Gamma(v)}{\Gamma(u+v+1)} \cdot \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} = \frac{u}{u+v}.$$

Analogously,

$$E\{X^2\} = \frac{B(u+2, v)}{B(u, v)} = \frac{(u+1)u}{(u+v+1)(u+v)},$$

so the variance is given by

$$\begin{aligned} V\{X\} &= \frac{u}{u+v} \left\{ \frac{u+1}{u+v+1} - \frac{u}{u+v} \right\} = \frac{u}{u+v} \left\{ \left(1 - \frac{v}{u+v+1}\right) - \left(1 - \frac{v}{u+v}\right) \right\} \\ &= \frac{uv}{u+v} \left\{ \frac{1}{u+v} - \frac{1}{u+v+1} \right\} = \frac{uv}{(u+v)^2(u+v+1)}. \end{aligned}$$

If  $u = v = \frac{1}{2}$ , then

$$E\{X\} = \frac{1}{2} \quad \text{and} \quad V\{X\} = \frac{1}{8}.$$

If  $u = v = 1$ , then

$$E\{X\} = \frac{1}{2} \quad \text{and} \quad V\{X\} = \frac{1}{12}.$$

**Example 6.3** A two-dimensional random variable  $(X, Y)$  has the frequency

$$f(x, y) = \begin{cases} \frac{1}{\Gamma(\mu)\Gamma(\nu)} x^{\mu-1}(y-x)^{\nu-1}e^{-y}, & 0 < x < y, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\mu$  and  $\nu$  denote positive constants.

- 1) Find the marginal frequencies of the random variables  $X$  and  $Y$ .
- 2) Find mean and variance of the random variables  $X$  and  $Y$ .
- 3) Compute the correlation coefficient  $\varrho(X, Y)$ .

1) When  $x > 0$ , then

$$f_X(x) = \frac{x^{\mu-1}}{\Gamma(\mu)\Gamma(\nu)} \int_y^\infty (y-x)^{\nu-1} e^{-y} dy = \frac{1}{\Gamma(\mu)} x^{\mu-1} \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-t-x} dt = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x},$$

so  $X$  is Gamma distributed,  $X \in \Gamma(\mu, 1)$ .

When  $y > 0$ , then

$$\begin{aligned} f_Y(y) &= \frac{e^{-y}}{\Gamma(\mu)\Gamma(\nu)} \int_0^y x^{\mu-1}(y-x)^{\nu-1} dx = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-y} \cdot y^{\mu+\nu-1} \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt \\ &= \frac{1}{\Gamma(\mu+\nu)} y^{\mu+\nu-1} e^{-y}, \end{aligned}$$

so  $Y$  is also Gamma distributed,  $Y \in \Gamma(\mu + \nu, 1)$ .

2) Since  $\alpha = 1$  in both cases, it follows by a known formula that

$$E\{X\} = V\{X\} = \mu \quad \text{and} \quad E\{Y\} = V\{Y\} = \mu + \nu.$$

3) Compute,

$$\begin{aligned} E\{XY\} &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty y e^{-y} \left\{ \int_0^y x^\mu (y-x)^{\nu-1} dx \right\} dy \\ &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty y e^{-y} \cdot y^{\mu+\nu} \left\{ \int_0^1 t^\mu (1-t)^{\nu-1} dt \right\} dy \\ &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \cdot \frac{\Gamma(\mu+1) \cdot \Gamma(\nu)}{\Gamma(\mu+\nu+1)} \int_0^\infty y^{\mu+\nu+1} e^{-y} dy \\ &= \frac{\mu}{\Gamma(\mu+\nu+1)} \cdot \Gamma(\mu+\nu+2) = \mu(\mu+\nu+1). \end{aligned}$$

It follows that

$$\text{Cov}(X, Y) = \mu(\mu + \nu + 1) - \mu(\mu + \nu) = \mu,$$

hence

$$\varrho = \frac{\mu}{\sqrt{\mu(\mu + \nu)}} = \sqrt{\frac{\mu}{\mu + \nu}}.$$

**Example 6.4** Let  $X_1$  and  $X_2$  be mutually independent random variables,

$$X_i \in \Gamma(\mu_i, \alpha), \quad i = 1, 2,$$

and let

$$Z = \frac{X_1}{X_1 + X_2}.$$

Compute the frequency of  $Z$

1) by finding the frequency of  $Y = X_1/X_2$  and then use that

$$Z = \frac{Y}{1 + Y}.$$

2) by finding the frequency of the two-dimensional random variable  $(Z, X_1 + X_2)$ .

1) In general, the frequency of  $Y = X_1/X_2$  is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_1(yx)f_2(x) |x| dx.$$

Since  $X_1$  and  $X_2$  are positive,  $Y$  is also positive. Then for  $y > 0$ ,

$$\begin{aligned} f_Y(y) &= \frac{1}{\Gamma(\mu_1) \alpha^{\mu_1}} \cdot \frac{1}{\Gamma(\mu_2) \alpha^{\mu_2}} \cdot \int_0^{\infty} (yx)^{\mu_1-1} \exp\left(-\frac{yx}{\alpha}\right) \cdot x^{\mu_2-1} \exp\left(-\frac{x}{\alpha}\right) x dx \\ &= \frac{y^{\mu_1-1}}{\Gamma(\mu_1) \Gamma(\mu_2) \alpha^{\mu_1+\mu_2}} \cdot \int_0^{\infty} x^{\mu_1+\mu_2-1} \exp\left(-\frac{y+1}{\alpha} x\right) dx \\ &= \frac{y^{\mu_1-1}}{\Gamma(\mu_1) \Gamma(\mu_2) \alpha^{\mu_1+\mu_2}} \left(\frac{\alpha}{y+1}\right)^{\mu_1+\mu_2-1} \cdot \frac{\alpha}{y+1} \int_0^{\infty} t^{\mu_1+\mu_2-1} e^{-t} dt \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1) \Gamma(\mu_2)} \cdot \frac{y^{\mu_1-1}}{(y+1)^{\mu_1+\mu_2}}. \end{aligned}$$

Now, consider the mapping

$$z = \varphi(y) = \frac{y}{1+y} = 1 - \frac{1}{1+y}, \quad \varphi : \mathbb{R}_+ \rightarrow ]0, 1[.$$

Its inverse is given by

$$y = \varphi^{-1}(z) = \frac{z}{1-z} = -1 + \frac{1}{1-z},$$

from which we in particular derive that

$$y + 1 = \frac{1}{1-z} \quad \text{and} \quad \frac{dy}{dz} = \frac{1}{(1-z)^2}, \quad z \in ]0, 1[.$$

Hence, for  $z \in ]0, 1[$ ,

$$\begin{aligned}
 f_Z(z) &= f_Y(\varphi^{-1}(z)) \cdot |(\varphi^{-1})'(z)| = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \cdot \frac{\left(\frac{z}{1-z}\right)^{\mu_1-1}}{\left(\frac{1}{1-z}\right)^{\mu_1+\mu_2}} \cdot \frac{1}{(1-z)^2} \\
 &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} z^{\mu_1-1} (1-z)^{\mu_1+\mu_2-(\mu_1-1)-2} = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} z^{\mu_1-1} (1-z)^{\mu_2-1},
 \end{aligned}$$

and  $f_Z(z) = 0$  otherwise.

2) Alternatively, consider the mapping

$$\tau(x_1, x_2) = \left( \frac{x_1}{x_1 + x_2}, x_1 + x_2 \right) = (z_1, z_2), \quad (x_1, x_2) \in \mathbb{R}_+^2.$$

It follows that

$$x_1 = z_1(x_1 + x_2) = z_1 z_2 \quad \text{and} \quad x_2 = z_2 - x_1 = z_2(1 - z_1),$$

hence

$$x_1 = z_1 z_2 \quad \text{and} \quad x_2 = z_2(1 - z_1).$$

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The image of  $\tau$  is  $\tau(\mathbb{R}_+^2) = ]0, 1[ \times \mathbb{R}_+$ .

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(z_1, z_2)} = \begin{vmatrix} z_2 & z_1 \\ -z_2 & 1 - z_1 \end{vmatrix} = \begin{vmatrix} z_2 & z_1 \\ 0 & 1 \end{vmatrix} = z_2 > 0,$$

thus the frequency of  $(Z_1, Z_2) = (Z, X_1 + X_2)$  for  $(z_1, z_2) \in ]0, 1[ \times \mathbb{R}_+$  is given by

$$\begin{aligned} f(z_1, z_2) &= f_1(z_1 z_2) f_2(z_2(1 - z_1)) \cdot z_2 \\ &= \frac{1}{\Gamma(\mu_1)} \cdot \frac{1}{\alpha^{\mu_1}} \cdot (z_1 z_2)^{\mu_1 - 1} \exp\left(-\frac{z_1 z_2}{\alpha}\right) \times \\ &\quad \times \frac{1}{\Gamma(\mu_2)} \cdot \frac{1}{\alpha^{\mu_2}} (z_2(1 - z_1))^{\mu_2 - 1} \exp\left(-\frac{z_2 - z_1 z_2}{\alpha}\right) z_2 \\ &= \frac{1}{\Gamma(\mu_1) \Gamma(\mu_2)} \cdot \frac{1}{\alpha^{\mu_1 + \mu_2}} z_1^{\mu_1 - 1} (1 - z_1)^{\mu_2 - 1} \cdot z_2^{\mu_1 - 1 + \mu_2 - 1 + 1} \exp\left(-\frac{z_2}{\alpha}\right) \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1) \Gamma(\mu_2)} z_1^{\mu_1 - 1} (1 - z_1)^{\mu_2 - 1} \cdot \frac{1}{\Gamma(\mu_1 + \mu_2) \alpha^{\mu_1 + \mu_2}} z^{\mu_1 + \mu_2 - 1} \exp\left(-\frac{z_2}{\alpha}\right), \end{aligned}$$

hence  $Z_2$  is  $\Gamma(\mu_1 + \mu_2, \alpha)$  distributed, and  $Z = \frac{X_1}{X_1 + X_2}$  has the frequency

$$f_Z(z) = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1) \Gamma(\mu_2)} z^{\mu_1 - 1} (1 - z)^{\mu_2 - 1} \quad \text{for } z \in ]0, 1[,$$

which is seen by integrating with respect to  $z_2$  and then putting  $z_1 = z$ , and of course  $f_Z(z) = 0$  for  $z \notin ]0, 1[$ .

**Example 6.5** Let  $X_1, X_2, \dots$  be mutually independent random variables, all of the frequency

$$f(x) = \begin{cases} x e^{-x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

and let  $S_n = \sum_{i=1}^n X_i$ ,  $n \in \mathbb{N}$ , and  $S_0 = 0$ .

1. Find the distribution of  $S_n$ .

Let  $t$  be a fixed positive number. We define a random variable  $N$  of values in  $\mathbb{N}_0$  by

$$N = n, \quad \text{if } S_n \leq t \text{ and } S_{n+1} > t.$$

2. Prove that

$$P\{N = n\} = \frac{t^{2n+1}e^{-t}}{(2n+1)!} + \frac{t^{2n}e^{-t}}{(2n)!}, \quad n \in \mathbb{N}_0.$$

3. Compute the mean  $E\{N\}$ .

1) Every  $X_i$  is  $\Gamma(2, 1)$  distributed, so  $S_n$  is  $\Gamma(2n, 1)$  distributed, and  $S_n$  has the frequency

$$f_n(s) = \begin{cases} \frac{1}{(2n-1)!} x^{2n-1} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases} \quad n \in \mathbb{N}.$$

If  $n = 0$ , then  $S_0$  is causally distributed.

2) We see that

$$\begin{aligned} P\{N = 0\} &= P\{S_0 \leq t, S_1 > t\} = P\{S_1 > t\} \\ &= \int_t^\infty x e^{-x} dx = [-x e^{-x} - e^{-x}]_t^\infty = t e^{-t} + e^{-t}, \end{aligned}$$

in agreement with the formula.

If  $n \in \mathbb{N}$ , then

$$\begin{aligned} P\{N = n\} &= P\{S_n \leq t \wedge S_{n+1} > t\} = P\{S_n \leq t\} - P\{S_n \leq t \wedge S_{n+1} \leq t\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = \int_0^t \frac{x^{2n-1}}{(2n-1)!} e^{-x} dx - \int_0^t \frac{x^{2n+1}}{(2n+1)!} e^{-x} dx, \end{aligned}$$

thus

$$\begin{aligned} P\{N = n\} &= \int_0^t \frac{x^{2n-1}}{(2n-1)!} e^{-x} dx - \int_0^t \frac{x^{2n+1}}{(2n+1)!} e^{-x} dx \\ &= \int_0^t \frac{x^{2n-1}}{(2n-1)!} e^{-x} dx + \left[ \frac{x^{2n+1}}{(2n+1)!} e^{-x} \right]_0^t - \int_0^t \frac{x^{2n}}{(2n)!} e^{-x} dx \\ &= \int_0^t \frac{x^{2n-1}}{(2n-1)!} e^{-x} dx + \frac{t^{2n+1}}{(2n+1)!} e^{-t} + \left[ \frac{x^{2n}}{(2n)!} e^{-x} \right]_0^t - \int_0^t \frac{x^{2n-1}}{(2n-1)!} e^{-x} dx \\ &= \frac{t^{2n+1}}{(2n+1)!} e^{-t} + \frac{t^{2n}}{(2n)!} e^{-t}, \end{aligned}$$

and the claim is proved.

3) The mean is

$$\begin{aligned}
 E\{N\} &= e^{-t} \sum_{n=1}^{\infty} \frac{n \cdot t^{2n+1}}{(2n+1)!} + e^{-t} \sum_{n=1}^{\infty} \frac{n \cdot t^{2n}}{(2n)!} \\
 &= \frac{1}{2} e^{-t} \left\{ \sum_{n=1}^{\infty} (2n+1) \frac{t^{2n+1}}{(2n+1)!} - \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n-1)!} \right\} \\
 &= \frac{1}{2} e^{-t} \left\{ t \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} + t \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} - \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} + t \right\} \\
 &= \frac{1}{2} e^{-t} \left\{ t \sum_{n=0}^{\infty} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \right\} = \frac{1}{2} e^{-t} \{t e^{-t} - \sinh t\} = \frac{1}{2} t - \frac{1}{4} (1 - e^{-2t}).
 \end{aligned}$$

**Example 6.6** Given a sequence of random variables  $(X_n)$ , where  $X_n$  has the frequency

$$f_n(x) = \begin{cases} \frac{1}{(n-1)!} a^n x^{n-1} e^{-ax} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where  $a$  denotes a positive constant.

- 1) Find the mean  $E\{X_n\}$  and the variance  $V\{X_n\}$  of the random variable  $X_n$ .
- 2) Compute the probability  $P\{X_4 > E\{X_4\}\}$ .
- 3) Assuming that  $X_2$  and  $X_n$  are independent, find the frequency of  $X_n + X_2$ .
- 4) Assuming that  $Y_1, Y_2, \dots, Y_n$  are mutually independent random variables, all of the same distribution as  $X_2$ , prove that the frequency of the random variable

$$Z_n = Y_1 + Y_2 + \dots + Y_n$$

is given by  $f_{2n}$ .

- 5) Apply the Central Limit Theorem to prove that

$$P\{X_{2n} > E\{X_{2n}\}\} \rightarrow \frac{1}{2} \quad \text{for } n \rightarrow \infty.$$

- 1) It follows from  $X_n \in \Gamma\left(n, \frac{1}{a}\right)$  that

$$E\{X_n\} = \frac{n}{a} \quad \text{and} \quad V\{X_n\} = \frac{n}{a^2}.$$

2) By a computation,

$$\begin{aligned} P\{X_4 > E\{X_4\}\} &= P\left\{X_4 > \frac{4}{a}\right\} = \int_{\frac{4}{a}}^{\infty} \frac{1}{3!} a^4 x^3 e^{-ax} dx = \frac{1}{6} \int_4^{\infty} t^3 e^{-t} dt \\ &= \frac{1}{6} \left[-(t^3 + 3t^2 + 6t + 6)e^{-t}\right]_4^{\infty} = \frac{64 + 48 + 24 + 6}{6} e^{-4} = \frac{71}{3} e^{-4}. \end{aligned}$$

3) Now,  $X_n + X_2 \in \Gamma\left(n + 2, \frac{1}{a}\right)$ , so the frequency is

$$f(x) = f_{n+2}(x) = \begin{cases} \frac{1}{(n+1)!} a^{n+2} x^{n+1} e^{-ax} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

4) Since every  $Y_k \in \Gamma\left(2, \frac{1}{a}\right)$ , we have

$$Z_n = Y_1 + Y_2 + \cdots + Y_n \in \Gamma\left(2n, \frac{1}{a}\right),$$

and the frequency is again  $f_{2n}(x)$ , thus  $Z_n = X_{2n}$ .

5) Now,

$$X_{2n} = Z_n = Y_1 + Y_2 + \cdots + Y_n,$$

and all the  $Y_k$  have the same distribution and are mutually independent. Hence, we can apply the Central Limit Theorem,

$$\begin{aligned} P\{X_{2n} > E\{X_{2n}\}\} &= 1 - P\{X_{2n} - E\{X_{2n}\} \leq 0\} \\ &= 1 - P\left\{\frac{X_{2n} - E\{X_{2n}\}}{\sqrt{V\{X_{2n}\}}} \leq 0\right\} \rightarrow 1 - \Phi(0) = 1 - \frac{1}{2} = \frac{1}{2}, \end{aligned}$$

for  $n \rightarrow \infty$ .

**Example 6.7** We assume that the lifetime of an instrument is exponentially distributed of parameter  $a$ . If the instrument is ruined, it is repaired, and the lifetime of the repaired instrument is assumed to have the same distribution as the lifetime of a new instrument. When the instrument is ruined for the  $k$ -th time, it is rejected. We assume that the lifetimes of the  $k$  periods of function are mutually independent random variables, thus they are all exponentially distributed of parameter  $a$ . Let  $Y_k$  denote the total lifetime of the instrument.

1) Find  $\mu_k = E\{Y_k\}$  and  $\sigma_k^2 = V\{Y_k\}$ .

2) Prove that for every  $\varepsilon > 0$ ,

$$P\left\{\left|\frac{Y_k}{\mu_k} - 1\right| \geq \varepsilon\right\} \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

3) What is the distribution of  $Y_k$ ?

The Gamma distribution is reproductive in the form parameter, when the scale parameter is kept fixed. It therefore follows that  $Y_k \in \Gamma\left(k, \frac{1}{a}\right)$ . Hence

1)

$$\mu_k = E\{Y_k\} = \frac{k}{a} \quad \text{and} \quad \sigma_k^2 = V\{Y_k\} = \frac{k}{a^2}.$$

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2) We get by Chebyshev's inequality,

$$P \left\{ \left| \frac{Y_k}{\mu_k} - 1 \right| \geq \varepsilon \right\} = P \{ |Y_k - \mu_k| \geq \varepsilon \mu_k \} \leq \frac{\sigma_k^2}{\varepsilon^2 \mu_k^2} = \frac{k a^2}{a^2 \varepsilon^2 k^2} = \frac{1}{\varepsilon^2 k} \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

3) This has already been proved:  $Y_k \in \Gamma \left( k, \frac{1}{a} \right)$ .

**Example 6.8** Let  $X_1, X_2, \dots, X_n$  be mutually independent random variable, all  $N(\mu, \sigma^2)$ -distributed. As usual we introduce the random variables

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Find the variance of  $S^2$ .

HINT: Apply properties of the  $\chi^2$  distribution.

Let

$$Y_n = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu)^2.$$

Then

$$Y_n \in \Gamma \left( \frac{n}{2}, 2 \right) = \chi^2(n),$$

hence

$$E \{ Y_n \} = n \quad \text{and} \quad V \{ Y_n \} = 2n.$$

Then

$$\begin{aligned} S^2 &= \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \mu + \mu - \bar{X})^2 \\ &= \frac{1}{n-1} \sum_{i=1}^n (X_i - \mu)^2 + \frac{2}{n-1} \sum_{i=1}^n (X_i - \mu) (\mu - \bar{X}) + \frac{1}{n-1} \sum_{i=1}^n (\bar{X} - \mu)^2 \\ &= \frac{\sigma^2}{n-1} Y_n - 2 \frac{n}{n-1} (\bar{X} - \mu)^2 + \frac{n}{n-1} (\bar{X} - \mu)^2 = \frac{\sigma^2}{n-1} Y_n - \frac{n}{n-1} (\bar{X} - \mu)^2. \end{aligned}$$

It follows that we can assume that  $\mu = 0$ , which will simplify the setup. Thus

$$Y_n = \frac{1}{\sigma^2} \sum_{i=1}^n X_i^2,$$

and

$$\begin{aligned}
 S^2 &= \frac{\sigma^2}{n-1} Y_n - \frac{n}{n-1} \bar{X}^2 = \frac{\sigma^2}{n-1} Y_n - \frac{n}{n-1} \left\{ \frac{1}{n} \sum_{i=1}^n X_i \right\}^2 \\
 &= \frac{\sigma^2}{n-1} Y_n - \frac{1}{n-1} \cdot \frac{1}{n} \left\{ \sum_{i=1}^n X_i^2 + 2 \sum_{1 \leq i < j \leq n} X_i X_j \right\} \\
 &= \frac{\sigma^2}{n-1} Y_n - \frac{\sigma^2}{n-1} \cdot \frac{1}{n} Y_n - \frac{1}{n-1} \cdot \frac{2}{n} \sum_{1 \leq i < j \leq n} X_i X_j \\
 &= \begin{cases} \frac{\sigma^2}{n} Y_n - \frac{2}{(n-1)n} \sum_{1 \leq i < j \leq n} X_i X_j, \\ \frac{1}{n} \sum_{i=1}^n X_i^2 - \frac{2}{(n-1)n} \sum_{1 \leq i < j \leq n} X_i X_j, \end{cases}
 \end{aligned}$$

where we shall use both expressions in the following.

First consider

$$\text{Cov} \left( \sum_{i=1}^n X_i^2, \sum_{1 \leq j < k \leq n} X_j X_k \right).$$

Since

$$\begin{aligned}
 \text{Cov} (X_i^2, X_j X_k) &= E \{ X_i^2 X_j X_k \} - (E \{ X_i^2 \} E \{ X_j X_k \})^2 \\
 &= E \{ X_\ell \} \cdots - (E \{ X_i^2 \} \cdot E \{ X_j \} \cdot E \{ X_k \})^2 = 0,
 \end{aligned}$$

where  $\ell = j$  or  $\ell = k$ , and  $E \{ X_m \} = \mu = 0$ , the covariance is zero.

This implies that

$$V \{ S^2 \} = \frac{\sigma^4}{n^2} V \{ Y_n \} + \left( \frac{2}{(n-1)n} \right)^2 V \left\{ \sum_{1 \leq i < j \leq n} X_i X_j \right\}.$$

Analogously,

$$\begin{aligned}
 V \left\{ \sum_{1 \leq i < j \leq n} X_i X_j \right\} &= \text{Cov} \left( \sum_{j=2}^n \sum_{i=1}^{j-1} X_i X_j, \sum_{k=2}^n \sum_{\ell=1}^{k-1} X_k X_\ell \right) = \sum_{j=2}^n \sum_{i=1}^{j-1} \text{Cov} \left( X_i X_j, \sum_{k=2}^n \sum_{\ell=1}^{k-1} X_k X_\ell \right) \\
 &= \sum_{j=2}^n \sum_{i=1}^{j-1} \text{Cov} (X_i X_j, X_i X_j) = \sum_{j=2}^n \sum_{i=1}^{j-1} V \{ X_i X_j \},
 \end{aligned}$$

where

$$\begin{aligned}
 V \{ X_i X_j \} &= E \{ X_i^2 X_j^2 \} - (E \{ X_i \} E \{ X_j \})^2 = E \{ X_i^2 \} E \{ X_j^2 \} - 0^2 \\
 &= V \{ X_i \} V \{ X_j \} = \sigma^4.
 \end{aligned}$$

The sum contains in total  $\frac{1}{2} n(n-1)$  terms, hence by insertion,

$$V \{ S^2 \} = \frac{\sigma^4}{n^2} \cdot 2n + \left( \frac{2}{(n-1)n} \right) \cdot \frac{1}{2} (n-1)n \sigma^4 = \frac{2\sigma^4}{n} \left( 1 + \frac{1}{n-1} \right) = \frac{2\sigma^4}{n-1}.$$

ALTERNATIVELY (and somewhat easier) we see that

$$\frac{n-1}{\sigma^2} S^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \in \chi^2(n-1),$$

hence

$$V \left\{ \frac{n-1}{\sigma^2} S^2 \right\} = 2(n-1).$$

Then

$$V \{S^2\} = 2(n-1) \cdot \frac{\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}.$$

**Example 6.9** Let  $X_1$  and  $X_2$  be independent random variables of the frequencies

$$f_{X_1}(x_1) = \begin{cases} x_1 e^{-x_1}, & x_1 > 0, \\ 0, & x_1 \leq 0, \end{cases} \quad f_{X_2}(x_2) = \begin{cases} \frac{1}{2} x_2^2 e^{-x_2}, & x_2 > 0, \\ 0, & x_2 \leq 0. \end{cases}$$

1. Find the means  $E \{X_1\}$  and  $E \{X_2\}$ .
2. Find the variances  $V \{X_1\}$  and  $V \{X_2\}$ .

Let the two-dimensional random variable  $(Y_1, Y_2) = \tau(X_1, X_2)$  be given by

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = X_1 + X_2.$$

3. Prove that  $\tau$  maps  $\mathbb{R}_+ \times \mathbb{R}_+$  bijectively onto  $]0, 1[ \times \mathbb{R}_+$ .
4. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .
5. Find the marginal frequencies of  $Y_1$  and  $Y_2$ .
6. Check if  $Y_1$  and  $Y_2$  are independent.
7. Find the means  $E \{Y_1\}$  and  $E \{Y_2\}$ .
8. Find  $P \{Y_1 Y_2 < 1\}$ .

It is obvious that  $X_1 \in \Gamma(2, 1)$  and  $X_2 \in \Gamma(3, 1)$  are Gamma distributed.

**1. and 2.** It is immediately seen that

$$E \{X_1\} = 2, \quad E \{X_2\} = 3 \quad \text{and} \quad V \{X_1\} = 2, \quad V \{X_2\} = 3.$$



3. When the equations are solved with respect to  $(x_1, x_2)$ , we get

$$x_1 = y_1 y_2 \quad \text{and} \quad x_2 = y_2 - x_1 = (1 - y_1) y_2.$$

The image is

$$\{(y_1, y_2) \mid y_1 y_2 > 0, y_1 > 0, (1 - y_1) y_2 > 0\},$$

i.e.  $]0, 1[ \times \mathbb{R}_+$  after a reduction.

The Jacobian is

$$\frac{\partial (x_1, x_2)}{\partial (y_1, y_2)} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2 > 0.$$


4. If  $0 < y_1 < 1$  and  $y_2 > 0$ , then

$$\begin{aligned} k(y_1, y_2) &= f_{X_1}(y_1 y_2) f_{X_2}(y_2 - y_1 y_2) \cdot y_2 \\ &= y_1 y_2 \exp(-y_1 y_2) \cdot \frac{1}{2} y_2^2 (1 - y_1)^2 \exp(-y_2 + y_1 y_2) \cdot y_2 = \frac{1}{2} y_1 (1 - y_1)^2 \cdot y_2^4 e^{-y_2} \\ &= 12 y_1 (1 - y_1)^2 \cdot \frac{1}{24} y_2^4 e^{-y_2}, \end{aligned}$$

and  $k(y_1, y_2) = 0$  otherwise.

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**5., 6. and 7.** Clearly,  $Y_1 \in Be(2, 3)$  is Beta distributed, and  $Y_2 \in \Gamma(5, 1)$  is Gamma distributed, and  $Y_1$  and  $Y_2$  are independent, and finally,

$$E\{Y_1\} = \frac{2}{2+3} = \frac{2}{5} \quad \text{and} \quad E\{Y_2\} = 5.$$

**8.** It follows from  $Y_1 Y_2 = X_1$  that

$$P\{Y_1 Y_2 < 1\} = P\{X_1 < 1\} = \int_0^1 x_1 e^{-x_1} dx_1 = \left[-(x_1 + 1)e^{-x_1}\right]_0^1 = 1 - \frac{2}{e}.$$

**Example 6.10** Let  $X_1$  and  $X_2$  be independent random variables of the frequencies

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{6} x_1^3 e^{-x_1}, & x_1 > 0, \\ 0, & x_1 \leq 0, \end{cases} \quad f_{X_2}(x_2) = \begin{cases} e^{-x_2}, & x_2 > 0, \\ 0, & x_2 \leq 0. \end{cases}$$

**1.** Find the means  $E\{X_1\}$  and  $E\{X_2\}$ .

Let the two-dimensional random variable  $(Y_1, Y_2) = \tau(X_1, X_2)$  be given by

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{2X_2}{X_1 + X_2}.$$

**2.** Prove that  $\tau$  maps  $\mathbb{R}_+ \times \mathbb{R}_+$  bijectively onto  $\mathbb{R}_+ \times ]0, 2[$ .

**3.** Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .

**4.** Find the marginal frequencies of  $Y_1$  and  $Y_2$ .

**5.** Check if  $Y_1$  and  $Y_2$  are independent.

**6.** Find the means  $E\{Y_1\}$  and  $E\{Y_2\}$ .

**1.** Since  $X_1 \in \Gamma(4, 1)$  and  $X_2 \in \Gamma(1, 1)$ , we get

$$E\{X_1\} = 4 \quad \text{and} \quad E\{X_2\} = 1.$$

**2.** The equations are solved,

$$\begin{cases} y_1 = x_1 + x_2, \\ y_2 = \frac{2x_2}{x_1 + x_2}, \end{cases} \quad \text{f\u00e5s} \quad \begin{cases} x_1 = y_1 \left(1 - \frac{1}{2} y_2\right), \\ x_2 = \frac{1}{2} y_1 y_2, \end{cases}$$

and we see that  $\tau$  is injective, and that  $\tau$  maps  $\mathbb{R}_+ \times \mathbb{R}_+$  onto

$$\left\{ (y_1, y_2) \mid y_1 \left(1 - \frac{1}{2} y_2\right) > 0, y_1 y_2 > 0 \right\} = \{(y_1, y_2) \mid y_1 > 0, 0 < y_2 < 2\},$$

thus

$$\tau : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \times ]0, 2[ \quad \text{bijectively.}$$

The corresponding Jacobian is

$$\frac{\partial (x_1, x_2)}{\partial (y_1, y_2)} = \begin{vmatrix} 1 - \frac{1}{2} y_2 & -\frac{1}{2} y_1 \\ \frac{1}{2} y_2 & \frac{1}{2} y_1 \end{vmatrix} = \frac{1}{2} y_1.$$

- 3.** It follows from the preparations in **2.** that the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$  for  $(y_1, y_2) \in \mathbb{R}_+ \times ]0, 2[$ , is given by

$$\begin{aligned} k(y_1, y_2) &= f_{X_1} \left( y_1 \left( 1 - \frac{1}{2} y_2 \right) \right) \cdot f_{X_2} \left( \frac{1}{2} y_1 y_2 \right) \cdot \frac{1}{2} y_1 \\ &= \frac{1}{6} y_1^3 \left( 1 - \frac{1}{2} y_2 \right)^3 \cdot \exp \left( -y_1 \left( 1 - \frac{1}{2} y_2 \right) \right) \cdot \exp \left( -\frac{1}{2} y_1 y_2 \right) \cdot \frac{1}{2} y_1 \\ &= \left\{ \frac{1}{24} y_1^4 e^{-y_1} \right\} \cdot 2 \left\{ 1 - \frac{1}{2} y_2 \right\}^3, \quad y_1 > 0, 0 < y_2 < 2, \end{aligned}$$

and  $k(y_1, y_2) = 0$  otherwise.

- 4. and 5.** Here **3.** immediately implies that  $Y_1$  and  $Y_2$  are independent random variables of the frequencies

$$k_{Y_1}(y_1) = \begin{cases} \frac{1}{24} y_1^4 e^{-y_1}, & y_1 > 0, \\ 0, & y_1 \leq 0, \end{cases} \quad \text{i.e. } Y_1 \in \Gamma(5, 1),$$

and

$$k_{Y_2}(y_2) = \begin{cases} 2 \left\{ 1 - \frac{1}{2} y_2 \right\}^3, & 0 < y_2 < 2, \\ 0, & \text{otherwise.} \end{cases}$$

- 6.** Clearly,  $E\{Y_1\} = 5$ .

Furthermore,

$$\begin{aligned} E\{Y_2\} &= \int_0^2 y_2 k_{Y_2}(y_2) dy_2 = 2 \int_0^2 y_2 \left( 1 - \frac{1}{2} y_2 \right)^3 dy_2 \\ &= 4 \int_0^2 \left\{ 1 - \left( 1 - \frac{1}{2} y_2 \right) \right\} \cdot \left( 1 - \frac{1}{2} y_2 \right)^3 dy_2 \\ &= 8 \int_0^2 \left\{ \left( 1 - \frac{1}{2} y_2 \right)^3 - \left( 1 - \frac{1}{2} y_2 \right)^4 \right\} \frac{1}{2} dy_2 \\ &= 8 \left[ -\frac{1}{4} \left( 1 - \frac{1}{2} y_2 \right)^4 + \frac{1}{5} \left( 1 - \frac{1}{2} y_2 \right)^5 \right]_0^2 \\ &= 8 \left\{ \frac{1}{4} - \frac{1}{5} \right\} = \frac{8}{20} = \frac{2}{5}. \end{aligned}$$

ALTERNATIVELY,  $2X_2 = Y_1Y_2$ . Since  $Y_1$  and  $Y_2$  are independent, we get

$$2 E \{X_2\} = E \{Y_1\} \cdot E \{Y_2\},$$

hence

$$E \{Y_2\} = \frac{2 E \{X_2\}}{E \{Y_1\}} = \frac{2}{5}.$$

**Example 6.11** Let  $(X_1, X_2)$  be a two-dimensional random variable of frequency

$$h(x_1, x_2) = \begin{cases} \frac{1}{2} (x_1 + x_2) e^{-(x_1+x_2)}, & x_1 > 0 \text{ og } x_2 > 0, \\ 0, & \text{ellers.} \end{cases}$$

1. Find the marginal frequencies of  $X_1$  and  $X_2$ .
2. Compute mean and variance of  $X_1$ .

Define the random variables  $Y_1$  and  $Y_2$  by

$$(Y_1, Y_2) = \tau(X_1, X_2) = \left( X_1 + X_2, \frac{X_1 - X_2}{X_1 + X_2} \right).$$

One may use without proof that the vector function  $\tau$  given by

$$\tau(x_1, x_2) = \left( x_1 + x_2, \frac{x_1 - x_2}{x_1 + x_2} \right),$$

maps  $\mathbb{R}_+ \times \mathbb{R}_+$  bijectively onto

$$D' = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0 \wedge -1 < y_2 < 1\}.$$

3. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .
4. Find the marginal frequencies of  $Y_1$  and  $Y_2$ , and check if  $Y_1$  and  $Y_2$  are independent.
5. Compute mean and variance of  $Y_1$ .
6. Compute  $\text{Cov}(X_1, X_2)$ .

1) We may without loss of generality restrict ourselves to the case where  $x_1 > 0$  and  $x_2 > 0$ . Then

$$\begin{aligned} f_{X_1}(x_1) &= \frac{1}{2} \int_0^\infty (x_1 + x_2) e^{-(x_1+x_2)} dx_2 = \frac{1}{2} x_1 e^{-x_1} \int_0^\infty e^{-x_2} dx_2 + \frac{1}{2} e^{-x_1} \int_0^\infty x_2 e^{-x_2} dx_2 \\ &= \frac{1}{2} (x_1 + 1) e^{-x_1}, \end{aligned}$$

thus

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{2} (x_1 + 1) e^{-x_1} & \text{for } x_1 > 0, \\ 0 & \text{for } x_1 \leq 0. \end{cases}$$

By the symmetry,

$$f_{X_2}(x_2) = \begin{cases} \frac{1}{2} (x_2 + 1) e^{-x_2} & \text{for } x_2 > 0, \\ 0 & \text{for } x_2 \leq 0. \end{cases}$$

2) The mean is

$$E\{X_1\} = \frac{1}{2} \int_0^\infty x_1^2 e^{-x_1} dx_1 + \frac{1}{2} \int_0^\infty x_1 e^{-x_1} dx_1 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Now,

$$E\{X_1^2\} = \frac{1}{2} \int_0^\infty x_1^3 e^{-x_1} dx_1 + \frac{1}{2} \int_0^\infty x_1^2 e^{-x_1} dx_1 = \frac{3!}{2} + \frac{2!}{2} = 4,$$

so the variance is


$$V\{X_1\} = 4 - \left(\frac{3}{2}\right)^2 = \frac{16 - 9}{4} = \frac{7}{4}.$$

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3) From

$$y_1 = x_1 + x_2 \quad \text{and} \quad y_2 = \frac{x_1 - x_2}{x_1 + x_2},$$

i.e.  $y_1 y_2 = x_1 - x_2$ , follows that

$$x_1 = \frac{1}{2} (y_1 + y_1 y_2) = \frac{1}{2} y_1 (1 + y_2)$$

and

$$x_2 = \frac{1}{2} (y_1 - y_1 y_2) = \frac{1}{2} y_1 (1 - y_2).$$

The Jacobian is

$$\frac{\partial (x_1, x_2)}{\partial (y_1, y_2)} = \begin{vmatrix} \frac{1}{2}(1+y_2) & \frac{1}{2}y_1 \\ \frac{1}{2}(1-y_2) & -\frac{1}{2}y_1 \end{vmatrix} = \frac{1}{4} y_1 \begin{vmatrix} 1+y_2 & 1 \\ 1-y_2 & -1 \end{vmatrix} = -\frac{1}{2} y_1,$$

thus if  $y_1 > 0$  and  $-1 < y_2 < 1$ , then

$$k(y_1, y_2) = h(x_1(y), x_2(y)) \cdot \frac{1}{2} y_1 = \frac{1}{2} y_1^2 e^{-y_1} \cdot \frac{1}{2} = \frac{1}{4} y_1^2 e^{-y_1}.$$

4) It follows immediately from the result of **3.** that  $Y_1$  and  $Y_2$  are independent and that  $Y_1 \in \Gamma(3, 1)$ , and  $Y_2$  is uniformly distributed over  $] -1, 1[$ .

The marginal frequencies are

$$g_{Y_1}(y_1) = \begin{cases} \frac{1}{2} y_1^2 e^{-y_1} & \text{for } y_1 > 0, \\ 0 & \text{for } y_1 \leq 0, \end{cases}$$

and

$$g_{Y_2}(y_2) = \begin{cases} \frac{1}{2} & \text{for } -1 < y_2 < 1, \\ 0 & \text{ellers.} \end{cases}$$

5) From  $Y_1 \in \Gamma(3, 1)$  follows that  $E\{Y_1\} = 3$  and  $V\{Y_1\} = 3$ .

6) The covariance can be found in several ways. We have e.g.

$$X_1 X_2 = \frac{1}{4} \left\{ (X_1 + X_2)^2 - (X_1 - X_2)^2 \right\} = \frac{1}{4} \{ Y_1^2 - Y_1^2 Y_2^2 \},$$

hence

$$\begin{aligned} \text{Cov}(X_1 X_2) &= E\{X_1 X_2\} - E\{X_1\} E\{X_2\} = \frac{1}{4} E\{Y_1^2\} (1 - E\{Y_2^2\}) - \frac{3}{2} \cdot \frac{3}{2} \\ &= \frac{1}{4} \left( V\{Y_1\} + (E\{Y_1\})^2 \right) \cdot \left( 1 - V\{Y_2\} - (E\{Y_2\})^2 \right) - \frac{9}{4} \\ &= \frac{1}{4} (3 + 3^3) \left\{ 1 - \frac{1}{2} \cdot 2^2 - 0^2 \right\} - \frac{9}{4} = \frac{12}{4} \left( 1 - \frac{1}{3} \right) - \frac{9}{4} = \frac{8}{4} - \frac{9}{4} = -\frac{1}{4}. \end{aligned}$$

ALTERNATIVELY,

$$\begin{aligned} \text{Cov}(X_1, X_2) &= \frac{1}{2} \int_0^\infty \int_0^\infty x_1 x_2 (x_1 + x_2) e^{-(x_1+x_2)} dx_1 dx_2 - \frac{9}{4} \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty (x_1^2 x_2 + x_1 x_2^2) e^{-(x_1+x_2)} dx_1 dx_2 - \frac{9}{4} \\ &= 2 \cdot \frac{1}{2} \int_0^\infty \int_0^\infty x_1^2 x_2 e^{-x_1-x_2} dx_1 dx_2 - \frac{9}{4} \\ &= \int_0^\infty x_1^2 e^{-x_1} dx_1 \cdot \int_0^\infty x_2 e^{-x_2} dx_2 - \frac{9}{4} = 2 \cdot 1 - \frac{9}{4} = -\frac{1}{4}. \end{aligned}$$

**Example 6.12** Let  $X_1$  and  $X_2$  be independent random variables of the frequencies

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{6} x_1^3 e^{-x_1}, & x_1 > 0, \\ 0, & \text{otherwise.} \end{cases} \quad f_{X_2}(x_2) = \begin{cases} \frac{2}{9} (3 - x_2), & 0 < x_2 < 3, \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the means  $E\{X_1\}$  and  $E\{X_2\}$ .

Let the two-dimensional random variable  $(Y_1, Y_2) = \tau(X_1, X_2)$  be given by

$$Y_1 = \frac{1}{3} X_1 X_2, \quad Y_2 = \frac{1}{3} X_1 (3 - X_2).$$

2. Prove that  $\tau$  maps  $\mathbb{R}_+ \times ]0, 3[$  bijectively onto  $\mathbb{R}_+ \times \mathbb{R}_+$ .
3. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .
4. Find the marginal frequencies of  $Y_1$  and  $Y_2$ .
5. Check if  $Y_1$  and  $Y_2$  are independent.
6. Find the means  $E\{Y_1\}$  and  $E\{Y_2\}$ .

1) Clearly,  $X_1$  is  $\Gamma(4, 1)$ -distributed, so  $E\{X_1\} = 4$ . Furthermore,

$$E\{X_2\} = \frac{2}{9} \int_0^3 (3x_2 - x_2^2) dx_2 = \frac{2}{9} \left[ \frac{3}{2} x_2^2 - \frac{1}{3} x_2^3 \right]_0^3 = 3 - 2 = 1.$$

2) It follows from

$$y_1 = \frac{1}{3} x_1 x_2 \quad \text{and} \quad y_2 = \frac{1}{3} x_1 (3 - x_2) = x_1 - \frac{1}{3} x_1 x_2$$

that

$$x_1 = y_1 + y_2 \quad \text{and} \quad x_2 = \frac{3y_1}{y_1 + y_2} = 3 - \frac{3y_2}{y_1 + y_2},$$

whence  $x_1 = y_1 + y_2 > 0$ , and thus  $y_1 > 0$  (because  $x_2 > 0$ ), and  $y_2 > 0$  (because  $\frac{y_1}{y_1 + y_2} < 1$ ), so  $\tau(\mathbb{R}_+ \times ]0, 3[) = \mathbb{R}_+ \times \mathbb{R}_+$ .

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} 1 & 1 \\ \frac{3y_2}{(y_1 + y_2)^2} & -\frac{3y_1}{(y_1 + y_2)^2} \end{vmatrix} = -\frac{3}{(y_1 + y_2)^2} (y_1 + y_2) = -\frac{3}{y_1 + y_2} < 0.$$

3) The simultaneous frequency of  $(Y_1, Y_2)$  is for  $y_1 > 0$  and  $y_2 > 0$  given by

$$\begin{aligned} k(y_1, y_2) &= f_{X_1}(y_1 + y_2) f_{X_2}\left(\frac{3y_1}{y_1 + y_2}\right) \cdot \frac{3}{y_1 + y_2} \\ &= \frac{1}{6} (y_1 + y_2)^3 \exp(-(y_1 + y_2)) \cdot \frac{2}{9} \left(3 - \frac{3y_1}{y_1 + y_2}\right) \cdot \frac{3}{y_1 + y_2} \\ &= \frac{1}{3} y_2 (y_1 + y_2) \exp(-(y_1 + y_2)), \end{aligned}$$

for  $y_1 > 0$  and  $y_2 > 0$ , and  $k(y_1, y_2) = 0$  otherwise.

4) When  $y_1 > 0$  and  $y_2 > 0$  the marginal frequencies are given by

$$k_{Y_1}(y_1) = \frac{1}{3} e^{-y_1} \int_0^\infty (y_2 y_1 + y_2^2) e^{-y_2} dy_2 = \frac{1}{3} (y_1 + 2) e^{-y_1},$$

and

$$k_{Y_2}(y_2) = \frac{1}{3} y_2 e^{-y_2} \int_0^\infty (y_1 + y_2) e^{-y_1} dy_1 = \frac{1}{3} y_2 (1 + y_2) e^{-y_2},$$

and  $= 0$  otherwise.

5) Since  $k_{Y_1}(y_1) k_{Y_2}(y_2) \neq k(y_1, y_2)$ , it follows that  $Y_1$  and  $Y_2$  are not independent.

6) The means are

$$E\{Y_1\} = \frac{1}{3} \int_0^\infty (y_1^2 + 2y_1) e^{-y_1} dy_1 = \frac{1}{3} (2! + 2 \cdot 1!) = \frac{4}{3}$$

and

$$E\{Y_2\} = \frac{1}{3} \int_0^\infty (y_2^2 + y_2^3) e^{-y_2} dy_2 = \frac{1}{3} (2! + 3!) = \frac{8}{3}.$$



**Example 6.13** *A shop is visited by both male and female customers, mutually independent of each other. The arrival times are measured from  $t = 0$  (the opening time). Let*

$$X_1, X_1 + X_2, X_1 + X_2 + X_3, \dots,$$

*denote the arrival times of the first, the second, the third, ... of the male customers, and let analogously*

$$Y_1, Y_1 + Y_2, Y_1 + Y_2 + Y_3, \dots$$

*denote the arrival times of the first, the second, the third ... of the female customers. We assume that the random variables  $X_i, i \in \mathbb{N}$ , are mutually independent and identically distributed of the frequency*

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

*and that the random variables  $Y_i, i \in \mathbb{N}$ , are mutually independent and identically distributed of the frequency*

$$g(y) = \begin{cases} \mu e^{-\mu y}, & y \geq 0, \\ 0, & y < 0, \end{cases}$$

*(here  $\lambda$  and  $\mu$  are positive constants).*

*We furthermore assume that the  $Y_i$  are independent of the  $X_i$ .*

- 1) *Find the frequency  $f_2(x)$  of  $X_1 + X_2$ .*
- 2) *Find the frequency  $f_3(x)$  of  $X_1 + X_2 + X_3$ .*
- 3) *Express by means of the random variables  $X_1, X_2$  and  $Y_1$  the event that at least two male customers arrive before the first female customer, and find the probability of this event.*
- 4) *Express by means of the random variables  $X_1, X_2, X_3$  and  $Y_1$  the event that at least three male customers arrive before the first female customer, and find the probability of this event.*
- 5) *Find the probability that precisely two male customers arrive before the first female customer.*
- 6) *Find the frequency of the random variable*

$$Z = \frac{X_1 + X_2}{Y_1}.$$

- 7) *Check if  $Z$  has a mean.*

**1. and 2.** Since  $X_i \in \Gamma\left(1, \frac{1}{\lambda}\right)$  is exponential and Gamma distributed, we have

$$X_1 + X_2 \in \Gamma\left(2, \frac{1}{\lambda}\right) \quad \text{og} \quad X_1 + X_2 + X_3 \in \Gamma\left(3, \frac{1}{\lambda}\right),$$

and

$$f_2(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

and

$$f_3(x) = \begin{cases} \frac{1}{2} \lambda^3 x^2 e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

3. and 6. and 7. The event is expressed by

$$\{X_1 + X_2 < Y_1\} = \left\{ \frac{X_1 + X_2}{Y_1} < 1 \right\},$$

because  $X_1$ ,  $X_2$  and  $Y_1$  only have positive values. When  $z > 0$ , then the frequency of

$$Z = \frac{X_1 + X_2}{Y_1}$$

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is given by

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_2(zx) \cdot g(x)|x| dx = \int_0^\infty \lambda^2 z x e^{-\lambda zx} \cdot \mu e^{-\mu x} \cdot x dx = \lambda^2 \mu z \int_0^\infty x^2 e^{-(\lambda z + \mu)x} dx \\ &= \frac{\lambda^2 \mu z}{(\lambda z + \mu)^3} \int_0^\infty t^2 e^{-t} dt = \frac{2\lambda^2 \mu z}{(\lambda z + \mu)^3}. \end{aligned}$$

Since  $z f_Z(z) \sim \frac{2\mu}{\lambda} \cdot \frac{1}{z}$ , and  $\int_0^\infty \frac{1}{z} dz$  is divergent,  $Z$  does not have a mean.

Finally,

$$\begin{aligned} P\{X_1 + X_2 < Y_1\} &= P\{Z < 1\} = \int_0^1 \frac{2\lambda^2 \mu z}{(\lambda z + \mu)^3} dz = 2\lambda\mu \int_0^1 \frac{\lambda z + \mu - \mu}{(\lambda z + \mu)^3} dz \\ &= 2\lambda\mu \int_0^1 \left\{ \frac{1}{(\lambda z + \mu)^2} - \frac{\mu}{(\lambda z + \mu)^3} \right\} dz = 2\mu \left[ -\frac{1}{\lambda z + \mu} + \frac{1}{2} \frac{\mu}{(\lambda z + \mu)^2} \right]_0^1 \\ &= 2\mu \left\{ -\frac{1}{\lambda + \mu} + \frac{1}{2} \frac{\mu}{(\lambda + \mu)^2} + \frac{1}{\mu} - \frac{1}{2} \frac{\mu}{\mu^2} \right\} = -2 \frac{\mu}{\lambda + \mu} + \left( \frac{\mu}{\lambda + \mu} \right)^2 + 1 \\ &= \left( 1 - \frac{\mu}{\lambda + \mu} \right)^2 = \left( \frac{\lambda}{\lambda + \mu} \right)^2. \end{aligned}$$

Analogously,

$$\{X_1 + X_2 + X_3\} = \left\{ \frac{X_1 + X_2 + X_3}{Y_1} < 1 \right\}.$$

The frequency of

$$U = \frac{X_1 + X_2 + X_3}{Y_1}$$

for  $u > 0$  is given by

$$\begin{aligned} f_U(u) &= \int_0^\infty f_3(ux) \cdot g(x) \cdot x dx = \int_0^\infty \frac{1}{2} \lambda^3 u^2 x^2 e^{-\lambda ux} \mu e^{-\mu x} x dx \\ &= \frac{1}{2} \lambda^3 \mu u^2 \int_0^\infty x^3 e^{-(\lambda u + \mu)x} dx = \frac{1}{2} \frac{\lambda^3 \mu u^2}{(\lambda u + \mu)^4} \int_0^\infty t^3 e^{-t} dt = \frac{3\lambda^3 \mu u^2}{(\lambda u + \mu)^4}. \end{aligned}$$

The probability of this event is

$$\begin{aligned}
 P\{X_1 + X_2 + X_3 < Y_1\} &= P\{U < 1\} = \int_0^1 \frac{3\lambda^3\mu u^2}{(\lambda u + \mu)^4} du \\
 &= 3\lambda\mu \int_0^1 \frac{\{(\lambda u + \mu) - \mu\}^2}{(\lambda u + \mu)^4} du = 3\lambda\mu \int_0^1 \frac{(\lambda u + \mu)^2 - 2\mu(\lambda u + \mu) + \mu^2}{(\lambda u + \mu)^4} du \\
 &= 3\mu \int_0^1 \left\{ \frac{1}{(\lambda u + \mu)^2} - \frac{2\mu}{(\lambda u + \mu)^3} + \frac{\mu^2}{(\lambda u + \mu)^4} \right\} \lambda du \\
 &= 3\mu \left[ -\frac{1}{\lambda u + \mu} + \frac{\mu}{(\lambda u + \mu)^2} - \frac{1}{3} \frac{\mu^2}{(\lambda u + \mu)^3} \right]_0^1 \\
 &= 3\mu \left\{ -\frac{1}{\lambda + \mu} + \frac{\mu}{(\lambda + \mu)^2} - \frac{1}{3} \frac{\mu^2}{(\lambda + \mu)^3} + \frac{1}{\mu} - \frac{\mu}{\mu^2} + \frac{1}{3} \frac{\mu^2}{\mu^3} \right\} \\
 &= 1 - 3 \frac{\mu}{\lambda + \mu} + 3 \left( \frac{\mu}{\lambda + \mu} \right)^2 - \left( \frac{\mu}{\lambda + \mu} \right)^3 = \left( 1 - \frac{\mu}{\lambda + \mu} \right)^3 = \left( \frac{\lambda}{\lambda + \mu} \right)^3.
 \end{aligned}$$

The wanted probability is

$$\begin{aligned}
 P\{X_1 + X_2 < Y_1 < X_1 + X_2 + X_3\} \\
 &= P\{X_1 + X_2 < Y_1\} - P\{X_1 + X_2 \leq Y_1 \wedge X_1 + X_2 + X_3 \leq Y_1\} \\
 &= P\{X_1 + X_2 < Y_1\} - P\{X_1 + X_2 + X_3 \leq Y_1\} \\
 &= \left( \frac{\lambda}{\lambda + \mu} \right)^2 - \left( \frac{\lambda}{\lambda + \mu} \right)^3 = \left( \frac{\lambda}{\lambda + \mu} \right) \left\{ 1 - \frac{\lambda}{\lambda + \mu} \right\} = \frac{\lambda^2\mu}{(\lambda + \mu)^3}.
 \end{aligned}$$

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**Example 6.14** *There are two telephone booths in a waiting room. At a given time three persons A, B and C arrive, all of them wanting to telephone. A and B go immediately into each their booth and start their call, while C is waiting, until either A or B has finished his call. We shall in the following assume that the length of the telephone calls are mutually of each other, and that this length is a random variable, which is exponentially distributed of the frequency*

$$f(x) = \begin{cases} 0, & x \leq 0, \\ \lambda e^{-\lambda x}, & x > 0, \end{cases}$$

where  $\lambda$  is a positive constant.

Let  $X_A$ ,  $X_B$  and  $X_C$  denote the lengths of the telephone calls of A, B and C, respectively.

- 1) Find the frequency and the distribution function of the random variable  $Y = X_A/X_B$ , and find  $P\{X_A < X_B\}$ .
- 2) Check if Y has a mean.
- 3) Find the distribution of the waiting time Z for C, and find the mean of Z.
- 4) Find the distribution of the random variable  $X_A + X_C$ .
- 5) Find the probability that A terminates his call before B, and that also C terminates his call before B, i.e. find

$$P\{X_A + X_C < X_B\}.$$

- 6) Find the probability that C is not the last one to terminate his call.

- 1) The frequency of  $Y = X_A/X_B$  for  $y > 0$  is given by

$$\begin{aligned} g_Y(y) &= \int_0^\infty f_A(yx)f_B(x) dx = \int_0^\infty \lambda e^{-\lambda yx} \cdot \lambda e^{-\lambda x} x dx \\ &= \lambda^2 \int_0^\infty x e^{-\lambda(1+y)x} dx = \frac{1}{(1+y)^2}, \quad y > 0, \end{aligned}$$

and  $g_Y(y) = 0$  for  $y \leq 0$ .

If  $y \leq 0$ , we get the distribution function  $G(y) = 0$ . If  $y > 0$ , then

$$G(y) = \int_0^y g(u) du = \int_0^y \frac{du}{(1+u)^2} = \left[ -\frac{1}{1+u} \right]_0^y = 1 - \frac{1}{1+y} \quad \left( = \frac{y}{1+y} \right).$$

It follows by the symmetry that

$$P\{X_A < X_B\} = \frac{1}{2}.$$

ALTERNATIVELY,

$$P\{X_A < X_B\} = P\{Y < 1\} = G(1) = \frac{1}{2}.$$

2) Since

$$\int_0^{\infty} \frac{y}{(1+y)^2} dy = \infty,$$

it follows that the mean of  $Y$  does not exist.

ALTERNATIVELY,

$$E\left\{\frac{1}{X_B}\right\} = \int_0^{\infty} \frac{1}{x} \lambda e^{-\lambda x} dx = \infty,$$

hence

$$E\{Y\} = E\{X_A\} \cdot E\left\{\frac{1}{X_B}\right\} = \infty.$$

3) Clearly,  $Z = \min\{X_A, X_B\}$ . We get for  $z > 0$ ,

$$P\{Z > z\} = P\{X_A > z\} \cdot P\{X_B > z\} = e^{-2\lambda z}.$$

This implies that  $Z$  is exponentially distributed of parameter  $2\lambda$ , i.e.

$$f_Z(z) = \begin{cases} 2\lambda e^{-2\lambda z}, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

Consequently,

$$E\{Z\} = \frac{1}{2\lambda}.$$

4) According to the properties of the Gamma distribution,  $X_A + X_C$  has the frequency

$$h(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

5) Since  $X_A + X_C$  has the frequency  $h(x)$ , and  $X_B$  has the frequency  $f_B(y)$ , we get

$$\begin{aligned} P\{X_A + X_C < X_B\} &= \int_{x=0}^{\infty} \left\{ \int_{y=x}^{\infty} h(x) f_B(y) dy \right\} dx = \int_0^{\infty} \lambda^2 x e^{-\lambda x} \left\{ \int_{y=x}^{\infty} \lambda e^{-\lambda y} dy \right\} dx \\ &= \int_0^{\infty} \lambda^2 x e^{-\lambda x} \cdot e^{-\lambda x} dx = \int_0^{\infty} \lambda^2 x e^{-2\lambda x} dx = \frac{1}{4}. \end{aligned}$$

6) If C is not the last one to terminate his call, then

$$\text{either } X_A + X_C < X_B \text{ or } X_B + X_C < X_A,$$

hence

$$p = P\{X_A + X_C < X_B\} + P\{X_B + X_C < X_A\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

**Example 6.15** In a shop one first serves one customer with a serving time  $X_1$  which is a random variable of frequency

$$f_1(x_1) = \begin{cases} 2e^{-2x_1}, & x_1 > 0, \\ 0, & x_1 \leq 0. \end{cases}$$

Then a family of 3 members is served, and the total serving time  $X_2$  of this family is a random variable of frequency

$$f_2(x_2) = \begin{cases} 4x_2^2 e^{-2x_2}, & x_2 > 0, \\ 0, & x_2 \leq 0. \end{cases}$$

We assume that the random variables  $X_1$  and  $X_2$  are independent.

1. Find the means  $E\{X_1\}$  and  $E\{X_2\}$ .

Define the random variables  $Y_1$  and  $Y_2$  by

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{X_1}{X_2}.$$

Here,  $Y_1$  is the total serving time of all 4 customers, and  $Y_2$  is the quotient between the serving times of the single customer and the family. One may use without proof that the vector function  $\tau$  given by

$$\tau(x_1, x_2) = \left( x_1 + x_2, \frac{x_1}{x_2} \right)$$

maps  $\mathbb{R}_+ \times \mathbb{R}_+$  bijectively onto itself.

2. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .
3. Find the marginal frequencies of  $Y_1$  and  $Y_2$ . (This question may be answered with and without using the answer of 2.).
4. Check if  $Y_1$  and  $Y_2$  are independent.
5. Find the means  $E\{Y_1\}$  and  $E\{Y_2\}$ .
6. Find the median of  $Y_2$ .

1. It follows from  $X_1 \in \Gamma\left(1, \frac{1}{2}\right)$  and  $X_2 \in \Gamma\left(3, \frac{1}{2}\right)$  that

$$E\{X_1\} = \frac{1}{2} \quad \text{and} \quad E\{X_2\} = \frac{3}{2}.$$

2. It follows from

$$y_1 = x_1 + x_2 \quad \text{and} \quad y_2 = \frac{x_1}{x_2}, \quad \text{i.e. } x_1 = y_2 x_2,$$

that  $y_1 = y_2 x_2 + x_2 = (y_2 + 1) x_2$ , thus  $x_2 = y_1 / (y_2 + 1)$ , and hence

$$x_1 = \frac{y_1 y_2}{y_2 + 1} = y_1 - \frac{y_1}{y_2 + 1} \quad \text{and} \quad x_2 = \frac{y_1}{y_2 + 1}.$$

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{y_2}{y_2 + 1} & \frac{y_1}{(y_2 + 1)^2} \\ 1 & -\frac{y_1}{(y_2 + 1)^2} \end{vmatrix} = \frac{y_1}{(y_2 + 1)^3} \begin{vmatrix} y_2 & 1 \\ 1 & -1 \end{vmatrix} = -\frac{y_1}{(y_2 + 1)^2}.$$

If therefore  $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ , then

$$\begin{aligned} k(y_1, y_2) &= f_1\left(\frac{y_1 y_2}{y_2 + 1}\right) f_2\left(\frac{y_1}{y_2 + 1}\right) \cdot \frac{y_1}{(y_2 + 1)^2} \\ &= 2 \exp\left(-2y_1 + \frac{2y_1}{y_2 + 1}\right) \cdot 4 \left(\frac{y_1}{y_2 + 1}\right)^2 \exp\left(-\frac{2y_1}{y_2 + 1}\right) \cdot \frac{y_1}{(y_2 + 1)^2} \\ &= 8 y_1^3 e^{-2y_1} \frac{1}{(y_2 + 1)^4} = \frac{1}{3} 2^4 y_1^3 e^{-2y_1} \cdot \frac{3}{(y_2 + 1)^4}. \end{aligned}$$

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**3. and 4.** It follows immediately from the expression of  $k(y_1, y_2)$  that  $Y_1$  and  $Y_2$  are independent and that  $Y_1 \in \Gamma\left(4, \frac{1}{2}\right)$ , hence

$$k_{Y_1}(y_1) = \begin{cases} \frac{1}{3} 2^4 y_1^3 e^{-2y_1} & \text{for } y_1 > 0, \\ 0 & \text{for } y_1 \leq 0, \end{cases}$$

and

$$k_{Y_2}(y_2) = \begin{cases} \frac{3}{(y_2 + 1)^4} & \text{for } y_2 > 0, \\ 0 & \text{for } y_2 \leq 0. \end{cases}$$

**5.** Clearly,

$$E\{Y_1\} = 4 \cdot \frac{1}{2} = 2.$$

Furthermore,

$$\begin{aligned} E\{Y_2\} &= 3 \int_0^\infty \frac{y_2 + 1 - 1}{(y_2 + 1)^4} dy_2 = 3 \int_0^\infty \left\{ \frac{1}{(y_2 + 1)^3} - \frac{1}{(y_2 + 1)^4} \right\} dy_2 \\ &= 3 \left[ -\frac{1}{2} \frac{1}{(y_2 + 1)^2} + \frac{1}{3} \frac{1}{(y_2 + 1)^3} \right]_0^\infty = 3 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{2}. \end{aligned}$$

**6.** The distribution function of  $Y_2$  for  $y_2 > 0$  is given by

$$F_{Y_2}(y_2) = \int_0^{y_2} \frac{3}{(t + 1)^4} dt = \left[ -\frac{1}{(t + 1)^3} \right]_0^{y_2} = 1 - \frac{1}{(y_2 + 1)^3}.$$

Hence the median is given by

$$1 - \frac{1}{(y_2 + 1)^3} = \frac{1}{2},$$

i.e.  $(y_2 + 1)^3 = 2$ , so

$$(Y_2) = \sqrt[3]{2} - 1.$$

**Example 6.16** An instrument  $A$  contains two components, the lifetimes of which  $X_1$  and  $X_2$  are independent random variables, both of frequency

$$f(x) = \begin{cases} 2a e^{-2ax}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $a$  is a positive constant.

We first use one of the components in the instrument  $A$ , and when this component is worn out, we shift immediately to the other component, thus the lifetime of  $A$  is  $X = X_1 + X_2$ .

Another instrument  $B$  only contains one component, the lifetime of which (which is also the lifetime of  $B$ ) is a random variable  $Y$  of frequency

$$g(y) = \begin{cases} a e^{-ay}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

- 1) Find mean and variance of  $Y$ .
- 2) Find frequency, mean and variance of  $X$ .
- 3) Let  $U = \frac{X}{Y}$  denote the quotient between the lifetimes of the two instruments. Find the frequency of  $U$ .
- 4) Given 200 instruments of the same type as  $A$ . Let the random variable  $Z$  denote the sum of the lifetimes for all 200 instruments. Use the Central Limit Theorem to find an approximate expression of

$$P \left\{ \frac{185}{a} < Z < \frac{215}{a} \right\}.$$

- 1) Since  $Y \in \Gamma \left( 1, \frac{1}{a} \right)$  is exponentially distributed, it follows by e.g. using a table that

$$E\{Y\} = \frac{1}{a} \quad \text{and} \quad V\{Y\} = \frac{1}{a^2}.$$

- 2) Since  $X_i \in \Gamma \left( 1, \frac{1}{2a} \right)$ ,  $i = 1, 2$ , are independent, the sum

$$X = X_1 + X_2 \in \Gamma \left( 2, \frac{1}{2a} \right)$$

is Gamma distributed, hence (e.g. by using a table)

$$f_X(x) = \begin{cases} 4a^2 x e^{-2ax} & \text{for } x > 0, \\ 0, & \text{for } x \leq 0, \end{cases}$$

and

$$E\{X\} = 2 \cdot \frac{1}{2a} = \frac{1}{a} \quad \text{and} \quad V\{X\} = 2 \cdot \left( \frac{1}{2a} \right)^2 = \frac{1}{2a^2}.$$

ALTERNATIVELY, the frequency of  $X$  (for  $x > 0$ ) is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X_1}(t)f_{X_2}(x-t) dt = \int_0^x 2a e^{-2at} \cdot 2a e^{-2a(x-t)} dt \\ &= 4a^2 e^{-2ax} \int_0^x 1 dt = 4a^2 x e^{-2ax}. \end{aligned}$$

Then it follows that s

$$E\{X\} = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} 4a^2 x^2 e^{-2ax} dx = \frac{1}{2a} \int_0^{\infty} u^2 e^{-u} du = \frac{1}{2a} \cdot 2 = \frac{1}{a},$$

and

$$E\{X^2\} = \int_0^{\infty} x^2 f_X(x) dx = \int_0^{\infty} 4a^2 x^3 e^{-2ax} dx = \frac{1}{4a^2} \int_0^{\infty} u^3 e^{-u} du = \frac{1}{4a^2} \cdot 6 = \frac{3}{2a^2},$$

hence

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{3}{2a^2} - \frac{1}{a^2} = \frac{1}{2a^2}.$$

3) The frequency of  $U = \frac{X}{Y}$  is 0 for  $u \leq 0$ . We get for  $u > 0$ ,

$$\begin{aligned} h_U(u) &= \int_0^{\infty} f_X(ux)g(x)|x| dx = \int_0^{\infty} 4a^2 ux e^{-2aux} \cdot a e^{-ax} \cdot x dx \\ &= 4a^3 u \int_0^{\infty} x^2 e^{-a(1+2u)x} dx = \frac{4a^3 u}{a^3(1+2u)^3} \int_0^{\infty} t^2 e^{-t} dt = \frac{8u}{(1+2u)^3}. \end{aligned}$$

Summing up,

$$h_U(u) = \begin{cases} \frac{8u}{(1+2u)^3} & \text{for } u > 0, \\ 0, & \text{for } u \leq 0. \end{cases}$$

ALTERNATIVELY, one may for  $u > 0$  start with computing the distribution function  $K(u)$  of  $U$ . This is given by

$$\begin{aligned} K(u) &= P\left\{\frac{X}{Y} \leq u\right\} = \int_A f_X(x)f_Y(y) dx dy = \int_{x=0}^{\infty} f_X(x) \left\{ \int_{y=\frac{x}{u}}^{\infty} f_Y(y) dy \right\} dx \\ &= \int_{x=0}^{\infty} 4a^2 x e^{-2ax} \left\{ \int_{y=\frac{x}{u}}^{\infty} a e^{-ay} dy \right\} dx = \int_{x=0}^{\infty} 4a^2 x e^{-2ax} \cdot e^{-ax} dx \\ &= \int_{x=0}^{\infty} 4a^2 x \cdot \exp\left(-ax\left(2 + \frac{1}{u}\right)\right) dx, \quad v = a\left(2 + \frac{1}{u}\right)x, \\ &= \int_{v=0}^{\infty} 4a^2 v \cdot \frac{1}{a^2\left(2 + \frac{1}{u}\right)^2} \cdot e^{-v} dv = \frac{4}{\left(2 + \frac{1}{u}\right)^2} = \frac{4u^2}{(2u+1)^2} \\ &= 1 - \frac{2}{2u+1} + \frac{1}{(2u+1)^2}. \end{aligned}$$

The we get the frequency (for  $u > 0$ ) by differentiation,

$$f(u) = \frac{4}{(2u+1)^2} - \frac{4}{(2u+1)^3} = 4 \cdot \frac{(2u+1) - 1}{(2u+1)^3} = \frac{8u}{(2u+1)^3}.$$

4) Since  $Z = \sum_{i=1}^{200} \tilde{X}_i$ , where the  $\tilde{X}_i$  are mutually stochastically independent, we get

$$E\{Z\} = \frac{200}{a} \quad \text{and} \quad V\{Z\} = \frac{100}{a^2}, \quad \text{i.e.} \quad \sqrt{V\{Z\}} = \frac{10}{a}.$$

Then it follows by the Central Limit Theorem that

$$\begin{aligned} P\left\{\frac{185}{a} < Z < \frac{215}{a}\right\} &= P\left\{\left|Z - \frac{200}{a}\right| < \frac{15}{a}\right\} \\ &= P\left\{\left|\frac{Z - \frac{200}{a}}{\frac{10}{a}}\right| < \frac{15}{a} \cdot \frac{1}{\frac{10}{a}}\right\} = P\left\{\frac{|Z - E\{Z\}|}{\sqrt{V\{Z\}}} < \frac{3}{2}\right\} \\ &\approx \Phi\left(\frac{3}{2}\right) - \Phi\left(-\frac{3}{2}\right) = 2\Phi\left(\frac{3}{2}\right) - 1 = 2 \cdot 0.9332 - 1 = 0.8664. \end{aligned}$$

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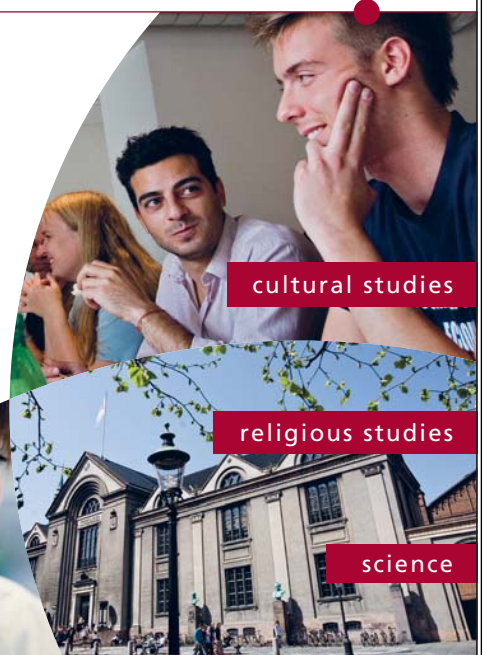


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## 7 The normal distribution and the Gamma distribution

**Example 7.1** Assume that  $(X_1, X_2)$  has the frequency

$$h(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} (x_1^2 + x_2^2)\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

(i.e.  $X_1$  and  $X_2$  are independent,  $N(0, \sigma^2)$  distributed random variables), and let  $(X_1^*, X_2^*)$  be given by

$$X_1^* = \min(X_1, X_2), \quad X_2^* = \max(X_1, X_2).$$

1. Prove that  $(X_1^*, X_2^*)$  has the frequency

$$h^*(x_1, x_2) = \begin{cases} 2h(x_1, x_2), & x_1 < x_2, \\ 0, & x_1 \geq x_2. \end{cases}$$

Define random variables  $Y_1$  and  $Y_2$  by

$$Y_1 = \frac{1}{2} (X_1^* + X_2^*) \quad \left[ = \frac{1}{2} (X_1 + X_2) \right],$$

$$Y_2 = \frac{1}{2} (X_1^* - X_2^*)^2 \quad \left[ = \frac{1}{2} (X_2 - X_1)^2 \right].$$

2. Prove that the vector function  $\tau$ , given by

$$\tau(x_1, x_2) = \left( \frac{1}{2} (x_1 + x_2), \frac{1}{2} (x_2 - x_1)^2 \right),$$

maps

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < x_2\}$$

bijectionally onto

$$D' = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}.$$

3. Find the simultaneous frequency  $k(y_1, y_2)$  of  $(Y_1, Y_2)$ .

4. Find the marginal frequencies of  $Y_1$  and  $Y_2$ .

5. Are  $Y_1$  and  $Y_2$  independent?

6. Find the means and variances of  $Y_1$  and  $Y_2$ .

1. If  $x_1 < x_2$ , then

$$P\{X_1^* > x_1 \wedge X_2^* \leq x_2\} = P\{x_1 < X_1 \leq x_2 \wedge x_1 < X_2 \leq x_2\} = \{F(x_2) - F(x_1)\}^2,$$

where  $F$  is the distribution function of an  $N(0, \sigma^2)$ -distribution. This implies that

$$\begin{aligned} H^*(x_1, x_2) &= P\{X_1^* \leq x_1 \wedge X_2^* \leq x_2\} = P\{X_2^* \leq x_2\} - P\{X_1^* > x_1 \wedge X_2^* \leq x_2\} \\ &= \{F(x_2)\}^2 - \{F(x_2) - F(x_1)\}^2 = -F(x_1)^2 + 2F(x_1)F(x_2). \end{aligned}$$

Then by differentiation,

$$h^*(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \left\{ -F(x_1)^2 + 2F(x_1) \cdot F(x_2) \right\} = 2h(x_1, x_2).$$

If  $x_2 \leq x_1$ , then of course  $h^*(x_1, x_2) = 0$ .

**2.** Clearly,  $Y_2 \geq 0$ . If  $y_1 \in \mathbb{R}$  and  $y_2 > 0$ , then it follows from

$$y_1 = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y_2 = \frac{1}{2}(x_2 - x_1)^2 \quad \text{and} \quad x_2 > x_1,$$

that

$$y_1 = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad \sqrt{\frac{y_2}{2}} = \frac{1}{2}(x_2 - x_1) \quad [\geq 0],$$

hence

$$x_1 = y_1 - \sqrt{\frac{y_2}{2}} \quad \text{and} \quad x_2 = y_1 + \sqrt{\frac{y_2}{2}}.$$

The solution is unique, and since the equation can always be solved, we find that

$$D' = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}.$$

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} 1 & -\frac{1}{2} \frac{1}{\sqrt{2y_2}} \\ 1 & \frac{1}{2} \frac{1}{\sqrt{2y_2}} \end{vmatrix} = \frac{1}{\sqrt{2y_2}} > 0.$$

**3.** The simultaneous frequency of  $(Y_1, Y_2)$  for  $y_2 > 0$  is given by

$$\begin{aligned} k(y_1, y_2) &= h^*\left(y_1 - \sqrt{\frac{y_2}{2}}, y_1 + \sqrt{\frac{y_2}{2}}\right) \cdot \frac{1}{\sqrt{2y_2}} \\ &= 2h\left(y_1 - \sqrt{\frac{y_2}{2}}, y_1 + \sqrt{\frac{y_2}{2}}\right) \cdot \frac{1}{\sqrt{2y_2}} \\ &= \frac{2}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}(2y_1^2 + y_2)\right) \cdot \frac{1}{\sqrt{2y_2}} \\ &= \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{y_1}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2\sigma^2}} \cdot \frac{1}{\sqrt{y_2}} \cdot \exp\left(-\frac{y_2}{2\sigma^2}\right). \end{aligned}$$

If  $y_2 \leq 0$ , then  $k(y_1, y_2) = 0$ .

**4. and 5.** It follows immediately from **3.** that  $Y_1$  and  $Y_2$  are independent and that

$$k_{Y_1}(y_1) = \frac{1}{\sqrt{\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2} y_1^2\right), \quad y_1 \in \mathbb{R}, \quad Y_1 \in N\left(0, \frac{\sigma^2}{2}\right),$$

and

$$k_{Y_2}(y_2) = \begin{cases} \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2\sigma^2}} \cdot \frac{1}{\sqrt{y_2}} \exp\left(-\frac{y_2}{2\sigma^2}\right), & y_2 > 0, \\ 0, & y_2 \leq 0, \end{cases} \quad Y_2 \in \Gamma\left(\frac{1}{2}, 2\sigma^2\right).$$

**6.** Then obviously,

$$\begin{aligned} E\{Y_1\} &= 0, & V\{Y_1\} &= \frac{\sigma^2}{2}, \\ E\{Y_2\} &= \sigma^2, & V\{Y_2\} &= \frac{1}{2} \cdot (2\sigma^2) = 2\sigma^4. \end{aligned}$$

**Remark 7.1** We shall in the following Example 7.2 treat this problem in an alternative way.  $\diamond$

**Example 7.2** Let  $X_1$  and  $X_2$  be independent  $N(0, \sigma^2)$ -distributed random variables, and let the random variables  $Y_1$  and  $Y_2$  be given by

$$Y_1 = \frac{1}{2} (X_1 + X_2), \quad Y_2 = \frac{1}{2} (X_2 - X_1)^2.$$

- 1) Prove that the random variables  $X_1 + X_2$  and  $X_2 - X_1$  are independent, e.g. by first finding the simultaneous frequency.
- 2) Prove that the random variables  $Y_1$  and  $Y_2$  are independent.
- 3) Find the frequencies of  $Y_1$  and  $Y_2$ .
- 4) Find the simultaneous frequency of  $(Y_1, Y_2)$ .
- 5) Find the means and variances of  $Y_1$  and  $Y_2$ .

1) From  $z_1 = x_1 + x_2$  and  $z_2 = x_2 - x_1$  follows that

$$x_1 = \frac{1}{2} (z_1 - z_2) \quad \text{and} \quad x_2 = \frac{1}{2} (z_1 + z_2)$$

of the Jacobian

$$\frac{\partial(x_1, x_2)}{\partial(z_1, z_2)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Since

$$x_1^2 + x_2^2 = \frac{1}{4} (z_1 - z_2)^2 + \frac{1}{4} (z_1 + z_2)^2 = \frac{1}{2} (z_1^2 + z_2^2),$$

the simultaneous frequency of  $(Z_1, Z_2) = (X_1 + X_2, X_2 - X_1)$  is given by

$$\begin{aligned} f_Z(z_1, z_2) &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} \cdot \left\{ \left(\frac{z_1}{\sqrt{2}}\right)^2 + \left(\frac{z_2}{\sqrt{2}}\right)^2 \right\}\right) \cdot \frac{1}{2} \\ &= \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{1}{2} \left\{ \frac{z_1}{\sqrt{2}\sigma} \right\}^2\right) \cdot \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{1}{2} \left\{ \frac{z_2}{\sqrt{2}\sigma} \right\}^2\right). \end{aligned}$$

It follows immediately that  $Z_1$  and  $Z_2$  are independent and that

$$f_{Z_i}(z_i) = \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{1}{2} \left\{ \frac{z_i}{\sqrt{2}\sigma} \right\}^2\right), \quad z_i \in \mathbb{R}, \quad i = 1, 2.$$

- 2) Since  $Y_1 = \frac{1}{2} Z_1$  and  $Y_2 = \frac{1}{2} Z_2$ , and  $Z_1$  and  $Z_2$  are independent, we conclude that  $Y_1$  and  $Y_2$  are also independent.
- 3) It follows from  $y_1 = \frac{1}{2} z_1$ , that  $z_2 = 2y_1$ , and  $\frac{dz_1}{dy_1} = 2$ , so

$$f_{Y_1}(y_1) = f_{Z_1}(2y_1) \cdot 2 = \frac{1}{\sqrt{\pi} \cdot \sigma} \exp\left(-\frac{1}{2} \left\{ \frac{\sqrt{2}y_1}{\sigma} \right\}^2\right), \quad y_1 \in \mathbb{R}.$$

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From  $y_2 = \frac{1}{2} z_2^2 \geq 0$  follows that  $|z_2| = \sqrt{2y_2}$ , thus

$$\begin{aligned} F_{Y_2}(y_2) &= P\{|Z_2| \leq \sqrt{2y_2}\} = P\left\{\left|\frac{Z_2}{\sqrt{2}\sigma}\right| \leq \frac{\sqrt{y_2}}{\sigma}\right\} \\ &= \Phi\left(\frac{\sqrt{y_2}}{\sigma}\right) - \Phi\left(-\frac{\sqrt{y_2}}{\sigma}\right) = 2\Phi\left(\frac{\sqrt{y_2}}{\sigma}\right) - 1. \end{aligned}$$

If  $y_2 > 0$ , then by differentiation,

$$f_{Y_2}(y_2) = 2 \cdot \varphi\left(\frac{\sqrt{y_2}}{\sigma}\right) \cdot \frac{1}{2\sqrt{y_2}\sigma} = \frac{1}{\sigma\sqrt{y_2}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_2}{2\sigma^2}\right),$$

and  $f_{Y_2}(y_2) = 0$  for  $y_2 \leq 0$ , i.e.

$$f_{Y_2}(y_2) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{1}{\sqrt{y_2}} \exp\left(-\frac{y_2}{2\sigma^2}\right) & \text{for } y_2 > 0, \\ 0 & \text{for } y_2 \leq 0. \end{cases}$$

4) The simultaneous frequency is given by

$$h(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = \frac{1}{\sqrt{2\pi}\sigma^2} \cdot \frac{1}{\sqrt{y_2}} \exp\left(-\frac{y_1^2}{\sigma^2} - \frac{y_2}{2\sigma^2}\right)$$

for  $y_1 \in \mathbb{R}$  and  $y_2 > 0$ , and  $h(y_1, y_2) = 0$  for  $y_2 \leq 0$ .

5) The mean and variance of  $Y_1$  are

$$E\{Y_1\} = 0 \quad \text{and} \quad V\{Y_1\} = \frac{\sigma^2}{2}.$$

For  $Y_2$  we get by the substitution  $t^2 = \frac{y_2}{\sigma^2}$  that

$$E\{Y_2\} = \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty y_2 \exp\left(-\frac{y_2}{2\sigma^2}\right) \frac{1}{\sqrt{y_2}} dy_2 = \sqrt{\frac{2}{\pi}} \int_0^\infty \sigma^2 t^2 \exp\left(-\frac{1}{2}t^2\right) dt = \sigma^2,$$

and

$$E\{Y_2^2\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sigma^4 t^4 \exp\left(-\frac{1}{2}t^2\right) dt = 3 \cdot 1 \cdot \sigma^4,$$

så

$$V\{Y_2\} = E\{Y_2^2\} - (E\{Y_2\})^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4.$$

## 8 Convergence in distribution

**Example 8.1** Given a sequence of random variables  $(U_n)_{n=1}^\infty$ , which converges in distribution towards a random variable  $U$  of distribution function  $F_U(u)$  and frequency  $f_U(u)$ .

1. Prove that the sequence  $(U_n^2)$  converges in distribution towards  $U^2$  of distribution function

$$F_U(\sqrt{u}) - F_U(-\sqrt{u})$$

and frequency

$$\frac{1}{2\sqrt{u}} \{f_U(\sqrt{u}) + f_U(-\sqrt{u})\}, \quad u > 0.$$

We perform a series of throws with an (honest) coin, where we assume that the throws are mutually independent. Define the random variables  $X_1, X_2, \dots$  by

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th throw results in } i \text{ heads,} \\ 0, & \text{if the } i\text{-th throw results in } i \text{ tails,} \end{cases}$$

and the random variables  $K_1, K_2, \dots$  by

$K_n =$  number of heads in the first  $n$  throws.

2. Express  $K_n$  by means of the  $X_i$  and find mean and variance of  $K_n$ .
3. Prove that the sequence  $\left(\frac{2}{\sqrt{n}} \left\{K_n - \frac{n}{2}\right\}\right)$  converges in distribution towards a normally distributed random variable.
4. Define the random variables  $Y_n$  by

$$Y_n = \frac{4}{n} \left(K_n - \frac{n}{2}\right)^2, \quad n \in \mathbb{N}.$$

Prove that the sequence  $(Y_n)$  converges in distribution towards a random variable  $Y$ , and find the distribution of  $Y$ .

- 1) It is given that

$$F_{U_n}(u) \rightarrow F_U(u) \quad \text{for } n \rightarrow \infty,$$

and that for  $u > 0$ ,

$$F_{U^2}(u) = P\{U^2 \leq u\} = P\{-\sqrt{u} \leq U \leq \sqrt{u}\} = F(\sqrt{u}) - F(-\sqrt{u}).$$

If  $u > 0$ , then

$$\begin{aligned} F_{U_n^2}(u) &= P\{U_n^2 \leq u\} = P\{-\sqrt{u} \leq U_n \leq \sqrt{u}\} \\ &= F_{U_n}(\sqrt{u}) - F_{U_n}(-\sqrt{u}) \rightarrow F_U(\sqrt{u}) - F_U(-\sqrt{u}) \\ &= F_{U^2}(u) \quad \text{for } n \rightarrow \infty, \end{aligned}$$

so  $(U_n^2)$  converges in distribution towards  $U^2$ .

The frequency is found by differentiation,

$$f_{U^2}(u) = \frac{d}{du} \{F_U(\sqrt{u}) - F_U(-\sqrt{u})\} = \frac{1}{2\sqrt{u}} \{f_U(\sqrt{u}) + f_U(-\sqrt{u})\}.$$

2) Clearly,  $K_n = \sum_{i=1}^n X_i$ , so

$$E\{K_n\} = \sum_{i=1}^n E\{X_i\} = n \cdot \frac{1}{2} = \frac{n}{2} = \mu_n.$$

Since the  $X_i$  are mutually independent, we get

$$V\{K_n\} = \sum_{i=1}^n V\{X_i\} = n \cdot \left\{ \frac{1}{2} - \left(\frac{1}{2}\right)^2 \right\} = \frac{n}{4} = s_n^2.$$

3) It follows from the Central Limit theorem that

$$P\left\{ \frac{2}{\sqrt{n}} \left(K_n - \frac{n}{2}\right) \leq x \right\} = P\left\{ \frac{K_n - \mu_n}{s_n} \leq x \right\} \rightarrow \Phi(x) \quad \text{for } n \rightarrow \infty.$$

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- 4) According to **1.** and **3.** the sequence  $(Y_n)$  converges for  $x > 0$  in distribution towards a random variable  $Y$  of the distribution function

$$\frac{1}{2\sqrt{x}} \{\Phi(x) - \Phi(-x)\} = \frac{1}{2\sqrt{x}} \{2\Phi(x) - 1\} = \frac{1}{\sqrt{x}} \Phi(x) - \frac{1}{2\sqrt{x}},$$

and  $= 0$  for  $x \leq 0$ .

The corresponding frequency for  $x > 0$  is given by

$$\frac{1}{2\sqrt{x}} \cdot \{\varphi(\sqrt{x}) + \varphi(-\sqrt{x})\} = \frac{\varphi(\sqrt{x})}{\sqrt{x}},$$

and  $= 0$  for  $x \leq 0$ .

**Example 8.2** Given a series of throws with an (honest) coin, where we assume that the throws are mutually independent. Define the random variables  $X_1, X_2, \dots$  and  $K_n$  and  $P_n$  by

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th throw results in a head,} \\ 0, & \text{if the } i\text{-th throw results in a tail,} \end{cases}$$

$K_n =$  number of heads in the first  $n$  throws,

$P_n =$  number of tails in the first  $n$  throws.

- 1.** Prove that  $K_n$  and  $P_n$  both have the mean  $\frac{n}{2}$ .

Define  $Z_n$  by

$$Z_n = \frac{2}{n} \left\{ \left( K_n - \frac{n}{2} \right)^2 + \left( P_n - \frac{n}{2} \right)^2 \right\}, \quad n \in \mathbb{N}.$$

- 2.** Express  $Z_n$  by the  $X_i$ .

- 3.** Find the mean of  $Z_n$ .

- 4.** Prove that the sequence  $(Z_n)$  converges in distribution towards a random variable  $Z$ , and find the distribution of  $Z$ .

- 1) We put

$$Y_i = 1 - X_i = \begin{cases} 0, & \text{if the } i\text{-th throw results in a head,} \\ 1, & \text{if the } i\text{-th throw results in a tail.} \end{cases}$$

Then

$$K_n = \sum_{i=1}^n X_i \quad \text{and} \quad P_n = \sum_{i=1}^n Y_i = n - \sum_{i=1}^n X_i,$$

hence

$$E\{K_n\} = \sum_{i=1}^n E\{X_i\} = \frac{n}{2} \quad \text{and} \quad E\{P_n\} = n - \sum_{i=1}^n E\{X_i\} = \frac{n}{2}.$$

2) It follows from the above that

$$\begin{aligned} Z_n &= \frac{2}{n} \left\{ \left( K_n - \frac{n}{2} \right)^2 + \left( P_n - \frac{n}{2} \right)^2 \right\} = \frac{2}{n} \left\{ \left( \sum_{i=1}^n X_i - \frac{n}{2} \right)^2 + \left( n - \sum_{i=1}^n X_i - \frac{n}{2} \right)^2 \right\} \\ &= \frac{4}{n} \left( \sum_{i=1}^n \left\{ X_i - \frac{1}{2} \right\} \right)^2 = \frac{4}{n} \sum_{i=1}^n \left( X_i - \frac{1}{2} \right)^2 + \frac{2}{n} \sum_{1 \leq i < j \leq n} \left( X_i - \frac{1}{2} \right) \left( X_j - \frac{1}{2} \right). \end{aligned}$$

3) Now,  $E\{X_i\} = \frac{1}{2}$ , and the  $X_i$  are mutually independent. Hence

$$\begin{aligned} E\{Z_n\} &= \frac{4}{n} E \left\{ \sum_{i=1}^n \left( X_i - \frac{1}{2} \right)^2 \right\} + \frac{8}{n} \sum_{1 \leq i < j \leq n} E \left\{ X_i - \frac{1}{2} \right\} \cdot E \left\{ X_j - \frac{1}{2} \right\} \\ &= \frac{4}{n} \sum_{i=1}^n E \left\{ \left( X_i - \frac{1}{2} \right)^2 \right\} + 0 = \frac{4}{n} \sum_{i=1}^n V\{X_i\} = \frac{4}{n} \cdot n \cdot \frac{1}{4} = 1. \end{aligned}$$

4) Since

$$V \left\{ \sum_{i=1}^n \left( X_i - \frac{1}{2} \right) \right\} = \sum_{i=1}^n V\{X_i\} = \frac{n}{4} = s_n^2,$$

we get for  $x > 0$  be the Central Limit Theorem that

$$\begin{aligned} P\{Z_n \leq x\} &= P \left\{ \frac{4}{n} \left( \sum_{i=1}^n \left\{ X_i - \frac{1}{2} \right\} \right)^2 \leq x \right\} = P \left\{ -\sqrt{x} \leq \frac{\sum_{i=1}^n \left( X_i - \frac{1}{2} \right)}{\frac{\sqrt{n}}{2}} \leq \sqrt{x} \right\} \\ &\rightarrow \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = 2\Phi(\sqrt{x}) - 1 \quad \text{for } n \rightarrow \infty. \end{aligned}$$

When instead  $x \leq 0$ , then

$$P\{Z_n \leq x\} = 0 \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Consequently,  $(Z_n)$  converges in distribution towards a random variable  $Z$ , where

$$F_Z(z) = \begin{cases} 2\Phi(\sqrt{z}) - 1 & \text{for } z > 0, \\ 0 & \text{for } z \leq 0. \end{cases}$$

If  $z > 0$ , then the corresponding frequency is given by

$$f_Z(z) = F'_Z(z) = \frac{1}{\sqrt{z}} \varphi(\sqrt{z}) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{z}} \exp\left(-\frac{z}{2}\right),$$

and  $Z$  is Gamma distributed of form parameter  $\mu = \frac{1}{2}$  and scale parameter  $\alpha = 2$ .

## 9 The $\chi^2$ distribution

**Example 9.1** Let  $X \in \chi^2(n)$ . Find the frequency of  $\sqrt{X}$ , and find the mean and variance of  $\sqrt{X}$ .

From  $X \in \chi^2(n) = \Gamma\left(\frac{n}{2}, 2\right)$ , follows that  $X$  has the frequency

$$f_X(x) = \begin{cases} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{2^{n/2}} \cdot x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

The map  $y = \sqrt{x}$  is bijective  $\mathbb{R}_+ \rightarrow \mathbb{R}$  with the inverse  $x = y^2$ , so if  $Y = \sqrt{X}$ , then the frequency  $f_Y(y)$  for  $Y$ , when  $y > 0$ , is given by

$$f_Y(y) = f_X(x(y)) \cdot \left| \frac{dx(y)}{dy} \right| = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{2^{n/2}} e^{-\frac{y^2}{2}} \cdot 2y = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1}} y^{n-1} \exp\left(-\frac{y^2}{2}\right),$$

and  $f_Y(y) = 0$  for  $y \leq 0$ .

The mean is given by

$$\begin{aligned} E\{\sqrt{X}\} &= \int_0^\infty \sqrt{x} f_X(x) dx = \frac{1}{\Gamma\left(\frac{n}{2}\right) \cdot 2^{\frac{n}{2}}} \int_0^\infty x^{\frac{n}{2}-1+\frac{1}{2}} e^{-\frac{x}{2}} dx \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} \int_0^\infty \frac{1}{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}} x^{\frac{n+1}{2}-1} e^{-\frac{x}{2}} dx = \sqrt{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned}$$

Using

$$E\left\{\left(\sqrt{X}\right)^2\right\} = E\{X\} = n,$$

we get

$$V\{\sqrt{X}\} = E\{X\} - \left(E\{\sqrt{X}\}\right)^2 = n - 2 \left\{ \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right\}^2.$$

## 10 The $F$ distribution

**Example 10.1** Let  $X \in F(n_1, n_2)$ . Prove that  $\frac{1}{X} \in F(n_2, n_1)$ .

If  $X \in F(n_1, n_2)$ , then  $f_X(x) = 0$  for  $x \leq 0$ , and if  $x > 0$ , then the frequency is

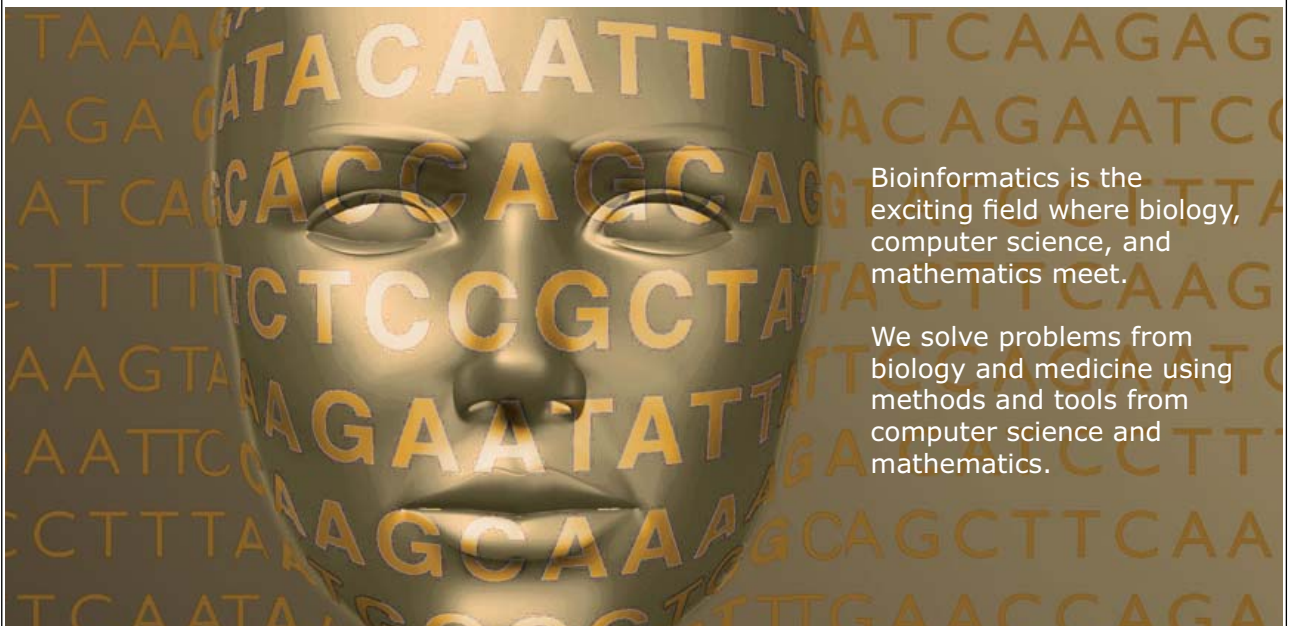
$$f_X(x) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{x^{\frac{1}{2}n_1-1}}{\left(1 + \frac{n_1}{n_2}x\right)^{\frac{1}{2}(n_1+n_2)}}.$$

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The map  $w = \frac{1}{x}$ ,  $x = \frac{1}{w}$ , is bijective,  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ , with  $\frac{dx}{dw} = -\frac{1}{w^2}$ . If we put  $W = \frac{1}{X}$ , then  $f_W(w) = 0$  for  $w \leq 0$ . If  $w > 0$ , then

$$\begin{aligned} f_W(w) &= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{\left(\frac{1}{w}\right)^{\frac{1}{2}n_1-1}}{\left(1 + \frac{n_1}{n_2} \cdot \frac{1}{w}\right)^{\frac{1}{2}(n_1+n_2)}} \cdot \frac{1}{w^2} \\ &= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1}}{B\left(\frac{n_1}{2}, \frac{n_1}{2}\right)} \cdot \frac{w^{\frac{1}{2}(n_1+n_2)}}{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1+\frac{1}{2}n_2}} \cdot \frac{w^{-\frac{1}{2}n_1+1-2}}{\left(1 + \frac{n_2}{n_1} w\right)^{\frac{1}{2}(n_1+n_2)}} \\ &= \frac{\left(\frac{n_2}{n_1}\right)^{\frac{1}{2}n_2}}{B\left(\frac{n_2}{2}, \frac{n_1}{2}\right)} \cdot \frac{w^{\frac{1}{2}n_2-1}}{\left(1 + \frac{n_2}{n_1} w\right)^{\frac{1}{2}(n_1+n_2)}}, \end{aligned}$$

and we conclude that  $W = \frac{1}{X} \in F(n_2, n_1)$ .

ALTERNATIVELY, an  $F(n_1, n_2)$ -distribution occurs as the distribution of

$$X = \frac{\frac{1}{n_1} X_1}{\frac{1}{n_2} X_2}, \quad \text{where } X_1 \in \chi^2(n_1) \text{ and } X_2 \in \chi^2(n_2),$$

and where  $X_1, X_2$  are independent. Then we immediately get

$$\frac{1}{X} = \frac{\frac{1}{n_2} X_2}{\frac{1}{n_1} X_1} \in F(n_2, n_1).$$

**Example 10.2** Let  $X \in F(n_1, n_2)$ . Prove that  $E\{X\} = \frac{n_2}{n_2 - 2}$  for  $n_2 > 2$ .

Let  $n_2 > 2$ . Introduce the change of variable

$$x = \frac{n_2}{n_1} \left( \frac{1}{y} - 1 \right), \quad y \in ]0, 1[,$$

i.e.

$$y = \frac{1}{1 + \frac{n_1}{n_2} x} \quad \text{og} \quad 1 - y = \frac{\frac{n_1}{n_2} x}{1 + \frac{n_1}{n_2} x}.$$



We get

$$\begin{aligned}
 E\{X\} &= \int_0^\infty \frac{\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}n_1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \cdot \frac{x^{\frac{1}{2}n_1}}{\left(1 + \frac{n_1}{n_2}x\right)^{\frac{1}{2}(n_1+n_2)}}} dx \\
 &= \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \left\{ \frac{\frac{n_1}{n_2}x}{1 + \frac{n_1}{n_2}x} \right\}^{\frac{1}{2}n_1} \cdot \left\{ \frac{1}{1 + \frac{n_1}{n_2}x} \right\}^{\frac{1}{2}n_2} dx \\
 &= \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^1 (1-y)^{\frac{1}{2}n_1} y^{\frac{1}{2}n_2} \cdot \frac{n_2}{n_1} \cdot \frac{1}{y^2} dy \\
 &= \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{n_2}{n_1} \int_0^1 y^{\frac{1}{2}(n_2-2)-1} (1-y)^{\frac{1}{2}(n_1+2)-1} dy \\
 &= \frac{n_2}{n_1} \cdot \frac{B\left(\frac{n_1}{2} + 1, \frac{n_2}{2} - 1\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} = \frac{n_2}{n_1} \cdot \frac{\Gamma\left(\frac{n_1}{2} + 1\right) \Gamma\left(\frac{n_2}{2} - 1\right)}{\Gamma\left(\frac{n_1}{2} + 1 + \frac{n_2}{2} - 1\right)} \cdot \frac{\Gamma\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \\
 &= \frac{n_2}{n_1} \cdot \frac{\frac{n_1}{2} \Gamma\left(\frac{n_1}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)} \cdot \frac{\Gamma\left(\frac{n_2}{2} - 1\right)}{\left(\frac{n_2}{2} - 1\right) \Gamma\left(\frac{n_2}{2} - 1\right)} = \frac{n_2}{n_1} \cdot \frac{n_1}{2} \cdot \frac{1}{\frac{n_2}{2} - 1} = \frac{n_2}{n_2 - 2}.
 \end{aligned}$$

ALTERNATIVELY,

$$X = \frac{n_2}{n_1} \frac{X_1}{X_2}, \quad \text{where } X_1 \in \chi^2(n_1) \text{ and } X_2 \in \chi^2(n_2),$$

and where  $X_1$  and  $X_2$  are independent. Then for  $n_2 > 2$ ,

$$\begin{aligned}
 E\{X\} &= \frac{n_2}{n_1} \cdot E\{X_1\} \cdot E\left\{\frac{1}{X_2}\right\} = n_2 \cdot E\left\{\frac{1}{X_2}\right\} \\
 &= n_2 \int_0^\infty \frac{1}{x} \cdot x^{(n_2/2)-1} e^{-(x/2)} dx \cdot \frac{1}{\Gamma\left(\frac{n_2}{2}\right) 2^{n_2/2}} \\
 &= n_2 \int_0^\infty x^{(n_2/2)-2} e^{-(x/2)} dx \cdot \frac{1}{\Gamma\left(\frac{n_2}{2}\right) 2^{n_2/2}} \quad (x = 2y) \\
 &= n_2 \int_0^\infty y^{(n_2/2)-2} e^{-y} dy \cdot \frac{2^{(n_2/2)-1}}{\Gamma\left(\frac{n_2}{2}\right) \cdot 2^{n_2/2}} \\
 &= n_2 \Gamma\left(\frac{n_2}{2} - 1\right) \cdot \frac{1}{\Gamma\left(\frac{n_2}{2}\right) \cdot 2} = n_2 \cdot \frac{1}{\left(\frac{n_2}{2} - 1\right) 2} = \frac{n_2}{n_2 - 2}.
 \end{aligned}$$

It follows from the computations above that  $E\{X\}$  does not exist for  $n_2 \leq 2$ .

## 11 The $F$ distribution and the $t$ distribution

**Example 11.1** Let  $X \in t(n)$ . Prove that  $X^2 \in F(1, n)$ . Prove that the mean  $E\{X\}$  exists, if and only if  $n > 1$ , and find  $E\{X\}$  for  $n > 1$ .  
 Prove that the variance  $V\{X\}$  exists, if and only if  $n > 2$ , and find  $V\{X\}$  for  $n > 2$ .

The random variable  $X \in t(n)$  has the frequency

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \quad x \in \mathbb{R}.$$

Let  $Y = X^2$  have the frequency  $g(y)$ . Then  $g(y) = 0$  for  $y \leq 0$ . For  $y > 0$  we obtain the distribution function

$$\begin{aligned} G(y) &= P\{Y < leq y\} = P\{X^2 \leq y\} = P\{-\sqrt{y} < leq X \leq \sqrt{y}\} \\ &= P\{X \leq \sqrt{y}\} - P\{X \leq -\sqrt{y}\} = F(\sqrt{y}) - F(-\sqrt{y}), \end{aligned}$$

hence

$$\begin{aligned} g(y) &= G'(y) = f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{y}} f(\sqrt{y}) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\left(1 + \frac{y}{n}\right)^{\frac{n+1}{2}}} \cdot \frac{1}{\sqrt{y}} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{n}\right)^{\frac{1}{2}-1} \cdot \frac{y^{\frac{1}{2}\cdot 1-1}}{\left(1 + \frac{1}{n}y\right)^{\frac{1}{2}(1+n)}} \in B(1, n), \end{aligned}$$

as required.

If  $n = 1$ , then

$$\int_{-\infty}^{\infty} \frac{\Gamma(1)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)} \frac{|x|}{1+x^2} dx = \infty,$$

and the mean does not exist.

If  $n > 1$ , then

$$\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \cdot |x| \sim |x|^{-n},$$

and  $\int |x|^{-n} dx$  is convergent. The integrand  $\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$   $x$  is an odd function, hence the mean is

$$E\{X\} = 0.$$

## 12 Estimation of parameters

**Example 12.1** A random variable  $X$  has its distribution function  $F(x)$  given by

$$F(x) = \begin{cases} 1 - \left(\frac{2}{x}\right)^\alpha & \text{for } x \geq 2, \\ 0 & \text{for } x < 2, \end{cases}$$

where  $\alpha$  is a positive constant.

- 1) Find all values of  $\alpha$ , for which  $E\{X\}$  exists, and find  $E\{X\}$  in these cases.
- 2) Find all values of  $\alpha$ , for which  $V\{X\}$  exists and find  $V\{X\}$  in these cases.
- 3) Given  $n$  observations  $x_1, x_2, \dots, x_n$  of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed as  $X$  above. Use the maximum method to find the best estimate of the parameter  $\alpha$ .

The distribution is called a Pareto distribution.

- 1) The corresponding frequency is found by differentiation,

$$f(x) = \begin{cases} \frac{\alpha \cdot 2^\alpha}{x^{\alpha+1}} & \text{for } x \geq 2, \\ 0 & \text{for } x < 2. \end{cases}$$

Since  $x \cdot f(x) \sim x^{-\alpha}$ , the mean exists, if and only if  $\alpha > 1$ . When this is the case, then

$$E\{X\} = \int_2^\infty x \cdot \alpha \cdot \frac{2^\alpha}{x^{\alpha+1}} dx = \frac{2^\alpha}{\alpha - 1} \int_2^\infty (\alpha - 1) \cdot \frac{2^{\alpha-1}}{x^{(\alpha-1)+1}} dx = \frac{2^\alpha}{\alpha - 1}.$$

- 2) Analogously,  $V\{X\}$  exists if and only if  $\alpha > 2$ . If so, we first compute

$$E\{X^2\} = \int_2^\infty x^2 \alpha \cdot \frac{2^\alpha}{x^{\alpha+1}} dx = \frac{4\alpha}{\alpha - 2} \int_2^\infty (\alpha - 2) \cdot \frac{2^{\alpha-2}}{x^{(\alpha-2)+1}} dx = \frac{4\alpha}{\alpha - 2},$$

hence

$$\begin{aligned} V\{X\} &= E\{X^2\} - (E\{X\})^2 = \frac{4\alpha}{\alpha - 2} - \frac{4\alpha^2}{(\alpha - 1)^2} \\ &= \frac{4\alpha}{(\alpha - 2)(\alpha - 1)^2} \{(\alpha - 1)^2 - \alpha(\alpha - 2)\} = \frac{4\alpha}{(\alpha - 1)^2(\alpha - 2)}. \end{aligned}$$

- 3) We shall find the maximum in  $\alpha$  of the function

$$h(\alpha) = \prod_{i=1}^n \alpha \cdot \frac{2^\alpha}{x_i^{\alpha+1}}, \quad \alpha > 0, \quad x_i > 2,$$

i.e. for

$$\begin{aligned} h_1(\alpha) &= \ln h(\alpha) = \sum_{i=1}^n \{ \ln \alpha + \alpha \ln 2 - (\alpha + 1) \ln x_i \} \\ &= n \ln \alpha + \alpha \cdot n \ln 2 - (\alpha + 1) \sum_{i=1}^n \ln x_i. \end{aligned}$$

Since

$$h'_1(\alpha) = \frac{n}{\alpha} + n \ln 2 - \sum_{i=1}^n \ln x_i = 0$$

for

$$\frac{1}{\alpha} = \frac{1}{n} \sum_{i=1}^n \ln x_i - \ln 2 = \frac{1}{n} \sum_{i=1}^n \ln \left( \frac{x_i}{2} \right),$$

the estimate of  $\alpha$  is given by

$$\alpha^* = \frac{1}{\sum_{i=1}^n \ln \left( \frac{x_i}{2} \right)} = \frac{n}{\ln \prod_{i=1}^n \left( \frac{x_i}{2} \right)},$$

where we check that  $h(\alpha) \rightarrow 0$  for  $\alpha \rightarrow 0+$  and for  $\alpha \rightarrow \infty$ , if  $x_i > 2$ .

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**Example 12.2** 1) Let  $a > 0$  be a positive constant. Prove that the function

$$f(x) = \begin{cases} \frac{x}{a} \exp\left(-\frac{x^2}{2a}\right), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

can be considered as a frequency of a random variable  $X$ .

- 2) Find mean and variance of the random variable  $X$ .
- 3) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$ , we shall apply the maximum method to find the best estimate of  $a$  and thus the best estimate of  $V\{X\}$ .
- 4) Prove that the estimator corresponding to the estimate of  $V\{X\}$  is a central estimator, and then prove that the corresponding sequence of estimators is a consistent sequence.

1) Since  $a > 0$ , we have  $f(x) \geq 0$ , and

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} \frac{x}{a} \exp\left(-\frac{x^2}{2a}\right) dx = \left[-\exp\left(-\frac{x^2}{2a}\right)\right]_0^{\infty} = 1,$$

hence  $f(x)$  is the frequency of a random variable  $X$ .

2) The mean is

$$\begin{aligned} E\{X\} &= \int_0^{\infty} x \cdot \frac{x}{a} \exp\left(-\frac{x^2}{2a}\right) dx = \left[-x \exp\left(-\frac{x^2}{2a}\right)\right]_0^{\infty} + \int_0^{\infty} \exp\left(-\frac{x^2}{2a}\right) dx \\ &= 0 + \sqrt{2\pi a} \cdot \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^0 \exp\left(-\frac{x^2}{2a}\right) dx = \sqrt{2\pi a} \cdot \Phi(0) = \sqrt{\frac{\pi a}{2}}. \end{aligned}$$

Using that

$$E\{X^2\} = \int_0^{\infty} x^2 \cdot \frac{x}{a} \exp\left(-\frac{x^2}{2a}\right) dx = 2a \int_0^{\infty} y e^{-y} dy = 2a,$$

we get

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = 2a - \frac{\pi a}{2} = \left(2 - \frac{\pi}{2}\right) a.$$

3) We shall find the maximum of the function (in  $a$ )

$$h(a) = \prod_{i=1}^n \frac{x_i}{a} \exp\left(-\frac{x_i^2}{2a}\right) = \frac{1}{a^n} \cdot \prod_{i=1}^n x_i \cdot \exp\left(-\frac{1}{2a} \sum_{i=1}^n x_i^2\right), \quad a > 0.$$

Since  $x_i > 0$ , we have  $h(a) \rightarrow 0$  for  $a \rightarrow 0+$  and for  $a \rightarrow \infty$ , so we shall find the maximum for  $a > 0$  of the auxiliary function

$$h_1(a) = \ln h(a) = -n \ln a + \sum_{i=1}^n \ln x_i - \frac{1}{2a} \sum_{i=1}^n x_i^2.$$

Now,

$$h'_1(a) = -\frac{n}{1} + \frac{1}{2a^2} \sum_{i=1}^n x_i^2 = \frac{n}{a^2} \left\{ -a + \frac{1}{2n} \sum_{i=1}^n x_i^2 \right\}.$$

this maximum is attained for the estimate

$$a^* = \frac{1}{2n} \sum_{i=1}^n x_i^2.$$

The best estimate of  $V\{X\}$  is

$$\left(2 - \frac{\pi}{2}\right) a = \left(2 - \frac{\pi}{2}\right) \cdot \frac{1}{2n} \sum_{i=1}^n x_i^2 = \frac{4 - \pi}{4n} \sum_{i=1}^n x_i^2,$$

corresponding to the estimator

$$Y_n = \left(1 - \frac{\pi}{4}\right) \frac{1}{n} \sum_{i=1}^n X_i^2$$

for  $V\{X\}$ .

4) We shall finally prove that  $E\{Y_n\} = \left(2 - \frac{\pi}{2}\right) a$ . We get

$$E\{Y_n\} = \left(1 - \frac{\pi}{4}\right) \frac{1}{n} \sum_{i=1}^n E\{C_i^2\} = \left(1 - \frac{\pi}{4}\right) E\{X^2\} = \left(1 - \frac{\pi}{4}\right) \cdot 2a = \left(2 - \frac{\pi}{2}\right) a,$$

from which we conclude that  $Y_n$  is a central estimator.

Since

$$V\{Y_n\} = \left(1 - \frac{\pi}{4}\right)^2 \cdot \frac{1}{n} V\{X^2\} = \frac{c}{n} = s_n^2,$$

where  $c > 0$  is some constant, which we do not need to find, it follows by Chebyshev's inequality that

$$P\left\{\left|Y_n - \left(2 - \frac{\pi}{2}\right) a\right| \geq \varepsilon\right\} \leq \frac{s_n^2}{\varepsilon^2} = \frac{c}{\varepsilon^2} \cdot \frac{1}{n} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

proving that  $(Y_n)$  is a consistent sequence of estimators.

**Example 12.3** A random variable  $X$  has its distribution given by

$$(6) \quad P\{X = k\} = \binom{k+4}{4} p^5 q^k, \quad k \in \mathbb{N}_0,$$

where  $p > 0$ ,  $q > 0$  and  $p + q = 1$ . The mean of  $X$  is given by

$$(7) \quad E\{X\} = \mu = \frac{5q}{p}.$$

(The proof is not required.)

- 1) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$ , apply the maximum method to find the best estimates of  $p$  and  $q$ , by also using (7) an estimate of  $\mu$ .
- 2) Prove that the estimator corresponding to the estimate of  $\mu$  is a central estimator.
- 3) Find for a distribution of the type (6) above with  $\mu = 5$ ,

$$P\{X < 5\}, \quad P\{X = 5\} \quad \text{and} \quad P\{X > 5\}.$$

- 1) We shall maximize the function (in  $p$ )

$$h(p) = \prod_{i=1}^n \binom{x_i+4}{4} p^5 (1-p)^{x_i}, \quad p \in ]0, 1[.$$

Since  $h(0) = h(1) = 0$ , this corresponds to an investigation of the solutions of the equation  $h'_1(p) = 0$ ,  $0 < p < 1$ , where

$$h_1(p) = \ln h(p) = \sum_{i=1}^n \ln \binom{x_i+4}{4} + 5n \ln p + \sum_{i=1}^n x_i \cdot \ln(1-p).$$

By differentiation,

$$\begin{aligned} h'_1(p) &= \frac{5n}{p} - \frac{1}{1-p} \sum_{i=1}^n x_i = \frac{1}{p(1-p)} \left\{ 5n - 5np - p \sum_{i=1}^n x_i \right\} \\ &= \frac{1}{p(1-p)} \left\{ 5n - p \left( 5n + \sum_{i=1}^n x_i \right) \right\}, \end{aligned}$$

thus

$$p^* = \frac{5n}{5n + \sum_{i=1}^n x_i}, \quad \text{corresponding to} \quad q^* = \frac{\sum_{i=1}^n x_i}{5n + \sum_{i=1}^n x_i}.$$

Using (7) we get the estimate of the mean

$$\mu^* = 5 \frac{q^*}{p^*} = \frac{1}{n} \sum_{i=1}^n x_i.$$

2) Clearly,

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

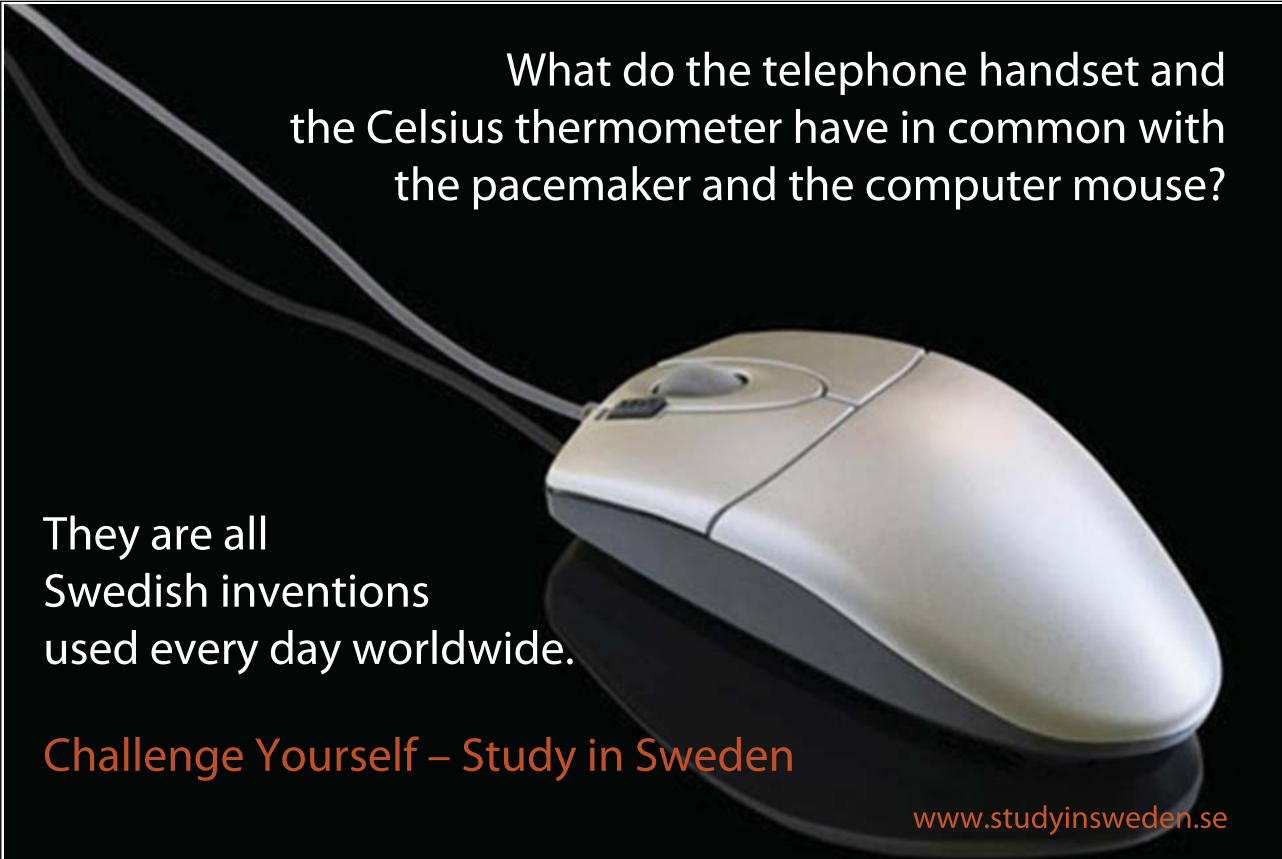
is a central estimator, because

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{X_i\} = E\{X\} = \mu.$$

3) If  $\mu = 5 \frac{q}{p} = 5$ , or  $p = q = \frac{1}{2}$ , then

$$P\{X = k\} = \binom{k+4}{4} \left\{\frac{1}{2}\right\}^{5+k}, \quad k \in \mathbb{N}_0.$$

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Then

$$\begin{aligned}
 P\{X < 5\} &= P\{X = 0\} + P\{X = 1\} + P\{X = 2\} + P\{X = 3\} \\
 &\quad + P\{X = 4\} \\
 &= \binom{4}{4} \frac{1}{2^5} + \binom{5}{4} \frac{1}{2^6} + \binom{6}{4} \frac{1}{2^7} + \binom{7}{4} \frac{1}{2^8} + \binom{8}{4} \frac{1}{2^9} \\
 &= \frac{1}{2^9} \left\{ 2^4 + 5 \cdot 2^3 + \frac{6 \cdot 5}{1 \cdot 2} \cdot 2^2 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot 2 + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \right\} \\
 &= \frac{1}{2^9} \{2^4 + 5 \cdot 2^3 + 15 \cdot 2^2 + 35 \cdot 2 + 35 \cdot 2\} \\
 &= \frac{1}{2^9} \{2^4 + 5 \cdot 2^3 + 15 \cdot 2^2 + 35 \cdot 2^2\} \\
 &= \frac{1}{2^7} \{2^2 + 5 \cdot 2 + 15 + 35\} = \frac{64}{128} = \frac{1}{2},
 \end{aligned}$$

and

$$P\{X = 5\} = \binom{9}{4} \frac{1}{2^{10}} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{1024} = \frac{63}{512},$$

and

$$P\{X > 5\} = 1 - P\{X < 5\} - P\{X = 5\} = 1 - \frac{1}{2} - \frac{63}{512} = \frac{193}{512}.$$

**Example 12.4** A random variable  $X$  has the frequency

$$f_X(x) = \begin{cases} k \cdot x^{a-1} \exp(-bx^a) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where  $a$  and  $b$  are positive constants.

1. Find, expressed by  $a$  and  $b$ , the constant  $k$ .
2. Find the median of  $X$ .

We assume in the following two questions that  $a = 4$ .

3. Find the mean  $E\{X^4\}$  and the variance  $V\{X^4\}$ .
4. Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$ , apply the maximum method the best estimate of the parameter  $b$ .

1) It is obvious that  $f_X(x) \geq 0$  for  $k > 0$ . Then we get the condition

$$1 = \int_0^\infty f_X(x) dx = k \int_0^\infty x^{a-1} \exp(-bx^a) dx = \frac{k}{a} \int_0^\infty \exp(-bx^a) d(x^a) = \frac{k}{ab},$$

from which we derive that  $k = ab$ .

2) Clearly,  $x > 0$ . The condition of the median  $x$  is

$$\frac{1}{2} = F_X(x) = \int_0^x f_X(t) dt = ab \int_0^x t^{a-1} \exp(-bt^a) dt = 1 - \exp(-bx^a),$$

hence

$$\exp(bx^a) = 2,$$

so  $bx^a = \ln 2$ , and the median is

$$(X) = \left\{ \frac{\ln 2}{b} \right\}^{\frac{1}{a}}.$$

3) If  $a = 4$ , then by the change of variable  $t = x^4$ ,

$$E\{X^4\} = 4b \int_0^\infty x^4 x^3 \exp(-bx^4) dx = b \int_0^\infty t e^{-bt} dt = \frac{1}{b} \int_0^\infty u e^{-u} du = \frac{1}{b},$$

and

$$E\{X^8\} = b \int_0^\infty t^2 e^{-bt} dt = \frac{1}{b^2} \int_0^\infty u^2 e^{-u} du = \frac{2}{b^2},$$

hence

$$V\{X^4\} = E\{X^8\} - (E\{X^4\})^2 = \frac{2}{b^2} - \frac{1}{b^2} = \frac{1}{b^2}.$$

4) We shall maximize the function

$$h(b) = \prod_{k=1}^n f_X(x_k) = 4^n b^n \prod_{k=1}^n x_k^3 \exp(-bx_k^4) = 4^n \left( \prod_{k=1}^n x_k^3 \right) b^n \exp\left(-b \sum_{k=1}^n x_k^4\right),$$

i.e.

$$h_1(b) = \ln h(b) = \ln \left\{ 4^n \prod_{k=1}^n x_k^3 \right\} + n \ln b - b \sum_{k=1}^n x_k^4,$$

where

$$h'_1(b) = \frac{h'(b)}{h(b)} = \frac{n}{b} - \sum_{k=1}^n x_k^4 = 0 \quad \text{for } b = \frac{n}{\sum_{k=1}^n x_k^4}.$$

Since  $h(b) > 0$  for  $b > 0$ , and  $h(b) \rightarrow 0$  for  $b \rightarrow 0+$  and for  $b \rightarrow \infty$ , we conclude that

$$b^* = \frac{n}{\sum_{k=1}^n x_k^4}$$

is the best estimat of  $b$ .

**Example 12.5** Let  $(X, Y)$  be a two-dimensional random variable, where  $X$  and  $Y$  are independent, and where  $X$  and  $Y$  have the same distribution, given by

$$P\{X = n\} = P\{Y = n\} = pq^{n-1}, \quad n \in \mathbb{N},$$

where  $p > 0$ ,  $q > 0$  and  $p + q = 1$ .

1) Find everyone of the following probabilities

$$P\{Y = X\}, \quad P\{Y < X\}, \quad P\{Y > X\} \quad \text{and} \quad P\{Y = 2X\}.$$

2) The random variables  $U$  and  $V$  are given by

$$(U, V) = (X + aY, X - aY),$$

where  $a$  is a real constant.

Find the correlation coefficient  $\rho(U, V)$ .

3) Assuming that  $m$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_m$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_m$ , apply the maximum method to find the best estimate of the parameter  $p$ .

4) Assuming that  $m$  observations of the mutually independent two-dimensional random variables  $(X_1, Y_1), (X_2, Y_2), \dots, (X_m, Y_m)$ , all distributed like  $(X, Y)$  above, have given the results  $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ .

What is a reasonable estimate of  $p$ ?

1) A straightforward summation gives

$$\begin{aligned} P\{Y = X\} &= \sum_{n=1}^{\infty} P\{X = n\} \cdot P\{Y = n\} = \sum_{n=1}^{\infty} p^2 q^{2n-2} = \frac{p^2}{1 - q^2} \sum_{n=1}^{\infty} (1 - q^2) (q^2)^{n-1} \\ &= \frac{p^2}{1 - q^2} = \frac{p^2}{(1 + q)(1 - q)} = \frac{p}{1 + q} = \frac{p}{2 - p}. \end{aligned}$$

By the symmetry,  $P\{Y < X\} = P\{X < Y\}$ . We then conclude from

$$1 = P\{Y = X\} + P\{Y < X\} + P\{X < Y\} = P\{Y = X\} + 2P\{Y < X\},$$

that

$$\begin{aligned} P\{Y < X\} &= P\{X < Y\} = \frac{1}{2} (1 - P\{Y = X\}) = \frac{1}{2} \left\{ 1 - \frac{p}{1 + q} \right\} = \frac{1}{2} \left\{ 1 - \frac{1 - q}{1 + q} \right\} \\ &= \frac{1}{2} \cdot \frac{1 + q - 1 + q}{1 + q} = \frac{q}{1 + q}. \end{aligned}$$

Finally,

$$\begin{aligned} P\{Y = 2X\} &= \sum_{n=1}^{\infty} P\{X = n\} \cdot P\{Y = 2n\} = \sum_{n=1}^{\infty} pa^{n-1} \cdot pq^{2n-1} = p^2 \sum_{n=1}^{\infty} q^{3n-2} \\ &= p^2 q \sum_{n=1}^{\infty} (q^3)^{n-1} = \frac{p^2 q}{1 - q^3} = \frac{pq}{q^2 + q + 1}. \end{aligned}$$

2) It follows from

$$E\{X\} = \sum_{n=1}^{\infty} pnq^{n-1} = p \cdot \frac{1}{(1-q)^2} = \frac{1}{p},$$

that

$$E\{U\} = E\{X\} + aE\{Y\} = (1+a)E\{X\} = \frac{1+a}{p}$$


and

$$E\{V\} = E\{X\} - aE\{Y\} = (1-a)E\{X\} = \frac{1-a}{p}.$$

Since  $X$  and  $Y$  are independent, and  $V\{X\} = \frac{q}{p^2}$ , we get

$$V\{U\} = V\{V\} = V\{X\} + a^2V\{Y\} = (1+a^2)V\{X\} = (1+a^2) \cdot \frac{p}{q^2}.$$

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Finally,

$$\begin{aligned} \text{Cov}(U, V) &= E\{UV\} - E\{U\} \cdot E\{V\} = \left( E\{X^2 - a^2Y\} - \frac{1-a^2}{p^2} \right) \\ &= \left[ V\{X\} + (E\{X\})^2 - a^2 (V\{Y\} + (E\{Y\})^2) - \frac{1-a^2}{p^2} \right] \\ &= (1-a^2) \left( V\{X\} + (E\{X\})^2 - \frac{1}{p^2} \right) = (1-a^2) \left( \frac{q}{p^2} + \frac{1}{p^2} - \frac{1}{p^2} \right) \\ &= (1-a^2) \frac{q}{p^2}, \end{aligned}$$

hence

$$\varrho(U, V) = \frac{\text{Cov}(U, V)}{\sqrt{V\{U\} \cdot V\{V\}}} = \frac{\text{Cov}(U, V)}{V\{U\}} = \frac{q}{p^2} \cdot (1-a^2) \cdot \frac{p^2}{q(1+a^2)} = \frac{1-a^2}{1+a^2}.$$

3) Using that  $q = 1 - p$ , it follows that we shall maximize the function

$$h(p) = \prod_{k=1}^m p q^{x_k-1} = p^m \prod_{k=1}^m (1-p)^{x_k-1},$$

or equivalently,

$$\ln h(p) = m \ln p + \left\{ \sum_{k=1}^m x_k - m \right\} \ln(1-p).$$

Now

$$\frac{d}{dp} \ln h(p) = \frac{h'(p)}{h(p)} = \frac{m}{p} - \frac{1}{1-p} \left\{ \sum_{k=1}^m x_k - m \right\}$$

is zero for

$$m(1-p) = p \left\{ \sum_{k=1}^m x_k - m \right\},$$

i.e. for

$$p \sum_{k=1}^m x_k = m,$$

and  $h(p) \rightarrow 0$  for  $p \rightarrow 0$  or for  $p \rightarrow 1$  [provided that at least one  $x_k > 1$ ]. We therefore conclude that the estimate of  $p$  is

$$p^* = \frac{m}{\sum_{k=1}^m x_k} = \frac{1}{\bar{x}}, \quad \text{with } \bar{x} = \frac{1}{m} \sum_{k=1}^m x_k.$$

4) Using the same method as in **3.**, it follows from the independency of  $X$  and  $Y$  that  $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$  can be considered as  $2m$  observations of  $X_1, X_2, \dots, X_m, M_{m+1}, \dots, X_{2m}$ , where  $Y_j = X_{m+j}$ . Then by **3.** the estimate becomes

$$p^{**} = \frac{2m}{\sum_{k=1}^m x_k + \sum_{k=1}^m y_k} = \frac{2m}{\sum_{k=1}^m (x_k + y_k)} = \frac{2}{\bar{x} + \bar{y}}.$$

**Example 12.6** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} a e^{-a\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $a$  is a positive constant.

- 1) Find  $E\{X\}$  and  $E\{\sqrt{X}\}$ .
- 2) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$ , apply the maximum method to find the best estimate of  $a$  and hence the best estimate of  $E\{\sqrt{X}\}$ .
- 3) Prove that the estimator corresponding to the estimate of  $E\{\sqrt{X}\}$  is a central estimator.

1) We get by the substitution  $t = \sqrt{x}$ ,

$$E\{X\} = a \int_0^\infty x \cdot e^{-a\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx = a \int_0^\infty t^2 e^{-at} dt = \frac{1}{a^2} \int_0^\infty u^2 e^{-u} du = \frac{2}{a^2}$$

and

$$E\{\sqrt{X}\} = a \int_0^\infty \sqrt{x} \cdot e^{-a\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx = a \int_0^\infty t e^{-at} dt = \frac{1}{a} \int_0^\infty u e^{-u} du = \frac{1}{a}.$$

2) We shall maximize the function

$$h(a) = a^n \cdot \exp\left(-a \sum_{i=1}^n \sqrt{x_i}\right) \cdot \frac{1}{2^n \sqrt{\prod_{i=1}^n x_i}}, \quad a > 0,$$

where we see that it suffices to maximize the function

$$h_1(a) = \ln h(a) = n \cdot \ln a - a \sum_{i=1}^n \sqrt{x_i} - \ln\left(2^n \sqrt{\prod_{i=1}^n x_i}\right), \quad a > 0.$$

Since  $x_i > 0$ , we have  $h(a) \rightarrow 0$  for  $a \rightarrow 0+$  and for  $a \rightarrow \infty$ , and  $h(a) > 0$  otherwise. From

$$h'_1(a) = \frac{h'(a)}{h(a)} = \frac{n}{a} - \sum_{i=1}^n \sqrt{x_i} = 0, \quad \text{only for } a = \frac{n}{\sum_{i=1}^n \sqrt{x_i}} = \frac{1}{\frac{\sum_{i=1}^n \sqrt{x_i}}{n}},$$

follows that the best estimate of  $a$  is given by

$$a^* = \frac{n}{\sum_{i=1}^n \sqrt{x_i}} = \frac{1}{\frac{\sum_{i=1}^n \sqrt{x_i}}{n}}.$$

3) The estimator of  $E\{\sqrt{X}\}$  is

$$Y_n = \frac{1}{n} \sum_{i=1}^n \sqrt{X_i}.$$

Its mean is

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{\sqrt{X_i}\} = \frac{1}{n} \sum_{i=1}^n \frac{1}{a} = \frac{1}{a} = E\{\sqrt{X}\},$$

hence  $Y_n$  is a central estimator of  $E\{\sqrt{X}\}$ .

**Example 12.7** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $\lambda$  is a positive constant.

1. Compute the mean  $\mu$ .

A Geiger counter is only recording every second particle, which arrives to the counter. The particles arrive according to a Poisson process of an (unknown) intensity  $\lambda$ . The difference in time between two successive recorded arrivals has the frequency  $f(x)$  given above (this shall not be proved), and the difference in time between the first and the second recorded particle, between the second and the third recorded particle ... are mutually independent random variables  $X_1, X_2, \dots$ , all of frequency  $f(x)$ .

2. Given the observed differences in time  $x_1, x_2, \dots, x_n$ , find by means of the maximum method the best estimate  $\lambda_n^*$  of the parameter  $\lambda$ , and hence the best estimate  $\mu_n^*$  of the mean  $\mu$ .

3. Prove that the estimator  $Y_n$  corresponding to the estimate  $\mu_n^*$  of  $\mu$  is a central estimator.

4. Allowing  $n$  to vary, prove that the sequence of estimators  $(Y_n)_{n=1}^\infty$  of  $\mu$  is a consistent sequence.

1) Since  $X \in \Gamma\left(2, \frac{1}{\lambda}\right)$ , the mean is  $\mu = \frac{2}{\lambda}$ .

2) We shall find the maximum of the function

$$h(\lambda) = \prod_{i=1}^n \lambda^2 x_i e^{-\lambda x_i} = \left(\prod_{i=1}^n x_i\right) \cdot \lambda^{2n} \exp\left(-\lambda \sum_{i=1}^n x_i\right),$$

which is equivalent to finding the maximum of the function

$$h_1(\lambda) = \lambda h(\lambda) = 2n \ln \lambda - \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n \ln x_i.$$

It follows from  $x_i > 0$  that  $h(\lambda) \rightarrow 0$  for  $\lambda \rightarrow 0+$  and for  $\lambda \rightarrow \infty$ . Since

$$h'_1(\lambda) = \frac{h'(\lambda)}{h(\lambda)} = \frac{2n}{\lambda} - \sum_{i=1}^n \frac{1}{x_i}$$

is only zero for

$$\lambda = \frac{2n}{\sum_{i=1}^n \frac{1}{x_i}} = \frac{2}{\bar{x}},$$

where  $\bar{x}$  as usual denotes the mean, this corresponds to our maximum. Hence the best estimate of  $\lambda$  is given by

$$\lambda_n^* = \frac{2n}{\sum_{i=1}^n \frac{1}{x_i}} = \frac{2}{\bar{x}},$$

and the best estimate of  $\mu = \frac{2}{\lambda}$  is

$$\mu_n^* = \frac{2}{\lambda_n^*} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}.$$

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3) The estimator  $Y_n$  corresponding to  $\mu_n^*$  is

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

Since

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{X_i\} = \frac{2}{\lambda} = \mu,$$

the estimator is central.

4) Finally,

$$V\{Y_n\} = \frac{1}{n} V\{X\} = \frac{2}{n\lambda^2} = s_n^2,$$

hence by Chebyshev's inequality,

$$P\{|Y_n - \mu| \geq \varepsilon\} \leq \frac{s_n^2}{\varepsilon^2} = \frac{2}{\varepsilon^2 \lambda^2} \cdot \frac{1}{n} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

It follows that the sequence of estimators is consistent.

**Example 12.8** A random variable  $X$  has its distribution given by

$$P\{X = k\} = (k - 1)p^2q^{k-2}, \quad k = 2, 3, 4, \dots,$$

where  $p > 0$ ,  $q > 0$  and  $p + q = 1$ , thus  $X \in \text{Pas}(2, p)$ .

1. Find mean  $\mu$  and variance  $\sigma^2$  of  $X$ .
2. Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$  (where not all  $x_i$  are equal to 2), apply the maximum method the best estimate of  $p$  and the best estimate of  $\mu$ .
3. Prove that the corresponding estimator of  $\mu$  is

$$\tilde{X}_n = \frac{1}{n} \sum_{i=1}^n X_i,$$

and that this is a central estimator.

We now perform  $m$  observations of the mutually independent random variables  $Y_1, Y_2, \dots, Y_m$ , all distributed like  $X$  above and furthermore all independent of the  $X_i$ . The results are called  $y_1, y_2, \dots, y_m$  (where not all  $y_j$  are equal to 2). We denote by  $\tilde{Y}_m$  the estimator of  $\mu$  which has been found by the maximum method from  $Y_1, Y_2, \dots, Y_m$ . Based on the two estimators  $\tilde{X}_n$  and  $\tilde{Y}_m$  we form another two estimators of  $\mu$ , namely

$$Z = \frac{1}{2} (\tilde{X}_n + \tilde{Y}_m) \quad \text{og} \quad U = \frac{n}{n+m} \tilde{X}_n + \frac{m}{n+m} \tilde{Y}_m.$$

4. Prove that  $Z$  and  $U$  are both central estimators of  $\mu$ .
5. Prove for  $m \neq n$  that the estimator  $U$  is more efficient than the estimator  $Z$ .

1) When  $X \in \text{Pas}(2, \mu)$ , then

$$E\{X\} = \mu = \frac{2}{p} \quad \text{and} \quad V\{X\} = \sigma^2 = \frac{2(1-p)}{p^2}.$$

2) We shall maximize the function

$$h(p) = \prod_{i=1}^n (x_i - 1) p^2 q^{x_i - 2} = \left\{ \prod_{j=1}^n (x_j - 1) \right\} \cdot p^{2n} \cdot (1-p)^{\sum_{i=1}^n x_i - 2n},$$

which is equivalent to maximizing the function

$$h_1(p) = \ln h(p) = \sum_{j=1}^n \ln(x_j - 1) + 2n \cdot \ln p + \left\{ \sum_{i=1}^n x_i - 2n \right\} \ln(1-p).$$

Since not every  $x_i$  is 2, we have  $h(0) = h(1) = 0$ , and  $h(p) > 0$  for  $0 < p < 1$ . Since

$$h'_p(p) = \frac{2n}{p} - \frac{1}{1-p} \left\{ \sum_{i=1}^n x_i - 2n \right\} = 0$$

for

$$(1-p) \cdot 2n - p \left\{ \sum_{i=1}^n x_i - 2n \right\} = (1-p) \cdot 2n - pn(\bar{x} - 2) = 0,$$

i.e. for

$$2n - p \sum_{i=1}^n x_i = n(2 - p\bar{x}) = 0,$$

we obtain the maximum (i.e. the best estimate for  $p$ ),

$$p^* = \frac{2n}{\sum_{i=1}^n x_i} = \frac{2}{\bar{x}}.$$

Since  $\mu = \frac{2}{p}$ , the best estimate of  $\mu$  is given by

$$\mu^* = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x},$$

i.e. by the mean of the observations.

3) The estimator corresponding to  $\mu^*$  is precisely

$$\tilde{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

It follows from

$$E \left\{ \tilde{X}_n \right\} = \frac{1}{n} \sum_{i=1}^n E \left\{ X_i \right\} = \frac{1}{n} \cdot n \cdot \frac{2}{p} = \mu,$$

that  $\tilde{X}_n$  is a central estimator.

4) Since both  $E \left\{ \tilde{X}_n \right\} = \mu$  and  $E \left\{ \tilde{Y}_m \right\} = \mu$ , we get

$$E \left\{ r \tilde{X}_n + (1-r) \tilde{Y}_m \right\} = \mu, \quad r \in \mathbb{R},$$

hence  $r \tilde{X}_n + (1-r) \tilde{Y}_m$  is central for every  $r \in \mathbb{R}$ . By choosing  $r = 2$ , we get  $Z$ . By choosing  $r = \frac{n}{n+m}$ , we get  $U$ .

5) Since

$$V \left\{ \tilde{X}_n \right\} = \frac{\sigma^2}{n} \quad \text{og} \quad V \left\{ \tilde{Y}_m \right\} = \frac{\sigma^2}{m},$$

and since  $\tilde{X}_n$  and  $\tilde{Y}_m$  are independent, we get

$$V \left\{ r \tilde{X}_n + (1-r) \tilde{Y}_m \right\} = \sigma^2 \left( \frac{r^2}{n} + \frac{(1-r)^2}{m} \right).$$

If we put

$$f(r) = \frac{r^2}{n} + \frac{(1-r)^2}{m},$$

then

$$f'(r) = \frac{2r}{n} - \frac{2(1-r)}{m} = \frac{2}{mn} (rm - n(1-r)) = \frac{2}{nm} (r(m+n) - n) = 0$$

for  $r = \frac{n}{n+m}$ , corresponding to a minimum. Hence the variance  $\text{er}$  is smallest for  $U$ , i.e.  $U$  is the most efficient estimator in the family.

**Example 12.9** A random variable  $X \in N(0, a)$  is normally distributed of mean 0 and variance  $a$ .

- 1) Compute  $E\{X^4\}$ .
- 2) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$ , apply the maximum method in order to find the best estimate of  $a$ .
- 3) Prove that the corresponding estimator  $Y_n$  of  $a$  is a central estimator.
- 4) Compute the variance of  $Y_n$ .
- 5) Prove that the corresponding sequence of estimators  $(Y_n)_{n=1}^\infty$  is a consistent sequence.

1) By a small computation,

$$\begin{aligned} E\{X^4\} &= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} x^4 \exp\left(-\frac{x^2}{2a}\right) dt = a^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^4 \exp\left(-\frac{1}{2}t^2\right) dt \\ &= a^2 \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t^3 \cdot \exp\left(-\frac{1}{2}t^2\right) t dt = 3a^2 \cdot \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t^2 \exp\left(-\frac{1}{2}t^2\right) dt \\ &= 3a^2. \end{aligned}$$

2) Since

$$\varphi_a(x) = \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{x^2}{2a}\right),$$

we shall find the maximum of the function

$$h(a) = \prod_{i=1}^n \varphi_a(x_i) = \left\{ \frac{1}{\sqrt{2\pi}} \right\}^n \cdot a^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{a} \sum_{i=1}^n x_i^2\right),$$

which is equivalent to finding the maximum of the function

$$h_1(a) = \ln h(a) = \ln \left( \left\{ \frac{1}{\sqrt{2\pi}} \right\}^n \right) - \frac{n}{2} \ln a - \frac{1}{2} \cdot \frac{1}{a} \sum_{i=1}^n x_i^2.$$

Clearly,  $h(a) \rightarrow 0$  for  $a \rightarrow \infty$ . If not all the  $x_i$  are 0, then  $h(a) \rightarrow 0$  for  $a \rightarrow 0+$  due to rule of magnitudes. From

$$h'_1(a) = -\frac{n}{2a} + \frac{1}{2} \frac{1}{a^2} \sum_{i=1}^n x_i^2 = 0 \quad \text{for } a = \frac{1}{n} \sum_{i=1}^n x_i^2,$$

follows that the maximum is attained at the estimate

$$a^* = \frac{1}{n} \sum_{i=1}^n x_i^2.$$

3) The estimator corresponding to  $a^*$  is

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i^2.$$

From  $X_i^2 \in \Gamma\left(\frac{a}{2}, 2\right)$ , follows that

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n \frac{a}{2} \cdot 2 = a,$$

and  $Y_n$  is a central estimator of  $a$ .

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4) Since the  $X_i$  are mutually independent, the variance is

$$V\{Y_n\} = \frac{1}{n^2} \sum_{i=1}^n V\{X_i\} = \frac{a}{n} = s_n^2.$$

5) By Chebyshev's inequality,

$$P\{|Y_n - a| \geq \varepsilon\} \leq \frac{s_n^2}{\varepsilon^2} = \frac{a}{\varepsilon^2} \cdot \frac{1}{n} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

and the sequence is consistent.

**Example 12.10** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-1)}, & x \geq 1, \\ 0, & x < 1, \end{cases}$$

where  $\lambda$  is a positive constant.

1. Find the mean  $\mu$  of  $X$ .
2. Assuming  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, of the results  $x_1, x_2, \dots, x_n$ , use the maximum method to find the best estimate  $\lambda_n^*$  of  $\lambda$  and hence also the best estimate  $\mu_n^*$  of  $\mu$ .

The estimator corresponding to the estimate  $\lambda_n^*$  is denoted by  $Y_n$ , and the estimator corresponding to the estimate  $\mu_n^*$  is denoted by  $Z_n$ .

3. Prove that  $Z_n$  is a central estimator of  $\mu$ .
4. Find the frequency of  $Z_n$ .  
HINT: Start by computing the frequency of

$$U = \sum_{i=1}^n (X_i - 1).$$

5. Check if  $Y_n$  is a central estimator of  $\lambda$ .

1) One may either translate  $X$ , or compute directly,

$$E\{X\} = \lambda \int_1^\infty x e^{-\lambda(x-1)} dx = \lambda \int_0^\infty (x+1) e^{-\lambda x} dx = \frac{1}{\lambda} + 1.$$

2) We shall maximize the function

$$h(\lambda) = \prod_{i=1}^n f(x_i) = \lambda^n \exp\left(-\lambda \left(\sum_{i=1}^n x_i - n\right)\right),$$

where  $h(\lambda) \rightarrow 0$  for  $\lambda \rightarrow 0+$  and for  $\lambda \rightarrow \infty$ . We get from the equation

$$h'(\lambda) = h(\lambda) \left\{ \frac{n}{\lambda} - \left( \sum_{i=1}^n x_i - n \right) \right\} = 0$$

the estimate

$$\lambda_n^* = \frac{n}{\sum_{i=1}^n x_i - n} = \frac{1}{\frac{1}{n} \sum_{i=1}^n x_i - 1} = \frac{1}{\bar{x} - 1}.$$

Hence

$$\mu_n^* = \frac{1}{\lambda_n^*} + 1 = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}.$$

3) If

$$Y_n = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i - 1} \quad \text{and} \quad Z_n = \frac{1}{n} \sum_{i=1}^n X_i,$$

then

$$E\{Z_n\} = \frac{1}{n} \sum_{i=1}^n E\{X_i\} = E\{X\} = \frac{1}{\lambda} + 1 = \mu,$$

and  $Z_n$  is central.

4) From  $U_i = X_i - 1 \in \Gamma\left(1, \frac{1}{\lambda}\right)$ , follows that

$$U = \sum_{i=1}^n U_i \in \Gamma\left(n, \frac{1}{\lambda}\right),$$

thus

$$f_U(u) = \begin{cases} \frac{\lambda^n}{(n-1)!} u^{n-1} \exp(-\lambda u) & \text{for } u > 0, \\ 0 & \text{for } u \leq 0. \end{cases}$$

From

$$Z_n = \frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{n} U + 1,$$

and  $a = \frac{1}{n}$  and  $b = 1$  follows for  $z > 1$  that the frequency of  $Z_n$  is given by

$$\begin{aligned} f_{Z_n}(z) &= n \cdot f_U(n(z-1)) = n \cdot \frac{\lambda^n}{(n-1)!} \cdot n^{n-1} (z-1)^{n-1} \cdot e^{-\lambda n(z-1)} \\ &= \frac{n^2}{(n-1)!} \cdot \lambda^n (z-1)^{n-1} e^{-\lambda n(z-1)}, \end{aligned}$$

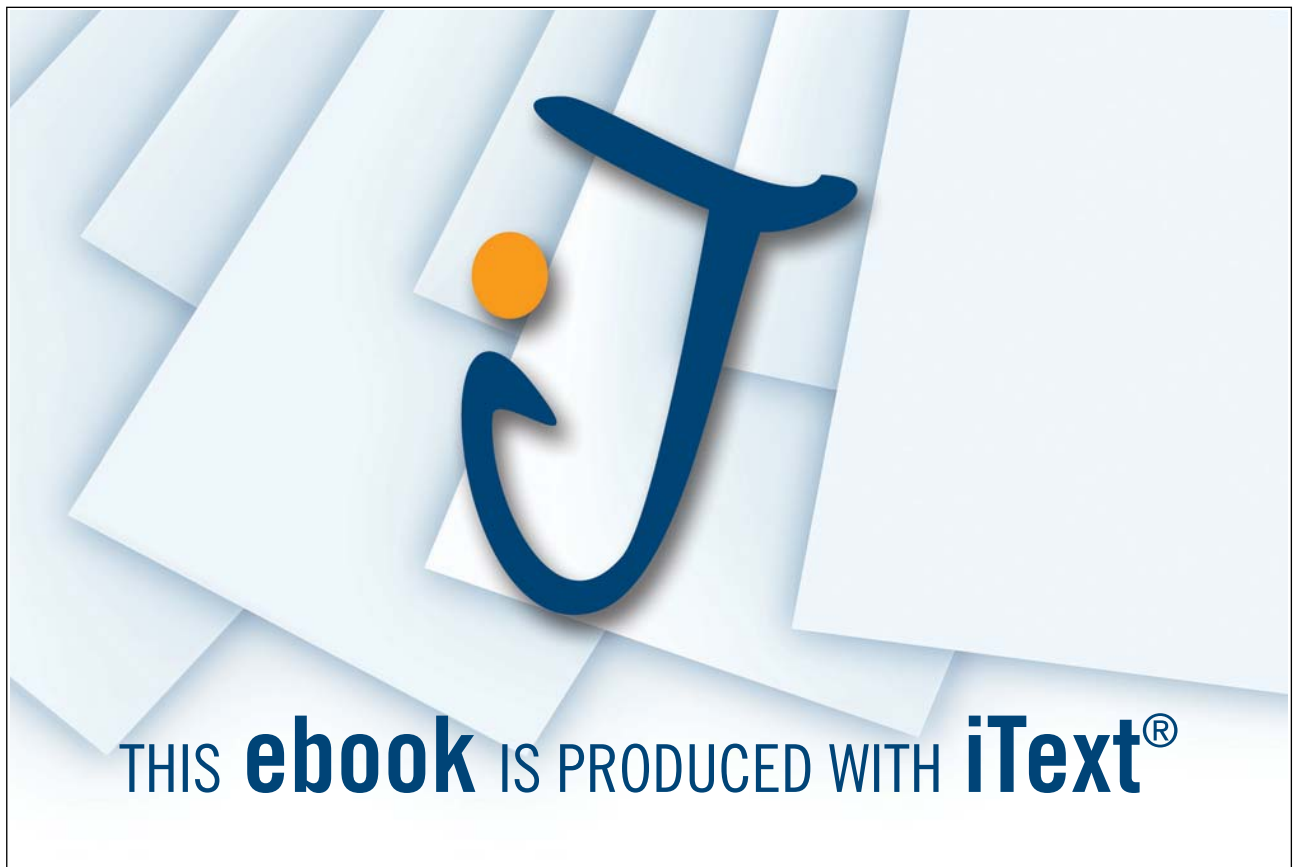
and of course  $f_{Z_n}(z) = 0$  for  $z \leq 1$ .

5) Since  $Y_n = \frac{n}{U}$ , where  $U \in \Gamma\left(n, \frac{1}{\lambda}\right)$ , we get for  $n > 1$ ,

$$\begin{aligned} E\{Y_n\} &= n E\left\{\frac{1}{U}\right\} = n \int_0^\infty \frac{1}{u} \cdot \frac{1}{(n-1)!} \lambda^n u^{n-1} e^{-\lambda u} du \\ &= \frac{n}{(n-1)!} \int_0^\infty u^{n-2} \lambda^n e^{-\lambda u} du \quad (t = \lambda u) \\ &= \frac{n}{(n-1)!} \lambda \int_0^\infty t^{n-2} e^{-t} dt = \frac{n}{(n-1)!} \cdot \lambda(n-2)! \\ &= \frac{n}{n-1} \lambda \neq \lambda, \end{aligned}$$

proving that  $Y_n$  is not a central estimator of  $\lambda$ .

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**Example 12.11** A random variable  $X$  has the frequency

$$f(x) = \begin{cases} \frac{1}{2} \lambda^3 x^2 e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where  $\lambda$  is a positive constant.

1. Compute the mean  $\mu$ .

Customers arrive to a shop according to a Poisson process of (unknown) intensity  $\lambda$ . One day a shop assistant has been asked to write down all the arrival times of the customers, but due to his laziness he is only recording the arrival times of every third customer. The time difference between two successive recorded arrivals has the frequency  $f(x)$  given above (a proof of this claim is not required), and the time differences between the first and the second recorded arrival, between the second and the third recorded arrival etc. are mutually independent random variables  $X_1, X_2, \dots$ , all of frequency  $f(x)$ .

2. Assuming that the time differences  $x_1, x_2, \dots, x_n$ , have been recorded, use the maximum method based on these observations to find the best estimate  $\lambda_n^*$  of  $\lambda$ , and hence the best estimate of the mean  $\mu$ .

3. Prove that the estimator  $Y_n$  corresponding to the estimate  $\mu_n^*$  of  $\mu$  is a central estimator.

4. Allowing  $n$  to vary, prove that the sequence of estimators  $(Y_n)_{n=1}^\infty$  is a consistent sequence.

5. Find for every value of  $n$  the distribution of  $Y_n$ .

1) Since  $X \in \Gamma\left(3, \frac{1}{\lambda}\right)$ , we have  $\mu = \frac{3}{\lambda}$ .

2) We shall maximize the function

$$h(\lambda) = \prod_{i=1}^n f(x_i) = \frac{1}{2^n} \prod_{i=1}^n x_i^2 \cdot \lambda^{3n} \exp\left(-\lambda \sum_{i=1}^n x_i\right),$$

where clearly  $h(\lambda) \rightarrow 0$  for  $\lambda \rightarrow 0+$  and for  $\lambda \rightarrow \infty$ . Since

$$h'(\lambda) = h(\lambda) \cdot \left\{ \frac{3n}{\lambda} - \sum_{i=1}^n x_i \right\} = 0$$

for

$$\lambda = \frac{3n}{\sum_{i=1}^n x_i} = \frac{3}{\bar{x}},$$

we get

$$\lambda_n^* = \frac{3n}{\sum_{i=1}^n x_i} = \frac{3}{\bar{x}} \quad \text{and} \quad \mu_n^* = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}.$$

3) It follows from

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

that

$$E\{Y_n\} = \frac{1}{2} \sum_{i=1}^n E\{X_i\} = E\{X\} = \mu = \frac{3}{\lambda},$$

hence  $Y_n$  is a central estimator.

4) Now,

$$V\{Y_n\} = \frac{1}{n} V\{X\} = \frac{9}{\lambda n} = s_n^2,$$

so it follows by Chebyshev's inequality that

$$P\{|Y_n - \mu| \geq \varepsilon\} \leq \frac{s_n^2}{\varepsilon^2} = \frac{9}{\lambda \varepsilon^2} \cdot \frac{1}{n} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

and the sequence  $(Y_n)$  is consistent.

5) Since  $X \in \Gamma\left(3, \frac{1}{\lambda}\right)$ , We get

$$Z_n = \sum_{i=1}^n X_i \in \Gamma\left(3n, \frac{1}{\lambda}\right).$$

Since  $Y_n = \frac{1}{n} Z_n$  and  $\frac{dz}{dy} = n$  for  $z = ny$ , we get for  $y > 0$  that

$$f_{Y_n}(y) = f_{Z_n}(ny) \cdot n = \frac{\lambda^{3n}}{\Gamma(3n)} n^{3n-1} y^{3n-1} \cdot \exp(-\lambda ny) \cdot n = \frac{(\lambda n)^{3n}}{\Gamma(3n)} y^{3n-1} \exp(-\lambda ny),$$

thus  $Y_n \in \Gamma\left(3n, \frac{1}{\lambda n}\right)$ .

**Example 12.12** A random variable  $X$  has the frequency

$$f(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right), \quad x \in \mathbb{R},$$

where  $a$  is a positive constant.

- 1) Compute  $E\{X\}$ ,  $E\{|X|\}$  and  $E\{X^2\}$ .
- 2) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$  (where not all the  $x_i$  are equal to 0), apply the maximal method to give the best estimate  $a_n^*$  of  $a$  and the corresponding estimator  $Y_n$  of  $a$ .
- 3) Prove that  $Y_n$  is a central estimator of  $a$ .
- 4) Check if the corresponding sequence of estimators  $(Y_n)_{n=1}^\infty$  is a consistent sequence.

1) Clearly,  $E\{X\}$ ,  $E\{|X|\}$  and  $E\{X^2\}$  all exist, and  $E\{X\} = 0$ . It follows by the symmetry that

$$E\{|X|\} = \frac{1}{2a} \cdot 2 \int_0^\infty x \cdot \exp\left(-\frac{x}{a}\right) dx = a \int_0^\infty t e^{-t} dt = a$$

and

$$E\{X^2\} = \frac{1}{2a} \cdot 2 \int_0^\infty x^2 \exp\left(-\frac{x}{a}\right) dx = a^2 \int_0^\infty t^2 e^{-t} dt = 2a^2.$$

2) We shall maximize the function

$$h(a) = \prod_{i=1}^n \frac{1}{2a} \exp\left(-\frac{|x_i|}{a}\right), \quad a > 0.$$

Since not all  $x_i$  are 0, we have  $h(a) \rightarrow 0$  for  $a \rightarrow 0$  and for  $a \rightarrow \infty$ . It therefore suffices to find the maximum of

$$h_1(a) = \ln h(a) = -n \ln 2 - n \ln a - \frac{1}{a} \sum_{i=1}^n |x_i|.$$

Since

$$h'_1(a) = -\frac{n}{a} + \frac{1}{a^2} \sum_{i=1}^n |x_i| = 0$$

only for  $a = \frac{1}{n} \sum_{i=1}^n |x_i|$ , we get

$$a_n^* = \frac{1}{n} \sum_{i=1}^n |x_i|,$$

with the corresponding estimator

$$Y_n = \frac{1}{n} \sum_{i=1}^n |X_i|.$$

3) Since

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{|X_i|\} = \frac{1}{n} \sum_{i=1}^n a = a,$$

it follows that  $Y_n$  is a central estimator.

4) Since

$$V\{|X|\} = E\{X^2\} - (E\{|X|\})^2 = 2a^2 - a^2 = a^2,$$

and all the  $X_i$  are mutually independent, we get

$$V\{Y_n\} = \frac{1}{n^2} \sum_{i=1}^n V\{|X_i|\} = \frac{a^2}{n} = s_n^2.$$

It follows from Chebyshev's inequality that

$$P\{|Y_n - a| \geq \varepsilon\} \leq \frac{s_n^2}{\varepsilon^2} = \frac{a^2}{\varepsilon^2} \cdot \frac{1}{n} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

thus the sequence of estimators  $(Y_n)$  is consistent.

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**Example 12.13** Let  $X_1, X_2, \dots, X_{2n+1}$  be mutually independent random variables of the same distribution function  $F(x)$  and frequency  $f(x)$ ,  $x \in \mathbb{R}$ . The random variables  $X_1, X_2, \dots, X_{2n+1}$  are sorted according to size, defining new random variables  $X_1^*, X_2^*, \dots, X_{2n+1}^*$ , which satisfy

$$X_1^* \leq X_2^* \leq \dots \leq X_{2n+1}^*$$

(i.e.  $X_1^*$  is the smallest one,  $X_2^*$  the second smallest one, etc.).

1. Find, expressed by means of  $F(x)$  and  $f(x)$ , the distribution functions and frequencies of the random variables  $X_1^*$  and  $X_{2n+1}^*$ .

2. Find, for  $u < v$ ,

$$P \{X_1^* > u \wedge X_{2n+1}^* \leq v\},$$

and then derive the simultaneous frequency of  $(X_1^*, X_{2n+1}^*)$ .

3. Prove that  $X_{n+1}^*$  (the “middle one”) has the frequency

$$f_{n+1}^* = (2n + 1) \binom{2n}{n} \{F(x)\}^n \{1 - F(x)\}^n f(x), \quad x \in \mathbb{R}.$$

We assume in the following that  $X_1, X_2, \dots, X_{2n+1}$  are mutually independent and rectangularly distributed over the interval  $]0, a[$  (where  $a > 0$ ).

4. Prove that the three random variables

$$Y_{2n+1} = X_{n+1}^*, \quad Z_{2n+1} = \frac{1}{2n+1} \sum_{i=1}^{2n+1} X_i, \quad U_{2n+1} = \frac{1}{2} \{X_1^* + X_{2n+1}^*\}$$

are all central estimators of the mean  $\frac{a}{2}$ .

5. Prove that  $Z_{2n+1}$  is more efficient than  $Y_{2n+1}$ , and that  $U_{2n+1}$  is more efficient than  $Z_{n+1}$ .

1) It follows from

$$\begin{aligned} F_{X_1^*}(x) &= P \{X_1^* \leq x\} = P \left\{ \min_{i=1, \dots, 2n+1} X_i \leq x \right\} = 1 - P \{X_1 > x, \dots, X_{2n+1} > x\} \\ &= 1 - \{1 - F_X(x)\}^{2n+1}, \end{aligned}$$

that the frequency is given by

$$f_{X_1^*}(x) = (2n + 1) \{1 - F_X(x)\}^{2n} f_X(x).$$

Analogously,

$$F_{X_{2n+1}^*}(x) = P \{X_1 \leq x, \dots, X_{2n+1} \leq x\} = \{F_X(x)\}^{2n+1},$$

thus

$$f_{X_{2n+1}^*}(x) = (2n + 1) \cdot \{F_X(x)\}^{2n} f_X(x).$$

2) If  $u < v$ , then

$$\begin{aligned} P\{X_1^* > u \wedge X_{2n+1}^* \leq v\} &= P\{u < \min X_i \wedge \max X_i \leq v\} \\ &= P\{u < X_1 \leq v, u < X_2 \leq v, \dots, u < X_{2n+1} \leq v\} \\ &= P\{u < X_1 \leq v\} \cdot P\{u < X_2 \leq v\} \cdots P\{u < X_{2n+1} \leq v\} \\ &= \{F_X(v) - F_X(u)\}^{2n+1}, \end{aligned}$$

and the distribution function of  $(X_1^*, X_{2n+1}^*)$  for  $u < v$  is given by

$$\begin{aligned} G(u, v) &= P\{X_1^* \leq u \wedge X_{2n+1}^* \leq v\} \\ &= P\{X_{2n+1}^* \leq v\} - P\{X_1^* > u \wedge X_{2n+1}^* \leq v\} \\ &= \{F_X(v)\}^{2n+1} - \{F_X(v) - F_X(u)\}^{2n+1}, \end{aligned}$$

and  $G(u, v) = 0$  for  $u \geq v$ .

The simultaneous frequency  $g(u, v)$  for  $u < v$  is given by

$$\begin{aligned} g(u, v) &= \frac{\partial^2 G}{\partial u \partial v} = \frac{\partial}{\partial v} \left\{ +(2n+1) \{F_X(v) - F_X(u)\}^{2n} \cdot f_X(u) \right\} \\ &= (2n+1) \cdot 2n \{F_X(v) - F_X(u)\}^{2n-1} f_X(u) f_X(v), \end{aligned}$$

and  $g(u, v) = 0$  for  $u \geq v$ .

3) The distribution function of the “middle” random variable is

$$F_{X_{n+1}^*}(x) = P\{n+1 \text{ of the } X_i \leq x \text{ and the rest } > x\}.$$

We can choose  $n+1$  variables  $X_i \leq x$  in  $\binom{2n+1}{n+1}$  ways. If we consider a fixed set

$$\mathbf{U} = (X_{i(1)}, \dots, X_{i(n+1)})$$

among these without caring about the remaining  $n$  variables, then we get by **1.** that

$$F_{\mathbf{U}}(x) = \{F_X(x)\}^{n+1} \quad \text{with } f_{\mathbf{U}}(x) = (n+1) \{F_X(x)\}^n f_X(x).$$

For the full system we get a conditional frequency (conditional, because the remaining  $n$  variables are  $> x$ ),

$$f_{\mathbf{U}}^*(x) = f_{\mathbf{U}}(x) \cdot \{1 - F(x)\}^n.$$

Notice that  $\mathbf{U}$  and  $\mathbf{U}^*$  play different roles; only  $\mathbf{U}$  is referring directly to  $\leq x$ .

When we collect all contributions, we get

$$\begin{aligned} f_{X_{n+1}^*}^*(x) &= \binom{2n+1}{n+1} \cdot (n+1) \cdot \{F_X(x)\}^n f_X(x) \cdot \{1 - F_X(x)\}^n \\ &= (2n+1) \binom{2n}{n} \cdot \{F_X(x)\}^n \{1 - F_X(x)\}^n f_X(x), \quad x \in \mathbb{R}. \end{aligned}$$

ALTERNATIVELY, start by isolating  $X_{n+1}^*$ . This random variable can be chosen in  $2n + 1$  ways. There are  $\binom{2n}{n}$  ways to choose  $X_1^*, \dots, X_n^*$ , and then  $X_{n+2}^*, \dots, X_{2n+1}^*$  are given. Thus

$$f_{X_{n+1}^*}(x) dx \approx P\{X_{n+1}^* \in ]x, x + dx]\} \approx (2n + 1) \binom{2n}{n} \{F_X(x)\}^n \{1 - F_X(x)\}^n f_X(x) dx,$$

and the result follows.

By isolating  $x < X_{n+1}^* \leq x + dx$  it becomes more clear what in the first version is meant by “conditional probability”.

4) When  $X$  is rectangularly distributed over  $]0, a[$ , then

$$f_X(x) = \begin{cases} \frac{1}{a} & \text{for } x \in ]0, a[, \\ 0 & \text{ellers,} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{a} & \text{for } 0 < x < a, \\ 1 & \text{for } x > a. \end{cases}$$

Let  $0 < x < a$ . By insertion,

$$\begin{aligned} f_{Y_{2n+1}}(x) &= f_{X_{n+1}^*}(x) = (2n + 1) \binom{2n}{n} \left\{\frac{x}{a}\right\}^n \left\{1 - \frac{x}{a}\right\}^n \cdot \frac{1}{a} \\ &= \frac{1}{B(n + 1, n + 1)} \cdot \frac{1}{a} \left\{\frac{x}{a}\right\}^n \left\{1 - \frac{x}{a}\right\}^n, \end{aligned}$$

(a Beta distribution), so

$$\begin{aligned} E\{Y_{2n+1}\} &= \frac{1}{B(n + 1, n + 1)} \int_0^a x \left\{\frac{x}{a}\right\}^n \left\{1 - \frac{x}{a}\right\}^n \cdot \frac{1}{a} dx \\ &= \frac{a}{B(n + 1, n + 1)} \int_0^1 t^{n+1} (1 - t)^n dt \\ &= a \cdot \frac{B(n + 2, n + 1)}{B(n + 1, n + 1)} = a \cdot \frac{(n + 1)! n!}{(2n + 2)!} \cdot \frac{(2n + 1)!}{n! n!} = \frac{a}{2}, \end{aligned}$$

proving that  $Y_{n+1}$  is central.

Then we get that  $Z_{n+1}$  is central, because

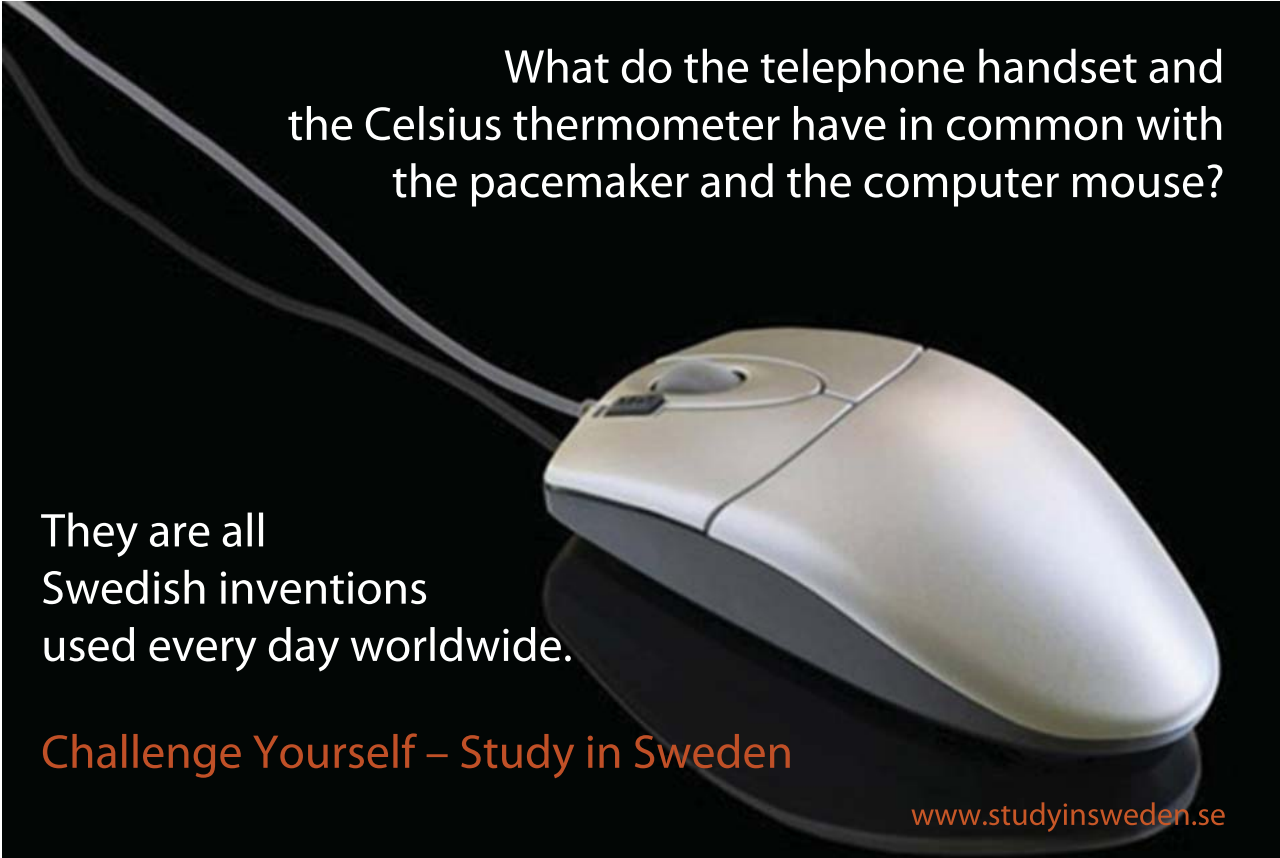
$$E\{Z_{2n+1}\} = \frac{1}{2n + 1} \sum_{i=1}^{2n+1} E\{X_i\} = \frac{a}{2}.$$

Finally,

$$\begin{aligned}
 E\{U_{2n+1}\} &= \frac{1}{2}E\{X_1^*\} + \frac{1}{2}E\{X_{2n+1}^*\} \\
 &= \frac{1}{2}(2n+1) \int_0^a x \left\{1 - \frac{x}{a}\right\}^{2n} \frac{1}{a} dx + \frac{1}{2}(2n+1) \int_0^a x \left\{\frac{x}{a}\right\}^{2n} \frac{1}{a} dx \\
 &= \frac{a}{2}(2n+1) \left\{ \int_0^1 t(1-t)^{2n} dt + \int_0^1 t^{2n+1} dt \right\} \\
 &= \frac{a}{2}(2n+1) \left\{ \int_0^1 (1-u)u^{2n} du + \int_0^1 t^{2n+1} dt \right\} \\
 &= \frac{a}{2}(2n+1) \int_0^1 u^{2n} du = \frac{a}{2},
 \end{aligned}$$

proving that  $U_{2n+1}$  is also central.

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5) We shall compute  $V\{Y_{2n+1}\}$ ,  $V\{Z_{2n+1}\}$  and  $V\{U_{2n+1}\}$ , and compare these expressions. The easiest computation is

$$V\{Z_{2n+1}\} = V\left\{\frac{1}{2n+1} \sum_{i=1}^{2n+1} X_i\right\} = \frac{1}{2n+1} \cdot \frac{a^2}{12}.$$

Then

$$\begin{aligned} E\{Y_{2n+1}^2\} &= (2n+1) \binom{2n}{n} \int_0^a x^2 \left\{\frac{x}{a}\right\}^n \left\{1 - \frac{x}{a}\right\}^n \frac{1}{a} dx \\ &= (2n+1) \binom{2n}{n} a^2 \int_0^1 t^{n+2} (1-t)^n dt \\ &= a^2 \cdot \frac{(2n+1)!}{n!n!} \cdot B(n+3, n+1) = a^2 \cdot \frac{(2n+1)!}{n!n!} \cdot \frac{(n+2)!n!}{(2n+3)!} \\ &= a^2 \frac{(n+2)(n+1)}{(2n+3)(2n+2)} = \frac{a^2}{2} \cdot \frac{n+2}{2n+3} \end{aligned}$$

implies that

$$V\{Y_{2n+1}\} = \frac{a^2}{2} \cdot \frac{n+2}{2n+3} - \frac{a^2}{4} = \frac{a^2}{4} \cdot \frac{1}{2n+3}.$$

From

$$V\{Z_{2n+1}\} = \frac{1}{2n+1} \cdot \frac{a^2}{12} = \frac{1}{6n+3} \cdot \frac{a^2}{4} < \frac{1}{2n+3} \cdot \frac{a^2}{4} = V\{Y_{2n+1}\},$$

follows that  $Z_{2n+1}$  is more efficient than  $Y_{2n+1}$ .

We still have to compute  $V\{U_{2n+1}\}$ . It follows from

$$U_{2n+1}^2 = \frac{1}{4} \{X_1^{*2} + X_{2n+1}^{*2} + 2X_1^* X_{2n+1}^*\},$$

and

$$(E\{U_{2n+1}\}) = \frac{a^2}{4},$$

that

$$V\{U_{2n+1}\} = \frac{1}{4} (E\{X_1^{*2}\} + E\{X_{2n+1}^{*2}\} + 2E\{X_1^* X_{2n+1}^{*2}\} - a^2).$$

The former two terms are simplest, cf. 1.,

$$\begin{aligned} E\{X_1^{*2}\} &= (2n+1) \int_0^a x^2 \left\{1 - \frac{x}{a}\right\}^{2n} \cdot \frac{1}{a} dx \\ &= a^2(2n+1) \int_0^1 t^2 (1-t)^{2n} dt = a^2(2n+1)B(3, 2n+1) \\ &= a^2(2n+1) \cdot \frac{2!(2n)!}{(2n+3)!} = \frac{a^2}{(2n+3)(n+1)!}, \end{aligned}$$

and

$$E \{X_{2n+1}^{*2}\} = (2n + 1) \int_0^a x^2 \left\{ \frac{x}{a} \right\}^{2n} \frac{1}{a} dx = a^2(2n + 1) \int_0^1 t^{2n+2} dt = \frac{2n + 1}{2n + 3} a^2.$$

According to **2.**, the simultaneous frequency of  $(X_1^*, X_{2n+1}^*)$  is

$$g(u, v) = \begin{cases} 2n(2n + 1) \left\{ \frac{v - u}{a} \right\}^{2n-1} \cdot \frac{1}{a^2} & \text{for } 0 < u < v < a, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E \{X_1^* X_{2n+1}^*\} &= \int_0^a \left\{ \int_0^v 2n(2n + 1) \left\{ \frac{v - u}{a} \right\}^{2n-1} \cdot uv \cdot \frac{1}{a^2} du \right\} dv \\ &= a^2 \int_0^1 \left\{ \int_0^v 2n(2n + 1)(v - u)^{2n-1} \cdot uv du \right\} dv \\ &= a^2 2n(2n + 1) \int_0^1 v \left\{ \int_0^v \{v(v - u)^{2n-1} - (v - u)^{2n}\} du \right\} dv \\ &= a^2 2n(2n + 1) \int_0^1 v \left\{ \int_0^v (vt^{2n-1} - t^{2n}) dt \right\} dv \\ &= a^2 2n(2n + 1) \int_0^1 v \left[ \frac{v^{2n+1}}{2n} - \frac{v^{2n+1}}{2n + 1} \right] dv \\ &= a^2 \int_0^1 v^{2n+2} dv = \frac{a^2}{2n + 3}. \end{aligned}$$

By insertion,

$$\begin{aligned} V \{U_{2n+1}\} &= \frac{a^2}{4} \cdot \left\{ \frac{1}{(2n + 3)(n + 1)} + \frac{2n + 1}{2n + 3} + \frac{2}{2n + 3} - 1 \right\} \\ &= \frac{1}{(2n + 3)(n + 1)} \cdot \frac{a^2}{4}. \end{aligned}$$

Since

$$(2n + 3)(n + 1) = 2n^2 + 5n + 3 > 6n + 3 = 3(2n + 1),$$

we get  $V \{U_{2n+1}\} < V \{Z_{2n+1}\}$ , hence  $U_{2n+1}$  is more efficient than  $Z_{2n+1}$ .

**Example 12.14** A random variable  $X$  has the frequency

$$f(x) = \frac{|x|}{2a^2} \exp\left(-\frac{|x|}{a}\right), \quad x \in \mathbb{R},$$

where  $a$  is a positive constant.

- 1) Compute  $E\{X\}$  and  $E\{|X|\}$ .
- 2) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$  (where all  $x_i$  are different from 0), apply the maximum method to find the best estimate  $a_n^*$  of  $a$  and hence also the best estimate  $b_n^*$  of  $E\{|X|\}$ .
- 3) Prove that the estimator  $Y_n$  corresponding to the estimate  $b_n^*$  is a central estimator.
- 4) We perform 10 observations. The results are

-3.3, -2.6, -3.6, 3.0, 3.2, -3.1, 3.5, -2.7, 2.6, -2.4.

What is the best estimate of  $E\{|X|\}$  in this case?

- 1) It follows by the symmetry that  $E\{X\} = 0$ , and

$$E\{|X|\} = 2 \cdot \frac{1}{2a^2} \int_0^\infty x^2 \exp\left(-\frac{x}{a}\right) dx = a \int_0^\infty t^2 e^{-t} dt = 2a.$$

- 2) We shall maximize

$$h_n(a) = \prod_{i=1}^n \frac{|x_i|}{2a^2} \exp\left(-\frac{|x_i|}{a}\right) = \frac{1}{2^n} \cdot \frac{1}{a^{2n}} \prod_{i=1}^n |x_i| \cdot \exp\left(-\frac{1}{a} \sum_{i=1}^n |x_i|\right).$$

Clearly,  $h_n(a) > 0$ , and  $h_n(a) \rightarrow 0$  for  $a \rightarrow 0+$ , and for  $a \rightarrow \infty$ .

We conclude from

$$\ln h_n(a) = -n \ln 2 - \sum_{i=1}^n \ln |x_i| - 2n \ln a - \frac{1}{a} \sum_{i=1}^n |x_i|,$$

that

$$\frac{h'_n(a)}{h_n(a)} = -\frac{2n}{a} + \frac{1}{a^2} \sum_{i=1}^n |x_i| = \frac{1}{a^2} \left\{ \sum_{i=1}^n |x_i| - 2na \right\},$$

so the unique maximum (the best estimate of  $a$ ) is

$$a_n^* = \frac{1}{2n} \sum_{i=1}^n |x_i|.$$

The best estimate  $b_n^*$  of  $E\{|X|\} = 2a$  is

$$b_n^* = \frac{1}{n} \sum_{i=1}^n |x_i|.$$

3) The estimator corresponding to  $b_n^*$  for  $E\{|X|\}$  is

$$Y_n = \frac{1}{n} \sum_{i=1}^n |X_i|.$$

We get from

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{|X_i|\} = \frac{1}{n} \cdot n \cdot E\{|X|\} = E\{|X|\},$$

that  $Y_n$  is a central estimator.

4) With the given measurements, the best estimate of  $E\{|X|\}$  is

$$b_{10}^* = \frac{1}{10} \{3.3 + 2.6 + 3.6 + 3.0 + 3.2 + 3.1 + 3.5 + 2.7 + 2.6 + 2.4\} = \frac{30.0}{10} = 3.$$

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**Example 12.15** A random variable  $X$  has its distribution given by

$$P\{X = k\} = (k + 1)p^2(1 - p)^k, \quad k \in \mathbb{N}_0,$$

where  $p$  is a constant,  $0 < p < 1$ , i.e.  $X \in \text{NB}(2, p)$ .

- 1) Find the mean  $\mu$  and variance  $\sigma^2$  of the random variable  $X$ .
- 2) Assuming that  $n$  observations of the mutually independent random variables  $X_1, X_2, \dots, X_n$ , all distributed like  $X$  above, have given the results  $x_1, x_2, \dots, x_n$  (where not all  $x_i$  are equal to 0), apply the maximal method to find the best estimate of  $p$  and the best estimate  $\mu_n^*$  of  $\mu$ .
- 3) Prove that the estimator  $Y_n$  corresponding to the estimate  $\mu_n^*$  is a central estimator of  $\mu$ .
- 4) We perform 10 observations of the results

4, 5, 1, 7, 0, 4, 2, 10, 3, 4.

What is the best estimate of  $p$  in this case?

- 1) Since  $X \in \text{NB}(2, p)$ , we get from a table that

$$\mu = 2 \cdot \frac{q}{p} = 2 \left( \frac{1}{p} - 1 \right) \quad \text{and} \quad \sigma^2 = 2 \frac{q}{p^2} \quad \left( = 2 \left( \frac{1}{p^2} - \frac{1}{p} \right) \right).$$

- 2) We shall maximize the function

$$h_n(p) = \prod_{i=1}^n P\{X_i = x_i\} = \prod_{i=1}^n (x_i + 1)p^2(1 - p)^{x_i}.$$

Clearly,

$$h_n(p) > 0 \quad \text{for } 0 < p < 1,$$

and  $h_n(p) \rightarrow 0$  for  $p \rightarrow 0+$ , and for  $p \rightarrow 1-$ , where we have assumed that not every  $x_i$  is 0. It follows from

$$\ln h_n(p) = \sum_{i=1}^n \ln(x_i + 1) + 2n \ln p + \ln(1 - p) \cdot \sum_{i=1}^n x_i,$$

that

$$\frac{h'_n(p)}{h_n(p)} = \frac{2n}{p} - \frac{1}{1 - p} \sum_{i=1}^n x_i.$$

This expression is equal to 0 for

$$2n(1 - p_n^*) = p_n^* \sum_{i=1}^n x_i,$$

thus for

$$p_n^* = \frac{2n}{2n + \sum_{i=1}^n x_i} = \frac{2}{2 + \bar{x}},$$

where  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ .

The best estimate of  $\mu$  is

$$\mu_n^* = 2 \left( \frac{1}{p_n^*} - 1 \right) = 2 + \frac{1}{n} \sum_{i=1}^n x_i - 2 = \bar{x}.$$

3) The estimator  $Y_n$  is

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i,$$

and it is obvious that

$$E\{Y_n\} = \frac{1}{n} \sum_{i=1}^n E\{X_i\} = E\{X\},$$

hence the estimator is central.

4) Since

$$\frac{1}{10} \sum_{i=1}^{10} x_i = \frac{1}{10} (4 + 5 + 1 + 7 + 0 + 4 + 2 + 10 + 3 + 4) = \frac{40}{10} = 4,$$

the best estimate of  $p$  is given by

$$p_n^* = \frac{2}{2 + \frac{1}{10} \sum_{i=1}^{10} x_i} = \frac{2}{2 + 4} = \frac{1}{3}.$$

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