18.435/2.111

QUIZ 2

1. Consider the dephasing operation represented in the operator sum notation by

$$\rho \to (1-p)\rho + p\sigma_z \rho \sigma_z^\dagger.$$

If this operation is applied to a qubit in the state

$$\alpha\,|\,0\rangle + \beta\,|\,1\rangle\,,$$

what is the density matrix of the resulting qubit?

2. Recall that CSS codes were derived from two classical linear binary codes with

$$C_2 \subset C_1.$$

A classical linear binary code is called t-error-detecting if the minimum Hamming weight (i.e., number of 1's) of a non-zero codeword is at least $t+1$. Let us say that a quantum CSS code is t-error-detecting if C_1 and $C_2^\perp$ are both t-error-detecting codes. Consider the linear encoding of 2 qubits into 4 qubits defined by mapping an orthonormal basis of the state space of 2 qubits to the following four codewords:

$$\frac{1}{\sqrt{2}}(|\,0000\rangle + |\,1111\rangle), \qquad \frac{1}{\sqrt{2}}(|\,0110\rangle + |\,1001\rangle),$$
$$\frac{1}{\sqrt{2}}(|\,1010\rangle + |\,0101\rangle), \qquad \frac{1}{\sqrt{2}}(|\,1100\rangle + |\,0011\rangle).$$

Show that this is a CSS code. What are the classical codes C_1 and C_2 that give rise to it? Is this a 1-error-detecting code?

3. Show that if an error σ_z or σ_x is applied to a single qubit in the code in problem 2, the resulting state is orthogonal to all four of the codewords. Show that this is also the case if a σ_x is applied to a single qubit *and* a σ_z is then applied to a single qubit (possibly the same one).

4. Suppose that Alice wants to send Bob two classical bits using superdense coding. Alice and Bob think they have the state

$$|\,\psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}(|\,00\rangle + |\,11\rangle)$$

but what they really have is a noisy version of this state, namely the following mixture of the four Bell states:

$$(1-p)\,|\,\psi_{\mathrm{EPR}}\rangle\langle\psi_{\mathrm{EPR}}\,| \;+\; (p/3)\sigma_x\,|\,\psi_{\mathrm{EPR}}\rangle\langle\psi_{\mathrm{EPR}}\,|\,\sigma_x^\dagger \tag{1}$$
$$+\;(p/3)\sigma_y\,|\,\psi_{\mathrm{EPR}}\rangle\langle\psi_{\mathrm{EPR}}\,|\,\sigma_y^\dagger \;+\; (p/3)\sigma_z\,|\,\psi_{\mathrm{EPR}}\rangle\langle\psi_{\mathrm{EPR}}\,|\,\sigma_z^\dagger. \tag{2}$$

This gives rise to a classical noisy channel. If Alice tries to send a two-bit message, $b_0 b_1$, to Bob, what possible messages can Bob receive, and with what probabilities does he receive them?