MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation

Problem 1. For the state $|\psi\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle_1 |g(x)\rangle_2$, where $g(x)$ is a 1-1 function, find the partial trace $\rho_1 \equiv tr_2(|\psi\rangle\langle\psi|)$ and calculate ${}^{\otimes n}\langle + | \rho_1 | + \rangle^{\otimes n}$.

Solution:

$$|\psi\rangle\langle\psi| = \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \otimes |g(x)\rangle_2 \langle g(y)|$$

Now, the fact that $g(x)$ is a 1-1 function implies that for $x \neq y$, we have $g(x) \neq g(y)$, and therefore,

$$\langle g(x) | g(y)\rangle = \delta_{xy} = tr(|g(x)\rangle\langle g(y)|)$$

Using the above relation, we have

$$\begin{aligned}
\rho_1 &\equiv tr_2(|\psi\rangle\langle\psi|) \\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \otimes tr\left(|g(x)\rangle_2 \langle g(y)|\right) \\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \delta_{xy} \\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)+f(x)} |x\rangle_1 \langle x| \\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} |x\rangle_1 \langle x| \\
&= I_1 / 2^n .
\end{aligned}$$

Now, the probability of measuring $|+\rangle^{\otimes n}$ is as follows

$$\begin{aligned}
tr(\rho_1 |+\rangle^{\otimes n}\, {}^{\otimes n}\langle +|) &= {}^{\otimes n}\langle + | \rho_1 | + \rangle^{\otimes n} \\
&= \frac{1}{2^n} {}^{\otimes n}\langle + | I_1 | + \rangle^{\otimes n} \\
&= \frac{1}{2^n} .
\end{aligned}$$

Problem 2. Find $H^{\otimes n} R_\alpha H^{\otimes n}$ and $H^{\otimes n} T_\alpha H^{\otimes n}$ in simpler terms, where

$$R_\alpha = \sum_{x=0}^{2^n-1} (-1)^{x\bullet\alpha} |x\rangle\langle x|$$

and

$$T_\alpha = \sum_{x=0}^{2^n-1} |x \oplus \alpha\rangle\langle x|.$$

Solution:

We know

$$H^{\otimes n} = \frac{1}{2^{n/2}} \sum_{a,b=0}^{2^n-1} (-1)^{a\bullet b} |b\rangle\langle a|$$

$\Rightarrow$

$$\begin{aligned}
H^{\otimes n} T_\alpha H^{\otimes n} &= \frac{1}{2^n} \sum_{a,b=0}^{2^n-1} \sum_{c,d=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a\bullet b + c\bullet d} |c\rangle\langle d | x \oplus \alpha\rangle\langle x | b\rangle\langle a| \\
&= \frac{1}{2^n} \sum_{a,b=0}^{2^n-1} \sum_{c,d=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a\bullet b + c\bullet d} |c\rangle \delta_{d,x\oplus\alpha} \delta_{xb} \langle a| \\
&= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a\bullet x + c\bullet(x\oplus\alpha)} |c\rangle\langle a| \\
&= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} (-1)^{c\bullet\alpha} |c\rangle\langle a| \sum_{x=0}^{2^n-1} (-1)^{a\bullet x + c\bullet x} \\
&= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} (-1)^{c\bullet\alpha} |c\rangle\langle a| 2^n \delta_{ac} \\
&= \sum_{a=0}^{2^n-1} (-1)^{a\bullet\alpha} |a\rangle\langle a| \\
&= R_\alpha .
\end{aligned}$$

Now, using $H^{\otimes n} = \left(H^{\otimes n}\right)^{-1}$, we have

$$H^{\otimes n} R_\alpha H^{\otimes n} = T_\alpha .$$

Hint: $\sum_{x=0}^{2^n-1} (-1)^{a\bullet x} = 2^n \delta(a)$.

Problem 3. Find $U_p R_p U_p^\dagger$ and $U_p T_p U_p^\dagger$ in simpler terms, where

$$R_p = \sum_{x=0}^{p-1} \exp(2\pi x i / p) |x\rangle\langle x|$$

$$T_p = \sum_{x=0}^{p-1} |x + 1 \text{ mod } p\rangle\langle x|$$

$$U_p = \frac{1}{\sqrt{p}} \sum_{x=0}^{p-1}\sum_{y=0}^{p-1} \exp(2\pi ixy / p)|y\rangle\langle x|$$

and p is a prime number.

Solution:

$$U_p T_\alpha U_p^\dagger = \frac{1}{p} \sum_{a,b=0}^{p-1} \sum_{c,d=0}^{p-1} \sum_{x=0}^{p-1} \exp(2\pi i(cd - ab)/p)|c\rangle\langle d|x+1\rangle\langle x|b\rangle\langle a|, \text{ addition mod } p$$
$$= \frac{1}{p} \sum_{a,b=0}^{p-1} \sum_{c,d=0}^{p-1} \sum_{x=0}^{p-1} \exp(2\pi i(cd - ab)/p)|c\rangle\delta_{d,x+1}\delta_{xb}\langle a|$$
$$= \frac{1}{p} \sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\sum_{x=0}^{p-1} \exp(2\pi i(cx + c - ax)/p)|c\rangle\langle a|$$
$$= \frac{1}{p} \sum_{a=0}^{p-1}\sum_{c=0}^{p-1} \exp(2\pi ic/p)|c\rangle\langle a| \sum_{x=0}^{p-1} \exp(2\pi i(cx - ax)/p)$$
$$= \frac{1}{p} \sum_{a=0}^{p-1}\sum_{c=0}^{p-1} \exp(2\pi ic/p)|c\rangle\langle a| p\delta_{ac}$$
$$= \sum_{a=0}^{p-1} \exp(2\pi ia/p)|a\rangle\langle a|$$
$$= R_p .$$

Similarly, you can show that

$$U_p^\dagger R_p U_p = T_p$$

and

$$U_p R_p U_p^\dagger = T_p^\dagger .$$

Hint: $\sum_{x=0}^{p-1} \exp(2\pi iax/p) = \begin{cases} p & a \equiv 0 \pmod{p} \\ \dfrac{\exp(2\pi ia) - 1}{\exp(2\pi ia/p) - 1} = 0 & \text{otherwise} \end{cases}$, for $a \in \mathbb{Z}$.

Problem 4. Show that $U_2 \otimes U_3 = PU_6P^{-1}$ where U_p is defined in Problem 3, and P is a permutation matrix (a matrix with only one nonzero element 1 in each row and column).

Solution: It can be seen that

$$U_2 \otimes U_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} U_3 & U_3 \\ U_3 & -U_3 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & 1 & 1 & \omega^3 & \omega^3 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^3 & \omega^5 & \omega \\ 1 & \omega^4 & \omega^2 & \omega^3 & \omega & \omega^5 \end{bmatrix}$$

and

$$U_6 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

where $\omega = \exp(2\pi i / 6)$.

There is no such P that satisfies $U_2 \otimes U_3 = P U_6 P^{-1}$. You can however find P_1 and P_2 such that $U_2 \otimes U_3 = P_1 U_6 P_2$. For instance,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also, you can verify that $U_2 \otimes U_3^\dagger = P_1 U_6 P_1^{-1}$.