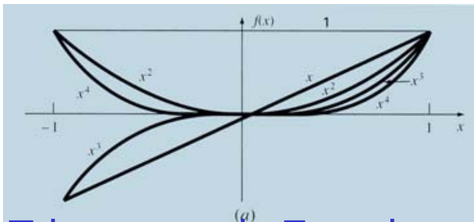
Numerical Marine Hydrodynamics

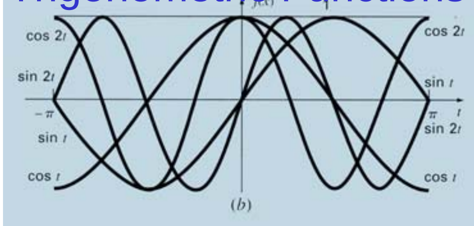
- Fourier Approximation
  - Continuous Fourier Series
  - Discrete Fourier Series
  - Trigonometric Polynomials
  - Frequency and Time Domains
    - Fourier Transform
    - Discrete Fourier Transform
    - Fast Fourier Transforms

Periodic Functions

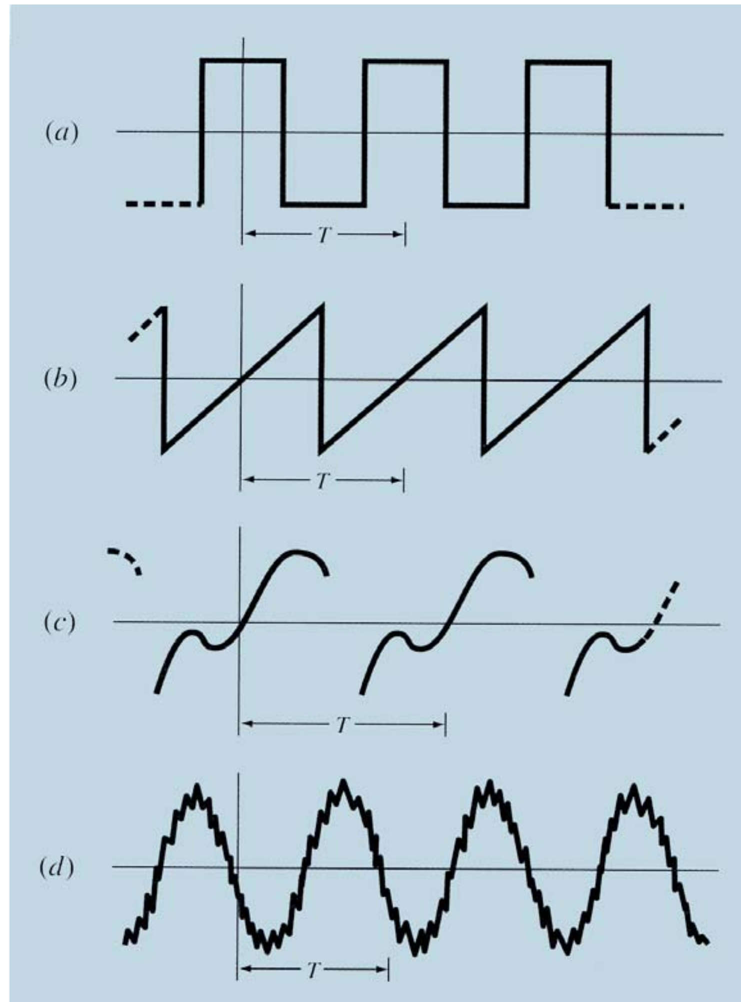
Monomials



Trigonometric Functions



Periodic Functions



Periodic Functions

$$g(x + P) = g(x)$$

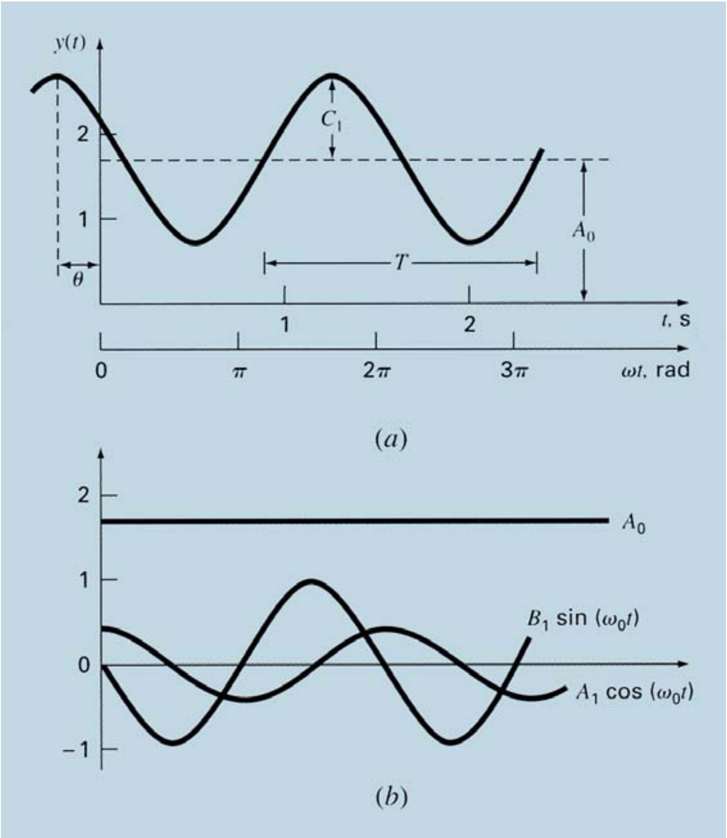
2π Periodic Functions

$$f(x) = g\left(\frac{Px}{2\pi}\right)$$

$$f(x + 2\pi) = g\left(\frac{Px}{2\pi} + P\right) = g\left(\frac{Px}{2\pi}\right) = f(x)$$

$$f(x + n2\pi) = f(x)$$

Continuous Fourier Series



Fourier Series Approximation

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos jx + b_j \sin jx)$$

Orthogonality Relations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos jx \ \cos kx \ dx = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin jx \ \sin kx \ dx = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin jx \ \cos kx \ dx = 0$$

Expansion Coefficients

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos jx \ dx$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin jx \ dx$$

Trigonometric Expansions

$$S_M(x) = \frac{a_0}{2} + \sum_{j=1}^{M} (a_j \cos jx + b_j \sin jx)$$

Continuous Fourier Series Example

$$f(x) = x/2\,, \quad -\pi < x < \pi$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\cos jx dx = \left[\frac{x\sin jx}{2\pi j} + \frac{\cos jx}{2\pi j^2}\right]_{-\pi}^{\pi} = 0\,, \quad j = 1,2,3,\ldots$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\sin jx dx = \left[-\frac{x\cos jx}{2\pi j} + \frac{\sin jx}{2\pi j^2}\right]_{-\pi}^{\pi} = \frac{(-1)^{j+1}}{j}\,, \quad j = 1,2,3,\ldots$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2} = \left[\frac{x^2}{4\pi}\right]_{-\pi}^{\pi} = 0$$

$$f(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\sin jx = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots$$

$$\begin{aligned} S_1 &= \sin x \\ S_2 &= \sin x - \frac{\sin 2x}{2} \\ S_3 &= \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \\ S_4 &= \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \end{aligned}$$

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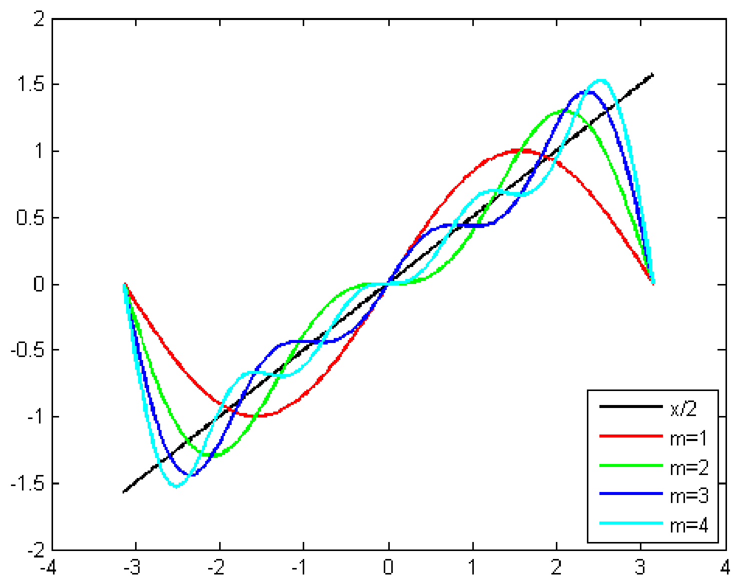


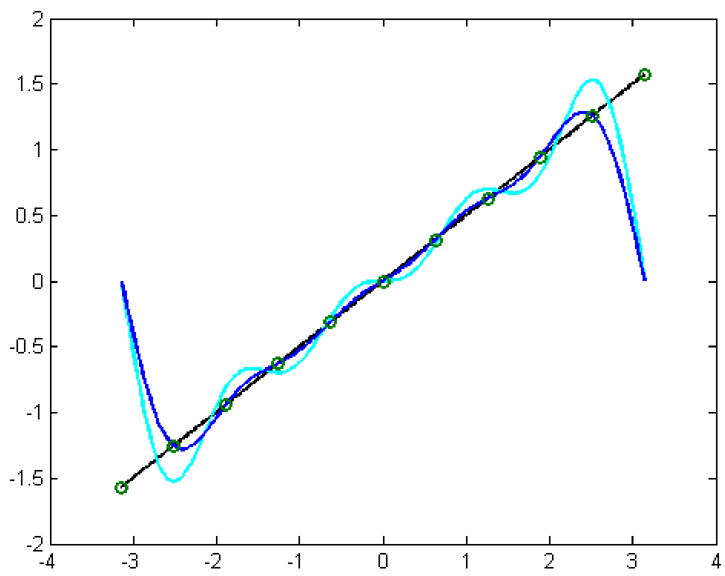
Discrete Fourier Series

Trigonometric Polynomials

$$T_M(x) = \frac{a_0}{2} + \sum_{j=1}^{M} (a_j \cos jx + b_j \sin jx)$$

Minimize

$$\epsilon^2 = \sum_{k=1}^{N} (f(x_k) - T_M(x_k))^2$$



$$M = 1$$

$$x_k = -\pi + \frac{2\pi k}{N}, \quad k = 0, 1, \ldots, N$$

$$\epsilon^2 = \sum_{k=1}^{N} \left(f(x_k) - \left(\frac{a_0}{2} + a_1 \cos x_k + b_1 \sin x_k \right) \right)^2$$

Normal Equation

$$\begin{bmatrix} N/2 & \Sigma_k \cos x_k & \Sigma_k \sin x_k \\ \Sigma_k \cos x_k/2 & \Sigma_k \cos^2 x_k & \Sigma_k \cos x_k \sin x_k \\ \Sigma_k \sin x_k/2 & \Sigma_k \cos x_k \sin x_k & \Sigma_k \sin_k^2 x \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \end{Bmatrix} = \begin{Bmatrix} \Sigma_k f(x_k) \\ \Sigma_k f(x_k) \cos x_k \\ \Sigma_k f(x_k) \sin x_k \end{Bmatrix}$$

Orthogonality

$$\begin{aligned} (\Sigma_k \sin x_k)/N &= 0 \\ (\Sigma_k \cos x_k)/N &= 0 \\ (\Sigma_k \sin^2 x_k)/N &= 1/2 \\ (\Sigma_k \cos^2 x_k)/N &= 1/2 \\ (\Sigma_k \cos x_k \sin x_k)/N &= 0 \end{aligned}$$

$/2$

$$\begin{bmatrix} N/2 & 0 & 0 \\ 0 & N/2 & 0 \\ 0 & 0 & N/2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \end{Bmatrix} = \begin{Bmatrix} \Sigma_k f(x_k) \\ \Sigma_k f(x_k) \cos x_k \\ \Sigma_k f(x_k) \sin x_k \end{Bmatrix}$$

Discrete Fourier Series

Solution M=1

$$\begin{Bmatrix} a_0 \\ a_1 \\ b_1 \end{Bmatrix} = \begin{bmatrix} 2/N & 0 & 0 \\ 0 & 2/N & 0 \\ 0 & 0 & 2/N \end{bmatrix} \begin{Bmatrix} \Sigma_k f(x_k) \\ \Sigma_k f(x_k) \cos x_k \\ \Sigma_k f(x_k) \sin x_k \end{Bmatrix}$$

$$\begin{aligned} a_0 &= \frac{2\Sigma_k f(x_k)}{N} \\ a_1 &= \frac{2\Sigma_k f(x_k) \cos x_k}{N} \\ a_2 &= \frac{2\Sigma_k f(x_k) \sin x_k}{N} \end{aligned}$$

General Case M>1

$$T_M(x) = \frac{a_0}{2} + \sum_{j=1}^{M} (a_j \cos jx + b_j \sin jx)$$

$$2M < N$$

$$\begin{aligned} a_j &= \frac{2}{N} \sum_{k=1}^{N} f(x_k) cos(jx_k) \ , \quad j = 0, 1, \ldots, M \\ b_j &= \frac{2}{N} \sum_{k=1}^{N} f(x_k) sin(jx_k) \ , \quad j = 1, \ldots, M \end{aligned}$$

Discrete Fourier Series
Trigonometric Polynomial Expansion

tpcoef.m

```
function [a,b] =tpcoef(x,y,m)
% x abissa of n+1 points
% y ordinates of n+1 points
% m order of trigonometric polynomial
% a cos coefficients
% b sin coefficients

n=length(x)-1;
max1=fix((n-1)/2);
if (m > max1)
    m = max1
end

a=zeros(1,m+1);
b=zeros(1,m+1);
y_ends = (y(1)+y(n+1))/2;
y(1) = y_ends;
y(n+1) = y_ends;

a(1) = sum(y);
for j=1:m
    a(j+1) = cos(j*x)*y';
    b(j+1) = sin(j*x)*y';
end

a = 2*a/n;
b = 2*b/n;
```

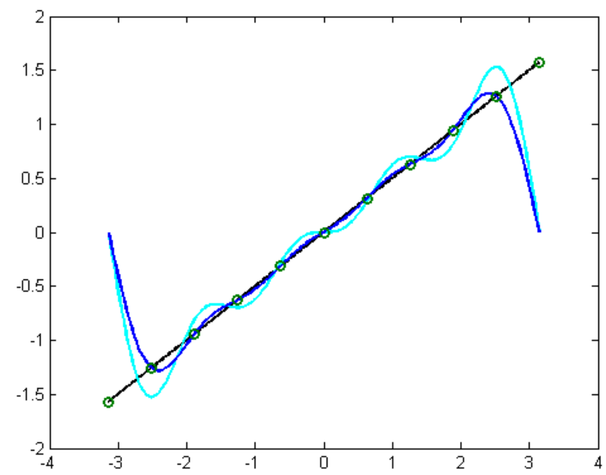
tpeval.m

```
function z = tpeval(x,a,b,m)
% evaluate trigonometric polynomial of order m at values x
z=a(1);
for j=1:m
    z = z+a(j+1)*cos(j*x)+b(j+1)*sin(j*x);
end
```

tptest.m

```
N=11;
X=[-pi:2*pi/(N-1):pi];
Y=X/2;
M=4;
[a,b]=tpcoef(X,Y,M);

n=101;
x=[-pi:pi/(n-1):pi];
y=tpeval(x,a,b,M);
h=plot(x,y,X,Y,'o');
set(h,'Linewidth',2);
```



Time-Frequency Analysis
Complex Fourier Series

Discrete Frequency Sampling

$$x = 2\pi t/T = \Delta\omega t\ , \quad -T/2 \le t \le T/2$$

$$\Delta\omega = 2\pi/T$$

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos j\Delta\omega t + b_j \sin j\Delta\omega t)$$

$$\begin{aligned}\cos x &= \frac{e^{ix}+e^{-ix}}{2}\\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i}\end{aligned}$$

$$f(t) = c_0 + \sum_{j=1}^{\infty}(c_j e^{ij\Delta\omega t} + c_{-j}e^{-ij\Delta\omega t})$$

$$\begin{aligned}c_0 &= a_0/2\\ c_j &= \frac{a_i - ib_i}{2}\\ c_{-j} &= \frac{a_i + ib_i}{2}\end{aligned}$$

Complex Fourier Series

$$\begin{aligned}f(t) &= \sum_{j=0}^{\infty} c_j e^{ij\Delta\omega t} + \sum_{j=1}^{\infty} c_{-j}e^{-ij\Delta\omega t}\\ &= \sum_{j=0}^{\infty} c_j e^{ij\Delta\omega t} + \sum_{j=-1}^{-\infty} c_j e^{ij\Delta\omega t}\\ &= \sum_{j=-\infty}^{\infty} c_j e^{ij\Delta\omega t}\end{aligned}$$

Expansion Coefficients

$$\begin{aligned}c_k &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos j\Delta\omega t\ - \sin j\Delta\omega t)\ dt\\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-ij\Delta\omega t}\ dt\end{aligned}$$

Time-Frequency Analysis
Fourier Transforms

Complex Fourier Series

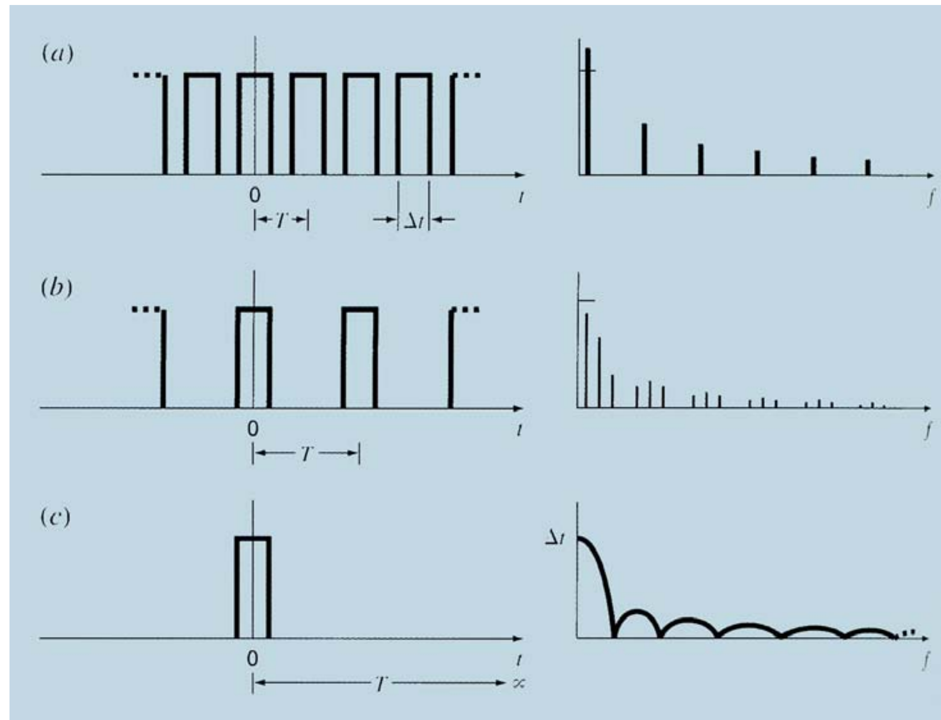
$$f(t) = \sum_{j=-\infty}^{\infty} c_j e^{ij\Delta\omega t}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-ij\Delta\omega t}\, dt$$

Fourier Transform Pair

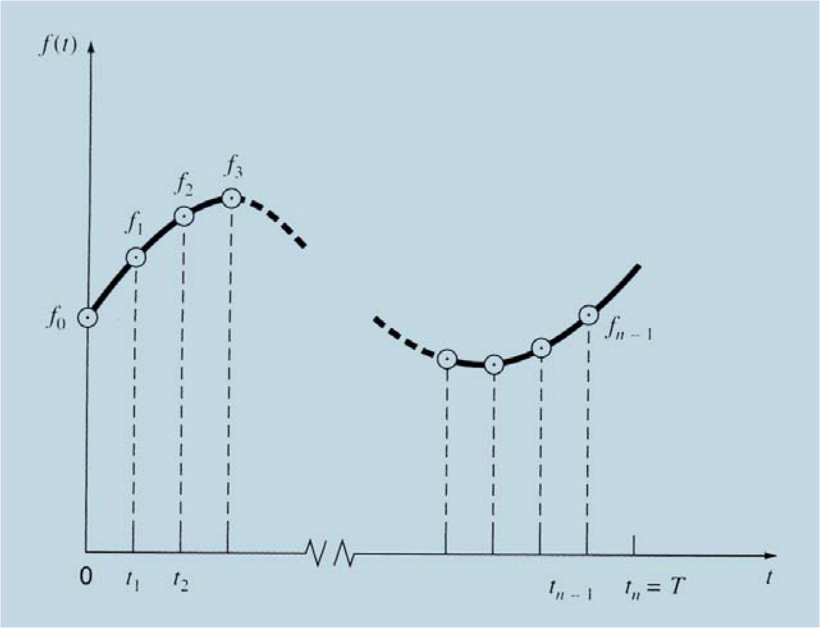
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$



Time-Frequency Analysis
Discrete Fourier Transforms

Disctrete Timeseries Sampling



Discrete Fourier Series

$$f(t) = \sum_{j=-\infty}^{\infty} c_j e^{ij\Delta\omega t}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-ij\Delta\omega t}\, dt$$

Time-Frequency Discretization

$$t_k = k\Delta t\ , \quad k = 0, 1, 2 \ldots N-1$$

$$\omega_j = (-N/2 + j)\Delta t\ , \quad j = 0, 1, 2 \ldots N-1$$

Periodicity

$$\Delta t = T/N$$

$$\Delta\omega = 2\pi/T$$

Sampling Constraint

$$\Delta t \Delta\omega = 2\pi/N$$

Discrete Fourier Transform

$$F_j = F(\omega_j) = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-ijk\Delta\omega\Delta t}$$

$$F_j = F(\omega_j) = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi jk/N}$$

Inverse Fourier Transform

$$f_k = \sum_{j=-N/2}^{N/2-1} F_j e^{ijk\Delta\omega\Delta t}$$

$$= \sum_{j=0}^{N-1} F_j e^{ijk\Delta\omega\Delta t}$$

$$= \sum_{j=0}^{N-1} F_j e^{i2\pi jk/N}$$

Time-Frequency Analysis
Fast Fourier Transforms

Power of 2 Sampling

$$N = 2^M$$

Exponential Power Formulation

$$\begin{aligned} F_j &= \sum_{k=0}^{N-1} f_k e^{-i2\pi jk/N}\ , \quad k = 0,1,\dots N-1 \\ &= \sum_{k=0}^{N-1} f_k w^{jk}\ , \quad k = 0,1,\dots N-1 \end{aligned}$$

$$w = e^{-i2\pi/N}$$

Transform Splitting

$$\begin{aligned} F_j &= \sum_{k=0}^{N/2-1} f_k e^{-i2\pi jk/N} + \sum_{k=N/2}^{N-1} f_k e^{-i2\pi jk/N} \\ &= \sum_{k=0}^{N/2-1} f_k e^{-i2\pi jk/N} + \sum_{m=0}^{N/2-1} f_{m+N/2} e^{-i2\pi j(m+N/2)/N} \\ &= \sum_{k=0}^{N/2-1} (f_k + e^{-i\pi j} f_{k+N/2}) e^{-i2\pi jk/N} \end{aligned}$$

$$e^{-i\pi j} = (-1)^j$$

Even Frequency Numbers

$$\begin{aligned} F_{2j} &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi 2jk/N} \\ &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi jk/(N/2)} \\ &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) w^{2jk} \end{aligned}$$

Odd Frequency Numbers

$$\begin{aligned} F_{2j+1} &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi(2j+1)k/N} \\ &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi k/N} e^{-i2\pi jk/(N/2)} \\ &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) w^k w^{2jk} \end{aligned}$$

Half-size Sequences

$$\begin{aligned} g_k &= f_k + f_{k+N/2} \\ h_k &= (f_k - f_{k+N/2}) w^k \end{aligned}$$

Half-size Transforms

$$\begin{aligned} F_{2j} &= G_j \\ F_{2j+1} &= H_j \end{aligned}$$

Time-Frequency Analysis
Fast Fourier Transforms

Even Frequency Numbers

$$F_{2j} = \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi 2jk/N}$$
$$= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi jk/(N/2)}$$
$$= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) w^{2jk}$$

Odd Frequency Numbers

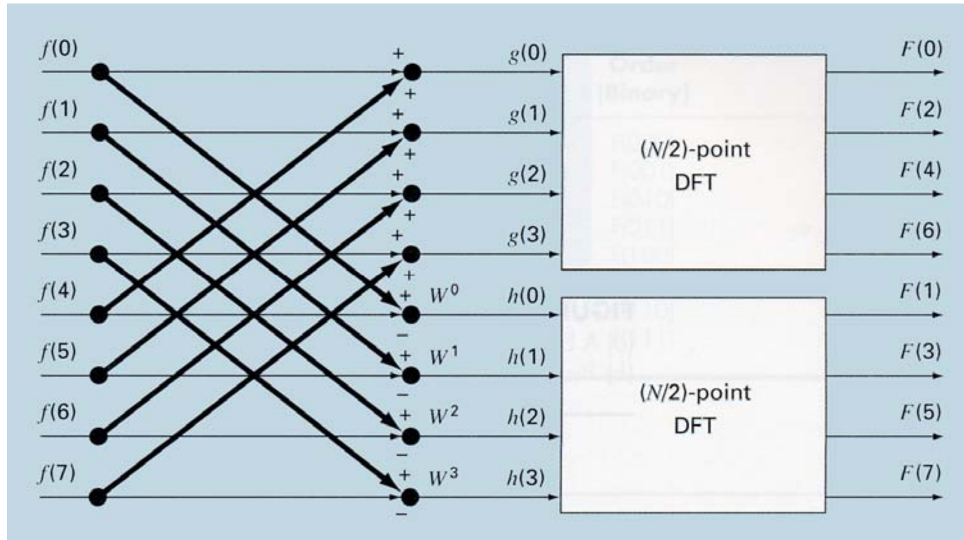
$$F_{2j+1} = \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi(2j+1)k/N}$$
$$= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi k/N} e^{-i2\pi jk/(N/2)}$$
$$= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) w^k w^{2jk}$$

Half-size Sequences

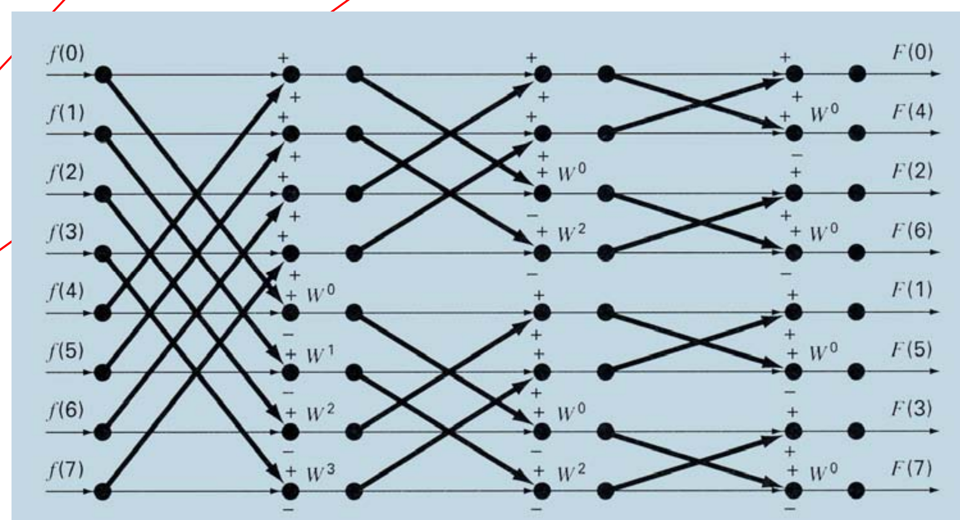
$$g_k = f_k + f_{k+N/2}$$
$$h_k = (f_k - f_{k+N/2}) w^k$$

Half-size Transforms

$$F_{2j} = G_j$$
$$F_{2j+1} = H_j$$

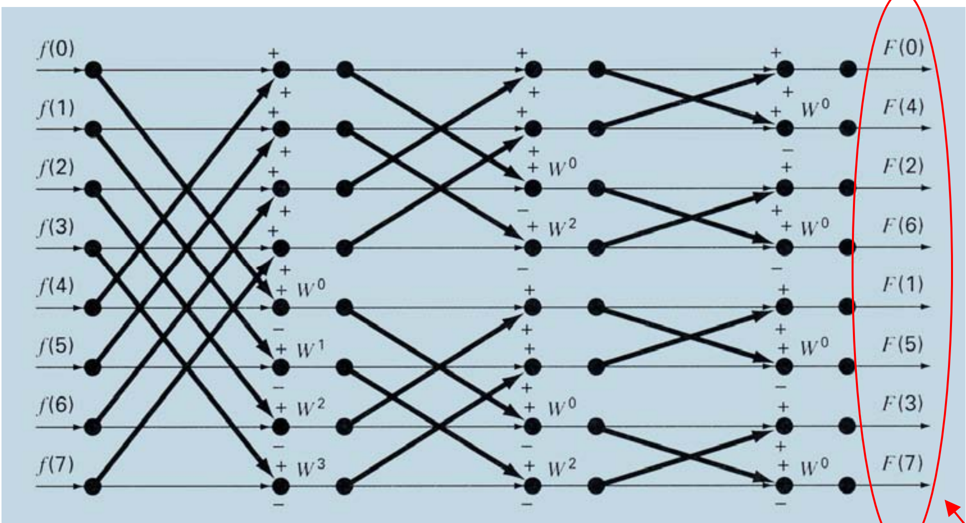


Sande-Tukey Algorithm – N=8

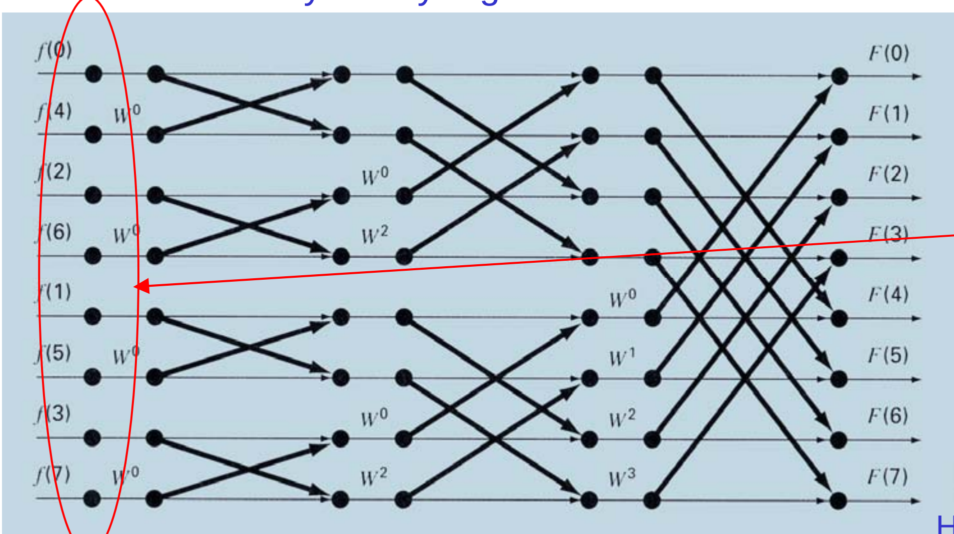


Time-Frequency Analysis
Fast Fourier Transforms

Sande-Tukey Algorithm – N=8



Cooley-Tukey Algorithm – N=8



Bit-reversal Uncrambling

Order	Binary	Bit-Reversal	Scrambled
0	0 0 0	0 0 0	0
1	0 0 1	1 0 0	4
2	0 1 0	0 1 0	2
3	0 1 1	1 1 0	6
4	1 0 0	0 0 1	1
5	1 0 1	1 0 1	5
6	1 1 0	0 1 1	3
7	1 1 1	1 1 1	7

Scrambled Order

Time-Frequency Analysis
Power Spectrum

Power Spectrum

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt$$

Discrete Fourier Series

$$f(t) = \sum_{-\infty}^{\infty} F_k e^{ij\Delta\omega t}$$

Orthogonality

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \sum_{-\infty}^{\infty} |F_k|^2$$

Total Power

$$p_j = 2|F_j|^2$$