

2 Stability of solutions to ODEs

How can we address the question of stability in general?

We proceed from the example of the pendulum equation. We reduce this second order ODE,

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0,$$

to two first order ODE's.

Write $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin x_1 \end{aligned}$$

The equilibrium points, or *fixed points*, are where the trajectories in phase space *stop*, i.e. where

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \vec{0}$$

For the pendulum, this requires

$$\begin{aligned} x_2 &= 0 \\ x_1 &= \pm n\pi, \quad n = 0, 1, 2, \dots \end{aligned}$$

Since $\sin x_1$ is periodic, the only distinct fixed points are

$$\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

Intuitively, the first is stable and the second is not.

How may we be more precise?

2.1 Linear systems

Consider the problem in general. First, assume that we have the linear system

$$\begin{aligned} \dot{u}_1 &= a_{11}u_1 + a_{12}u_2 \\ \dot{u}_2 &= a_{21}u_1 + a_{22}u_2 \end{aligned}$$

or

$$\dot{\vec{u}} = A\vec{u}$$

with

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Assume A has an inverse and that its eigenvalues are distinct. Then the only fixed point (where $\dot{\vec{u}} = 0$) is $\vec{u} = 0$.

The solution, in general, is

$$\vec{u}(t) = \alpha_1 e^{\lambda_1 t} \vec{c}_1 + \alpha_2 e^{\lambda_2 t} \vec{c}_2$$

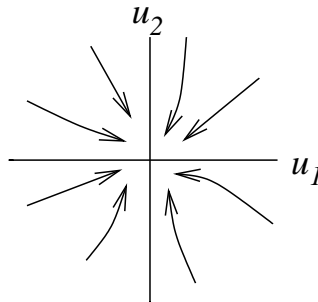
where

- λ_1, λ_2 are eigenvalues of A .
- \vec{c}_1, \vec{c}_2 are eigenvectors of A .
- α_1 and α_2 are constants (deriving from initial conditions).

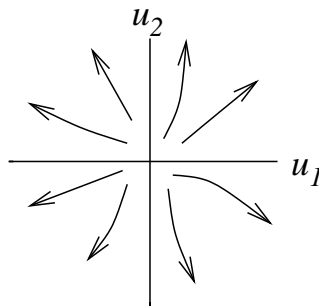
What are the possibilities for stability?

1. λ_1 and λ_2 are both real.

- (a) If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $u(t) \rightarrow 0$ as $t \rightarrow \infty$.
 \Rightarrow **stable**.



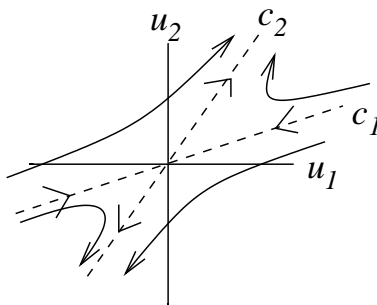
- (b) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $u(t) \rightarrow \infty$ as $t \rightarrow \infty$.
 \Rightarrow **unstable**.



- (c) If $\lambda_1 < 0 < \lambda_2$,

- If $\vec{u}(0)$ is a multiple of \vec{c}_1 , then $u(t) \rightarrow 0$ as $t \rightarrow \infty$.
- If $\vec{u}(0)$ is a multiple of \vec{c}_2 , then $u(t) \rightarrow \infty$ as $t \rightarrow \infty$.

\Rightarrow **unstable saddle**.



2. λ_1, λ_2 are both complex. Then

$$\lambda = \sigma \pm iq.$$

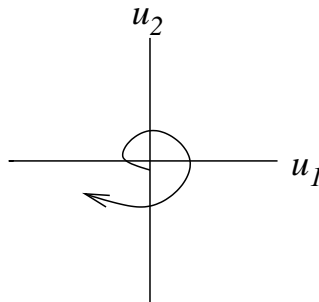
Assuming $\vec{u}(t)$ is real,

$$\vec{u}(t) = e^{\sigma t}(\vec{\beta}_1 \cos qt + \vec{\beta}_2 \sin qt)$$

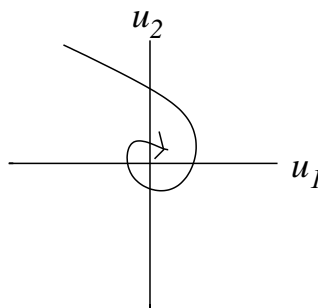
($\vec{\beta}_1, \vec{\beta}_2$ are formed from a linear combination of of A 's eigenvectors and the initial conditions).

There are three possibilities:

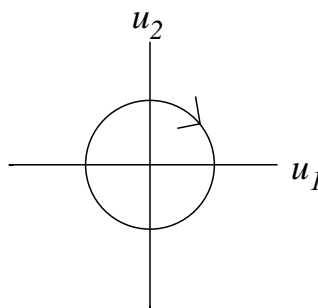
- (a) $\text{Re}\{\lambda\} = \sigma > 0 \implies$ **unstable**.



(b) $\sigma < 0 \implies$ **stable**.



(c) $\sigma = 0 \implies$ **marginally stable**.



We leave the case of repeated eigenvalues to Strogatz (pp. 135-6).

2.2 Nonlinear systems

We are interested in the qualitative behavior of systems like

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

where f_1 and f_2 are nonlinear functions of x_1 and x_2 .

Suppose $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a fixed point. Is it stable?

Define $u_i = x_i - x_i^*$ to be a small departure from the fixed point.

Perform a Taylor expansion around the fixed point.

For a one dimensional function $g(x)$ we would have

$$g(x^* + u) \simeq g(x^*) + g'(x^*) \cdot u$$

Here we obtain

$$f_i(x_1, x_2) = \underbrace{f_i(x_1^*, x_2^*)}_{=0 \text{ since fixed pt}} + \frac{\partial f_i}{\partial x_1}(x_1^*, x_2^*) u_1 + \frac{\partial f_i}{\partial x_2}(x_1^*, x_2^*) u_2 + O(u^2)$$

The first term vanishes since it is evaluated at the fixed point.

Also, since

$$u_i = x_i - x_i^*$$

we have

$$\dot{u}_i = \dot{x}_i = f_i(x_1, x_2)$$

Substituting $\dot{u}_i = f_i(x_1, x_2)$ above, we obtain

$$\dot{\vec{u}} \simeq A\vec{u}$$

where

$$A = \left(\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right) \bigg|_{\vec{x}=\vec{x}^*}$$

A is called the **Jacobian** matrix of f at \vec{x}^* .

We now apply these results to the pendulum. We have

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) = -\frac{g}{l} \sin x_1 \end{aligned}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ -g/l & 0 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

There is a different A for the case $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$. (The sign of g/l changes.)

The question of stability is then addressed just as in the linear case, via calculation of the eigenvalues and eigenvectors.