# 16 Period doubling route to chaos

We now study the "routes" or "scenarios" towards chaos.

We ask: How does the transition from periodic to strange attractor occur?

The question is analogous to the study of phase transitions: How does a solid become a melt; or a liquid become a gas?

We shall see that, just as in the study of phase transitions, there are *universal* ways in which systems become chaotic.

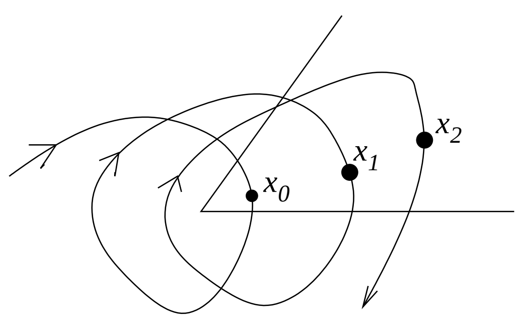
There are three universal routes:

- Period doubling
- Intermittency
- Quasiperiodicity

We shall focus the majority of our attention on period doubling.

## 16.1 Instability of a limit cycle

To analyze how a periodic regime may lose its stability, consider the Poincaré section:




The periodic regime is linearly unstable if

$$|\vec{x}_1 - \vec{x}_0| < |\vec{x}_2 - \vec{x}_1| < \ldots$$

or

$$|\delta\vec{x}_1| < |\delta\vec{x}_2| < \ldots$$

Recall that, to first order, a Poincaré map $T$ in the neighborhood of $\vec{x}_0$ is described by the Floquet matrix

$$M_{ij} = \frac{\partial T_i}{\partial X_j}.$$

In a periodic regime,

$$\vec{x}(t+\tau) = \vec{x}(t).$$

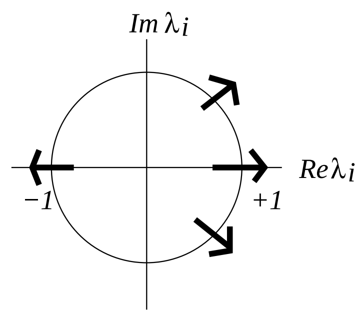
But the mapping $T$ sends

$$\vec{x}_0 + \delta\vec{x} \rightarrow \vec{x}_0 + M\delta\vec{x}.$$

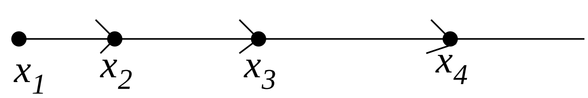
Thus stability depends on the 2 (possibly complex) eigenvalues $\lambda_i$ of $M$.

If $|\lambda_i| > 1$, the fixed point is unstable.

There are three ways in which $|\lambda_i| > 1$:

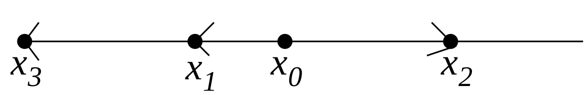



1. $\lambda = 1 + \varepsilon$, $\varepsilon$ real, $\varepsilon > 0$. $\delta\vec{x}$ is amplified is in the same direction:



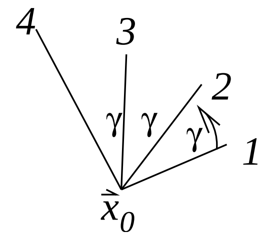

This transition is associated with Type 1 intermittency.

2. $\lambda = -(1 + \varepsilon)$. $\delta\vec{x}$ is amplified in *alternating* directions:




This transition is associated with period doubling.

3. $\lambda = \alpha \pm i\beta = (1+\varepsilon)e^{\pm i\gamma}$. $|\delta\vec{x}|$ is amplified, $\delta\vec{x}$ is rotated:



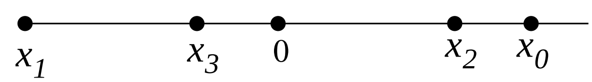

This transition is associated with quasiperiodicity.

In each of these cases, nonlinear effects eventually cause the instability to saturate.

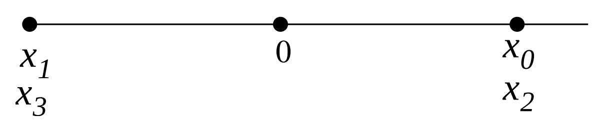
Let's look more closely at the second case, $\lambda \simeq -1$.

Just before the transition, $\lambda = -(1-\varepsilon)$, $\varepsilon > 0$.

Assume the Poincaré section goes through $x = -0$. Then an initial perturbation $x_0$ is damped with alternating sign:




Now vary the control parameter such that $\lambda = -1$. The iterations no longer converge:




We see that a new cycle has appeared with period twice that of the original cycle through $x = 0$.

This is a *period doubling* bifurcation.

## 16.2 Logistic map

We now focus on the simplest possible system that exhibits period doubling.

In essence, we set aside $n$-dimensional ($n \geq 3$) trajectories and focus only on the Poincaré section and the eigenvector whose eigenvalue crosses $(-1)$.

Thus we look at discrete intervals $T, 2T, 3T, \ldots$ and study the iterates of a transformation on an axis.

We therefore study first return maps

$$x_{k+1} = f(x_k)$$

and shall argue that these maps are highly relevant to $n$-dimensional flows.

For clarity, we adopt a biological interpretation.

Imagine an island with an insect population that breeds in summer and leaves egges that hatch the following summer.

Let $x_j$ = ratio of actual population in $j$th summer to some reference population.

Assume that next summer's population is determined by this summer's population according to

$$x_{j+1} = rx_j - sx_j^2.$$

The term $rx_{j+1}$ represents natural growth; if $r > 1$ the population grows (exponentially) by a factor $r$ each year.

The term $sx_j^2$ represents a reduction of natural growth due to crowding and competition for resources.

Now rescale $x_j \to (r/s)x_j$. Then

$$x_{j+1} = rx_j - rx_j^2.$$

Set $r = 4\mu$:

$$x_{j+1} = 4\mu x_j(1 - x_j).$$

This is called the *logistic map.*

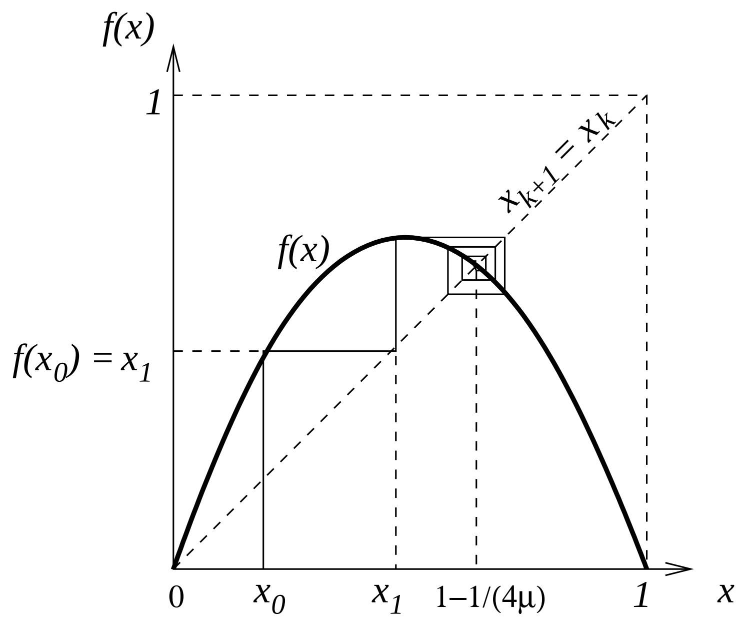
## 16.3 Fixed points and stability

We seek the long-term dependence of $x_j$ on the control parameter $\mu$. Remarkably, we shall see that $\mu$ plays a role not unlike that of the Rayleigh number in thermal convection.

So that $0 < x_j < 1$, we consider the range

$$0 < \mu < 1.$$

Recall that we have already discussed the graphical interpretation of such maps. Below is a sketch for $\mu = 0.7$:




The fixed points solve

$$x^* = f(x^*) = 4\mu x^*(1 - x^*),$$

which yields

$$x^* = 0 \quad \text{and} \quad x^* = 1 - \frac{1}{4\mu},$$

where the second fixed point exists only for $\mu > 1/4$.

Recall that stability requires

$$|f'(x^*)| < 1 \quad \Longrightarrow \quad |4\mu(1 - 2x^*)| < 1.$$

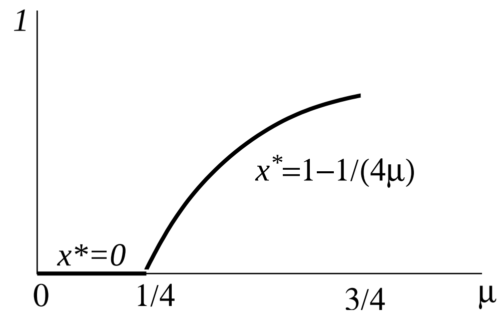
The stability condition for $x^* = 0$ is therefore

$$\mu < 1/4.$$

The non-trivial fixed point, $x^* = 1 - 1/(4\mu)$, is stable for

$$1/4 < \mu < 3/4.$$

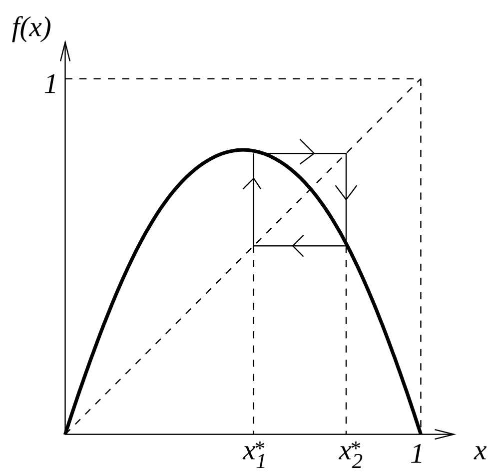
The long-time behavior of the insect population $x$ for $0 < \mu < 3/4$ then looks like




## 16.4 Period doubling bifurcations

What happens for $\mu > 3/4$?

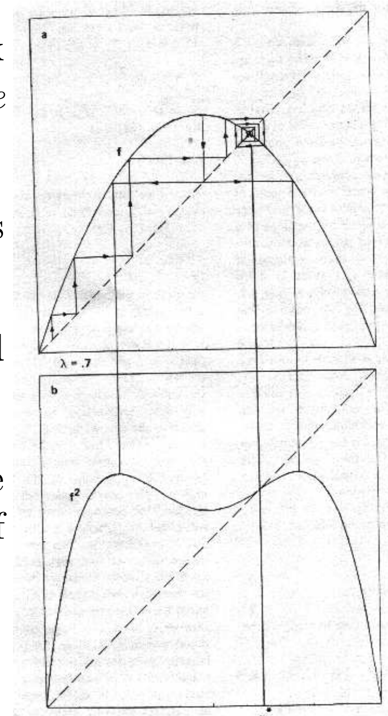
At $\mu = 3/4$, $x^* = 1 - 1/(4\mu)$ is marginally stable. Just beyond this point, the period of the asymptotic iterates doubles:



Let's examine this transition more closely. First, look at both $f(x)$ and $f^2(x) = f\big(f(x)\big)$ just *before the transition*, at $\mu = 0.7$.

- Since $f(x)$ is symmetric about $x = 1/2$, so is $f^2(x)$.
- If $x^*$ is a fixed point of $f(x)$, $x^*$ is also a fixed point of $f^2(x)$.

We shall see that period doubling depends on the relationship of the slope of $f^2(x^*)$ to the slope of $f(x^*)$.




Feigenbaum, Fig. 2.

The two slopes are related by the chain rule. By definition,

$$x_1 = f(x_0), x_2 = f(x_1) \implies x_2 = f^2(x_0).$$

Using the chain rule,

$$\begin{aligned} f^{2\prime}(x_0) &= \frac{\mathrm{d}}{\mathrm{d}x}\, f\big(f(x)\big)\big|_{x_0} \\ &= f'(x_0)\, f'\big(f(x_0)\big) \\ &= f'(x_0)\, f'(x_1) \end{aligned}$$

Thus, in general,

$$f^{n\prime}(x_0) = f'(x_0)\, f'(x_1) \ldots f'(x_{n-1}). \tag{32}$$

Now, suppose $x_0 = x^*$, a fixed point of $f$. Then

$$x_1 = x_0 = x^*$$

and

$$f^{2\prime}(x^*) = f'(x^*)\, f'(x^*) = |f'(x^*)|^2.$$

For the example of $\mu < 3/4$,

$$|f'(x^*)| < 1 \implies |f^{2'}(x^*)| < 1.$$

Moreover, if we start at $x_0 = 1/2$, the extremum of $f$, then equation (32) shows that

$$\begin{aligned} f'(1/2) = 0 \quad &\implies \quad f^{2'}(1/2) = 0 \\ &\implies \quad x = 1/2 \text{ is an extremum of } f^2. \end{aligned}$$

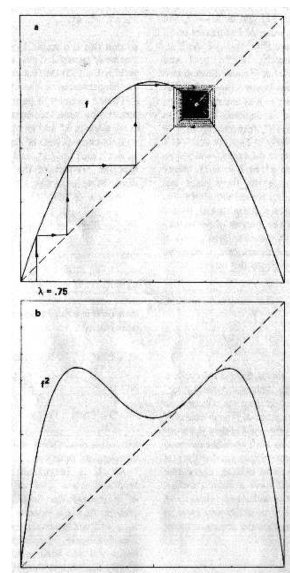
Equation (32) also shows us that $f^2$ has an extremum at the $x_0$ that iterates, under $f$, to $x = 1/2$. These *inverses* of $x = 1/2$ are indicated on the figure for $\mu = 0.7$.

What happens *at the transition*, where $\mu = 3/4$?

At $\mu = 3/4$,

$$f'(x^*) = -1 \implies f^2(x^*) = 1.$$

Therefore $f^2(x^*)$ is tangent to the identity map.




Feigenbaum, Fig. 3, $\mu = 0.75$.

Just *after the transition*, where $\mu > 3/4$, the peaks of $f^2$ increase, the minimum decreases, and

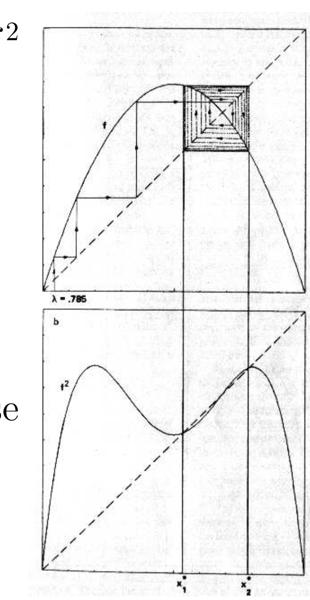
$$|f'(x^*)| > 1 \implies |f^{2'}(x^*)| > 1.$$

$f^2$ develops 2 new fixed points, $x_1^*$ and $x_2^*$, such that

$$x_1^* = f(x_2^*), \quad x_2^* = f(x_1^*).$$

We thus find a cycle of period 2. The cycle is stable because

$$|f^{2'}(x_1^*)| < 1 \quad \text{and} \quad |f^{2'}(x_2^*)| < 1.$$




Feigenbaum, Fig. 4, $\mu = 0.785$.

*Importantly*, the slopes at the fixed points of $f^2$ are *equal*:

$$f^{2\prime}(x_1^*) = f^{2\prime}(x_2^*).$$

This results trivially from equation (32), since the period-2 oscillation gives

$$f^{2\prime}(x_1^*) = f'(x_1^*)\, f'(x_2^*) = f'(x_2^*)\, f'(x_1^*) = f^{2\prime}(x_2^*).$$

In general, if $x_1^*$, $x_2^*$, ..., $x_n^*$ is a cycle of period $n$, such that

$$\begin{aligned} x_{r+1}^* &= f(x_r^*), \qquad r = 1, 2, \ldots, n-1 \\ \text{and} \quad x_1^* &= f(x_n^*) \end{aligned}$$

then each $x_r^*$ is a fixed point of $f^n$:

$$x_r^* = f^n(x_r^*), \qquad r = 1, 2, \ldots, n$$

and the slopes $f^{n\prime}(x_r^*)$ are all equal:

$$f^{n\prime}(x_r^*) = f'(x_1^*)\, f'(x_2^*) \ldots f'(x_n^*), \qquad r = 1, 2, \ldots, n.$$

This slope equality is a crucial observation:

- Just as the sole fixed point $x^*$ of $f(x)$ gives rise to 2 stable fixed points $x_1^*$ and $x_2^*$ of $f^2(x)$ as $\mu$ increases past $\mu = 3/4$, both $x_1^*$ and $x_2^*$ give rise to 2 stable fixed points of $f^4(x) = f^2\big(f^2(x)\big)$ as $\mu$ increases still further.
- The *period doubling* bifurcation derives from the equality of the fixed points—because each fixed point goes unstable for the same $\mu$.

We thus perceive a sequence of bifurcations at increasing values of $\mu$.

At $\mu = \mu_1 = 3/4$, there is a transition to a cycle of period $2^1$.

Eventually, $\mu = \bar{\mu}_1$, where the $2^1$-cycle is *superstable*, i.e.,

$$f^{2\prime}(x_1^*) = f^{2\prime}(x_2^*) = 0.$$

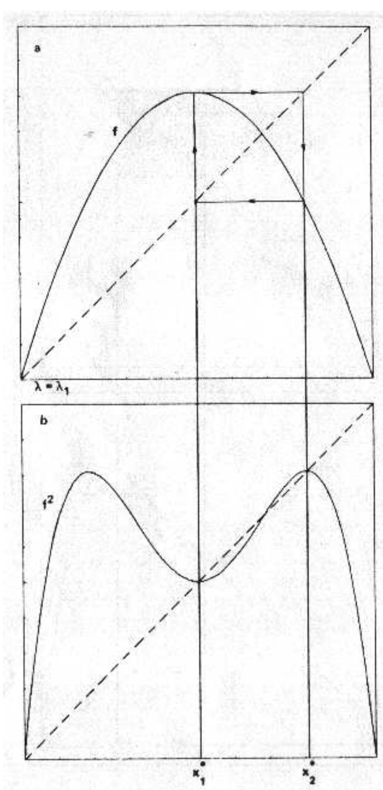
At $\mu = \mu_2$, the 2-cycle bifurcates to a $2^2 = 4$ cycle, and is superstable at $\mu = \bar{\mu}_2$.

We thus perceive the sequence

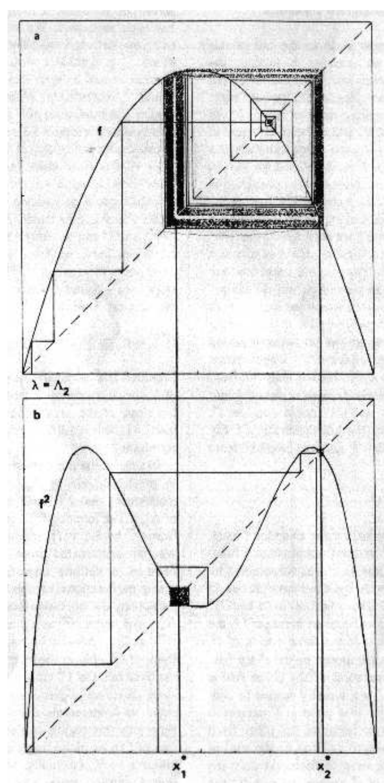
$$\mu_1 < \bar{\mu}_1 < \mu_2 < \bar{\mu}_2 < \mu_3 < \ldots$$

where

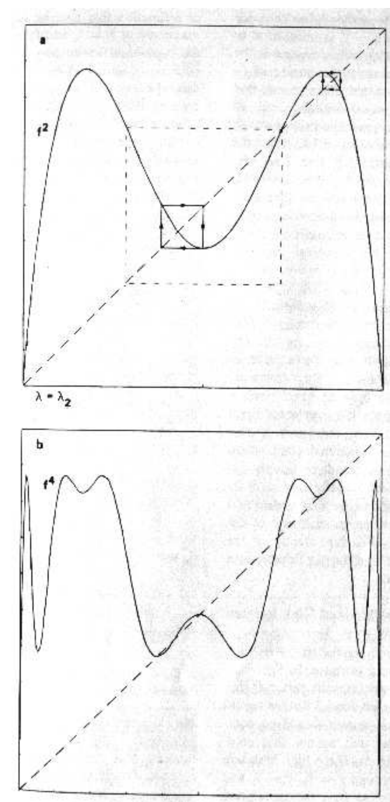
- $\mu_n$ = value of $\mu$ at transition to a cycle of period $2^n$.
- $\bar{\mu}_n$ = value of $\mu$ where $2^n$ cycle is *superstable.*
  Note that one of the superstable fixed points is always at $x = 1/2$.




$\mu = \bar{\mu}_1$, superstable 2-cycle (Feigenbaum, Fig. 5).




$\mu = \mu_2$, transition to period 4 (Feigenbaum, Fig. 6).




$\mu = \bar{\mu}_2$, superstable 4-cycle (Feigenbaum, Fig. 7).

Note that in the case $\mu = \bar{\mu}_2$, we consider the fundamental function to be $f_2$, and its doubling to be $f^4 = f^2(f^2)$.

In general, we are concerned with the functional compositions

$$f^{2^{n+1}} = f^{2^n}\left(f^{2^n}\right)$$

Cycles of period $2^{n+1}$ are always born from the instability of the fixed points of cycles of period $2^n$.

Period doubling occurs *ad infinitum*.

## 16.5 Scaling and universality

The period-doubling bifurcations obey a precise **scaling law**.

Define

$$\begin{aligned} \mu_\infty &= \text{value of } \mu \text{ when the iterates become aperiodic} \\ &= 0.892486\ldots \text{(obtained numerically, for the logistic map).} \end{aligned}$$

There is geometric convergence:

$$\boxed{\mu_\infty - \mu_n \propto \delta^{-n}} \quad \text{for large } n.$$

That is, each increment in $\mu$ from one doubling to the next is reduced in size by a factor of $1/\delta$, such that

$$\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} \to \delta \quad \text{for large } n.$$

The truly amazing result, however, is not the scaling law itself, but that

$$\boxed{\delta = 4.669\ldots}$$

is **universal**, valid for *any* unimodal map with quadratic maximum.

"Unimodal" simply means that the map goes up and then down.

The quadratic nature of the maximum means that in a Taylor expansion of $f(x)$ about $x_{\max}$, i.e.,

$$f(x_{\max} + \varepsilon) = f(x_{\max}) + \varepsilon f'(x_{\max}) + \frac{\varepsilon^2}{2} f''(x_{\max}) + \ldots$$

the leading order nonlinearity is quadratic, i.e.,

$$f''(x_{\max}) \neq 0.$$

(There is also a relatively technical requirement that the Schwartzian derivative of $f$ must be negative over the entire interval (Schuster))

This is an example of **universality**: if *qualitative* properties are present to enable periodic doubling, then *quantitative* properties are *predetermined.*

Thus we expect that *any* system—fluids, populations, oscillators, etc.— whose dynamics can be approximated by a unimodal map would undergo period doubling bifurcations in the same *quantitative* manner.

How may we understand the foundations of this universal behavior?

Recall that

- the $2^n$-cycle generated by $f^{2^n}$ is *superstable* at $\mu = \bar{\mu}_n$;
- superstable fixed points always include $x = 1/2$; and
- all fixed points have the same slope.

Therefore an understanding $f^{2^n}$ near its extremum at $x = 1/2$ will suffice to understand the period-doubling cascade.

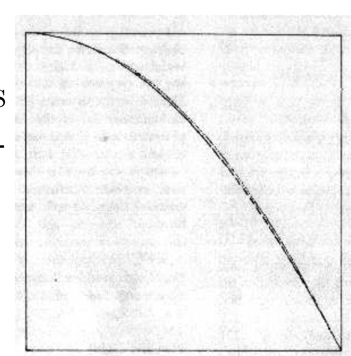
To see how this works, consider $f_{\bar{\mu}_1}(x)$ and $f^2_{\bar{\mu}_2}(x)$ (top of Figures 5 and 7).

The parabolic curve within the dashed (red) square, for $f^2_{\bar{\mu}_2}(x)$ looks just like $f_{\bar{\mu}_1}(x)$, after

- reflection through $x = 1/2$, $y = 1/2$; and
- magnification such that the squares are equal size.

The superposition of the first 5 such functions $(f, f^2, f^4, f^8, f^{16})$ rapidly converges to a single function.

Feigenbaum, Figure 8.



Thus as $n$ increases, a progressively smaller and smaller region near $f$'s maximum becomes relevant—so only the order of the maximum matters.

The composition of doubled functions therefore has a "stable fixed point" in the space of functions, in the infinite period-doubling limit.

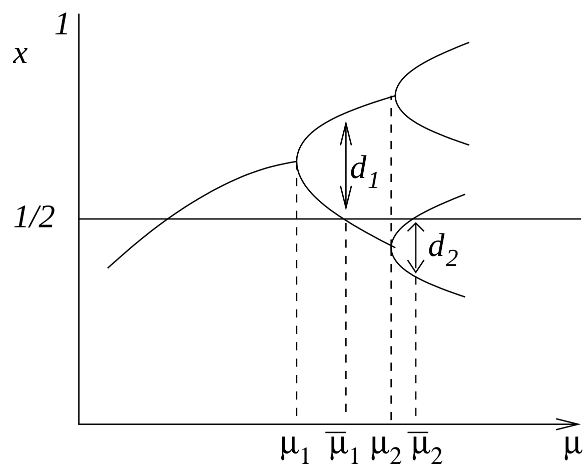
The scale reduction is based only on the functional composition

$$f^{2^{n+1}} = f^{2^n}\left(f^{2^n}\right)$$

which is the *same scale factor* for each $n$ ($n$ large).

This scale factor converges to a constant. What is it?

The bifurcation diagram looks like




Define $d_n$ = distance from $x = 1/2$ to nearest value of $x$ that appears in the superstable $2^n$ cycle (for $\mu = \bar{\mu}_n$).

From one doubling to the next, this separation is reduced by the same scale factor:

$$\frac{d_n}{d_{n+1}} \simeq -\alpha.$$

The negative sign arises because the adjacent fixed point is alternately greater than and less than $x = 1/2$.

We shall see that $\alpha$ is also universal:

$$\alpha = 2.502\ldots$$

### 16.6 Universal limit of iterated rescaled $f$'s

How may we describe the rescaling by the factor $\alpha$?

For $\mu = \bar{\mu}_n$, $d_n$ is the $2^{n-1}$ iterate of $x = 1/2$, i.e.,

$$d_n = f_{\bar{\mu}_n}^{2^{n-1}}(1/2) - 1/2.$$

For simplicity, shift the $x$ axis so that $x = 1/2 \rightarrow x = 0$. Then

$$d_n = f_{\bar{\mu}_n}^{2^{n-1}}(0).$$

The observation that, for $n \gg 1$,

$$\frac{d_n}{d_{n+1}} \simeq -\alpha \implies \lim_{n\to\infty} (-\alpha)^n d_{n+1} \equiv r_n \quad \text{converges.}$$

Stated differently,

$$\lim_{n\to\infty} (-\alpha)^n f_{\bar{\mu}_{n+1}}^{2^n}(0) \quad \text{must exist.}$$

Our superposition of successive plots of $f^{2^n}$ suggests that this result may be generalized to the whole interval.

Thus a rescaling of the $x$-axis describes convergence to the limiting function

$$g_1(x) = \lim_{n\to\infty} (-\alpha)^n f_{\bar{\mu}_{n+1}}^{2^n}\left[\frac{x}{(-\alpha)^n}\right].$$

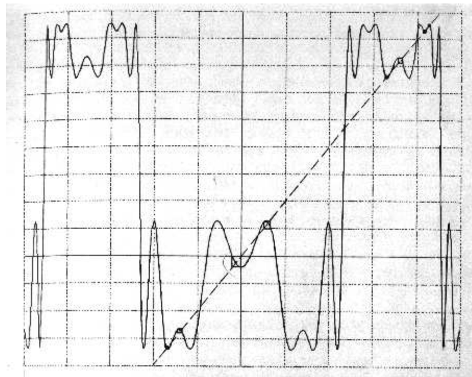
Here the $n$th interated function has its argument rescaled by $1/(-\alpha)^n$ and its value magnified by $(-\alpha)^n$.

The rescaling of the $x$-axis shows explicitly that only the behavior of $f_{\bar{\mu}_{n+1}}^{2^n}$ near $x = 0$ is important.

Thus $g_1$ should be *universal* for all $f$'s with quadratic maximum.

- Figure 5 (top), at $\bar{\mu}_1$, is $g_1$ for $n = 0$.
- Figure 7 (top), at $\bar{\mu}_2$, when rescaled by $\alpha$, is $g_1$ for $n = 1$.

- $g_1$ for $n$ large looks like (Feigenbaum, Figure 9)



The function $g_1$ is the universal limit of interated and rescaled $f$'s. Moreover, the location of the elements of the doubled cycles (the circulation squares) is itself universal.

## 16.7 Doubling operator

We generalize $g_1$ by introducing a family of functions

$$g_i = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{\mu}_{n+i}} \left[\frac{x}{(-\alpha)^n}\right], \qquad i = 0, 1, \ldots \tag{33}$$

Note that

$$\begin{aligned} g_{i-1} &= \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{\mu}_{n+i-1}} \left[\frac{x}{(-\alpha)^n}\right] \\ &= \lim_{n\to\infty} (-\alpha)(-\alpha)^{n-1} f^{2^{n-1+1}}_{\bar{\mu}_{n-1+i}} \left[\frac{1}{(-\alpha)}\ \frac{x}{(-\alpha)^{n-1}}\right] \end{aligned}$$

Set $m = n - 1$. Then

$$f^{2^{n-1+1}} = f^{2^{m+1}} = f^{2^m}\left(f^{2^m}\right)$$

and

$$\begin{aligned} g_{i-1} &= \lim_{m\to\infty}(-\alpha)(-\alpha)^m f^{2^m}_{\bar{\mu}_{m+i}}\left\{\frac{1}{(-\alpha)^m}\underbrace{(-\alpha)^m f^{2^m}_{\bar{\mu}_{m+i}}\left[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^m}\right]}_{g_i(\frac{x}{-\alpha})}\right\} \\ &= -\alpha g_i\left[g_i\left(\frac{x}{-\alpha}\right)\right] \end{aligned}$$

We thus define the *doubling operator* $T$ such that

$$g_{i-1}(x) = T\,g_i(x) = -\alpha g_i\left[g_i\left(\frac{x}{-\alpha}\right)\right]$$

Taking the limit $i \to \infty$, we also define

$$\begin{aligned} g(x) &\equiv \lim_{i\to\infty} g_i(x) \\ &= \lim_{n\to\infty}(-\alpha)^n f^{2^n}_{\bar{\mu}_\infty}\left[\frac{x}{(-\alpha)^n}\right] \end{aligned}$$

We therefore conclude that $g$ is a fixed point of $T$:

$$g(x) = T\,g(x) = -\alpha g\left[g\left(\frac{x}{-\alpha}\right)\right]. \tag{34}$$

$g(x)$ is the limit, as $n \to \infty$, of rescaled $f^{2^n}$, evaluated for $\mu_\infty$.

Whereas $g$ is a fixed point of $T$, $Tg_i$, where $i$ is finite, interates away from $g$.

Thus $g$ is an *unstable* fixed point of $T$.

## 16.8 Computation of $\alpha$

To determine $\alpha$, first write

$$g(0) = -\alpha g\left[g(0)\right].$$

We must set a scale, and therefore set

$$g(0) = 1 \implies g(1) = -1/\alpha.$$

There is no general theory that can solve equation (34) for $g$.

We can however obtain a unique solution for $\alpha$ by specifying the nature (order) of $g$'s maximum (at zero) and requiring that $g(x)$ be smooth.

We thus assume a quadratic maximum, and use the short power law expansion

$$g(x) = 1 + bx^2.$$

Then, from equation (34),

$$\begin{aligned} g(x) = 1 + bx^2 &= -\alpha g\left(1 + \frac{bx^2}{\alpha^2}\right) \\ &= -\alpha\left[1 + b\left(1 + \frac{bx^2}{\alpha^2}\right)^2\right] \\ &= -\alpha(1+b) - \frac{2b^2}{\alpha}x^2 + O(x^4) \end{aligned}$$

Equating terms,

$$\alpha = \frac{-1}{1+b}, \qquad \alpha = -2b$$

which yields,

$$b = \frac{-2 \pm \sqrt{12}}{4} \simeq -1.366 \quad \text{(neg root for max at } x = 0\text{)}$$

and therefore

$$\alpha \simeq 2.73,$$

which is within 10% of Feigenbaum's $\alpha = 2.5028\ldots$, obtained by using terms up to $x^{14}$.

## 16.9 Linearized doubling operator

We shall see that $\delta$ determines how quickly we move away from $g$ under application of the doubling operator $T$.

In essence, we shall calculate the eigenvalue that corresponds to instability of an unstable fixed point.

Thus our first task will be to linearize the doubling operator $T$. $\delta$ will then turn out to be one of its eigenvalues.

We seek to predict the scaling law

$$\bar{\mu}_n - \bar{\mu}_\infty \propto \delta^{-n},$$

now expressed in terms of $\bar{\mu}_i$ rather than $\mu_i$.

We first expand $f_{\bar{\mu}}(x)$ around $f_{\bar{\mu}_\infty}(x)$:

$$f_{\bar{\mu}}(x) \simeq f_{\bar{\mu}_\infty}(x) + (\bar{\mu} - \bar{\mu}_\infty)\, \delta f(x),$$

where the incremental change in function space is given by

$$\delta f(x) = \left. \frac{\partial f_{\bar{\mu}}(x)}{\partial \bar{\mu}} \right|_{\bar{\mu}_\infty}$$

Now apply the doubling operator $T$ to $f_{\bar{\mu}}$ and linearize with respect to $\delta f$:

$$\begin{aligned} T f_{\bar{\mu}} &= -\alpha f_{\bar{\mu}} \left[ f_{\bar{\mu}} \left( \frac{x}{-\alpha} \right) \right] \\ &\simeq -\alpha \left[ f_{\bar{\mu}_\infty} + (\bar{\mu} - \bar{\mu}_\infty)\, \delta f \right] \circ \left[ f_{\bar{\mu}_\infty} \left( \frac{x}{-\alpha} \right) + (\bar{\mu} - \bar{\mu}_\infty)\, \delta f \left( \frac{x}{-\alpha} \right) \right] \\ &= T f_{\bar{\mu}_\infty} + (\bar{\mu} - \bar{\mu}_\infty) L_{f_{\bar{\mu}_\infty}} \delta f + O(\delta f^2) \end{aligned}$$

where $L_f$ is the *linearized doubling operator* defined by

$$L_f\, \delta f = -\alpha \left\{ f' \left[ f \left( \frac{x}{-\alpha} \right) \right] \delta f \left( \frac{x}{-\alpha} \right) + \delta f \left[ f \left( \frac{x}{-\alpha} \right) \right] \right\}. \qquad (35)$$

The first term on the RHS derives from an expansion like $g[f(x) + \delta f(x)] \simeq g[f(x)] + g'[f(x)]\delta f(x)$.

A second application of the doubling operator yields

$$T\big(T(f_{\bar{\mu}})\big) = T^2 f_{\bar{\mu}_\infty} + (\bar{\mu} - \bar{\mu}_\infty) L_{T f_{\bar{\mu}_\infty}} L_{f_{\bar{\mu}_\infty}} \delta f + O\big((\delta f)^2\big).$$

Therefore $n$ applications of the doubling operator produce

$$T^n f_{\bar{\mu}} = T^n f_{\bar{\mu}_\infty} + (\bar{\mu} - \bar{\mu}_\infty)\, L_{T^{n-1} f_{\bar{\mu}_\infty}} \cdots L_{f_{\bar{\mu}_\infty}} \delta f + O\big((\delta f)^2\big). \qquad (36)$$

For $\bar{\mu} = \bar{\mu}_\infty$, we expect convergence to the fixed point $g(x)$:

$$T^n f_{\bar{\mu}_\infty} = (-\alpha)^n f_{\bar{\mu}_\infty}^{2^n}\left[\frac{x}{(-\alpha)^n}\right] \simeq g(x), \qquad n \gg 1.$$

Substituting $g(x)$ into equation (36) and assuming, similarly, that $L_{Tf_{\bar{\mu}_\infty}} \simeq L_g$,

$$T^n f_{\bar{\mu}}(x) \simeq g(x) + (\bar{\mu} - \bar{\mu}_\infty)\, L_g^n\, \delta f(x), \qquad n \gg 1. \tag{37}$$

We simplify by introducing the eigenfunctions $\phi_\nu$ and eigenvalues $\lambda_\nu$ of $L_g$:

$$L_g \phi_\nu = \lambda_\nu \phi_\nu, \qquad \nu = 1, 2, \ldots$$

Write $\delta f$ as a weighted sum of $\phi_\nu$:

$$\delta f = \sum_\nu c_\nu \phi_\nu$$

Thus $n$ applications of the linear operator $L_g$ may be written as

$$L_g^n\, \delta f = \sum_\nu \lambda_\nu^n c_\nu \phi_\nu.$$

Now *assume* that only one of $\lambda_\nu$ is greater than one:

$$\lambda_1 > 1, \quad \lambda_\nu < 1 \text{ for } \nu \neq 1.$$

(This conjecture, part of the original theory, was later proven.)

Thus for large $n$, $\lambda_1$ dominates the sum, yielding the approximation

$$L_g^n\, \delta f \simeq \lambda_1^n c_1 \phi_1, \qquad n \gg 1.$$

We can now simplify equation (36):

$$T^n f_{\bar{\mu}}(x) = g(x) + (\bar{\mu} - \bar{\mu}_\infty)\ \cdot \delta^n \cdot a \cdot h(x), \qquad n \gg 1$$

where

$$\delta = \lambda_1, \quad a = c_1, \quad \text{and} \quad h(x) = \phi_1.$$

Now note that when $x = 0$ and $\bar{\mu} = \bar{\mu}_n$,

$$T^n f_{\bar{\mu}_n}(0) = g(0) + (\bar{\mu}_n - \bar{\mu}_\infty) \cdot \delta^n \cdot a \cdot h(0).$$

Recall that $x = 0$ is a fixed point of $f^{2^n}_{\bar{\mu}_n}$ (due to the $x$-shift). Therefore

$$T^n f_{\bar{\mu}_n}(0) = (-\alpha)^n f^{2^n}_{\bar{\mu}_n}(0) = 0.$$

Recall also that we have scaled $g$ such that $g(0) = 1$. We thus obtain the Feigenbaum scaling law:

$$\lim_{n\to\infty} (\bar{\mu}_n - \bar{\mu}_\infty)\, \delta^n = \frac{-1}{a \cdot h(0)} = \text{constant!}$$

### 16.10 Computation of $\delta$

Recall that $\delta$ is the eigenvalue that corresponds to the eigenfunction $h(x)$.

Then applying the linearized doubling operator (35) to $h(x)$ yields

$$\begin{aligned} L_g h(x) &= -\alpha \left\{ g' \left[ g\left(\frac{x}{-\alpha}\right)\right] h\left(\frac{x}{-\alpha}\right) + h\left[ g\left(\frac{x}{-\alpha}\right)\right]\right\} \\ &= \delta \cdot h(x). \end{aligned}$$

Now approximate $h(x)$ by $h(0)$, the first term in a Taylor expansion about $x = 0$.

Seting $x = 0$, we obtain

$$-\alpha \left\{ g'\left[g(0)\right] h(0) + h\left[g(0)\right]\right\} = \delta \cdot h(0).$$

Note that the approximation

$$h(x) \simeq h(0) \implies h[g(0)] = h(1) \simeq h(0).$$

Thus $h(0)$ cancels in each term and, recalling that $g(0) = 1$,

$$-\alpha \left[g'(1) + 1\right] = \delta. \tag{38}$$

To obtain $g'(1)$, differentiate $g(x)$ twice:

$$\begin{aligned}
g(x) &= -\alpha g\left[g\left(\frac{-x}{\alpha}\right)\right] \\
g'(x) &= -\alpha\left\{g'\left[g\left(\frac{-x}{\alpha}\right)\right]\cdot\left(\frac{-1}{\alpha}\right)g'\left(\frac{-x}{\alpha}\right)\right\} \\
g''(x) &= \frac{-1}{\alpha}\left\{g''\left[g\left(\frac{x}{-\alpha}\right)\right]\left[g'\left(\frac{-x}{\alpha}\right)\right]^2 + g'\left[g\left(\frac{-x}{\alpha}\right)\right]g''\left(\frac{-x}{\alpha}\right)\right\}
\end{aligned}$$

Substitute $x = 0$. Note that

$$g'(0) = 0 \quad \text{and} \quad g''(0) \neq 0$$

because we have assumed a quadratic maximum at $x = 0$. Then

$$g''(0) = \frac{-1}{\alpha}\left[g'(1)g''(0)\right].$$

Therefore

$$g'(1) = -\alpha.$$

Substituting into equation (38), we obtain

$$\boxed{\delta = \alpha^2 - \alpha}.$$

This result derives from the crude approximation $h(0) = h(1)$. Better approximations yield greater accuracy (Feigenbaum, 1979).)

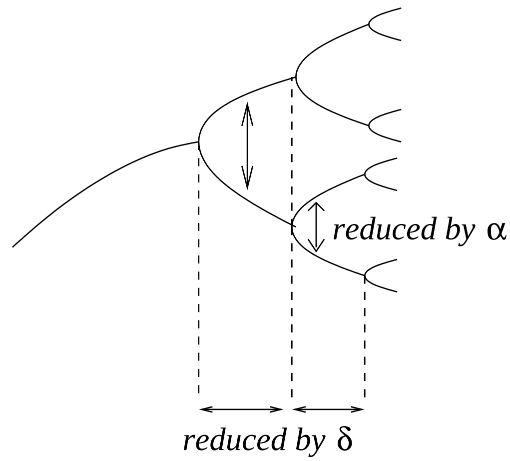
Recall that we previously estimated $\alpha \simeq 2.73$. Substituting that above, we obtain

$$\delta \simeq 4.72,$$

which is within 1% of the exact value $\delta = 4.669\ldots$.

## 16.11 Comparison to experiments

We have established the universality of $\alpha$ and $\delta$:

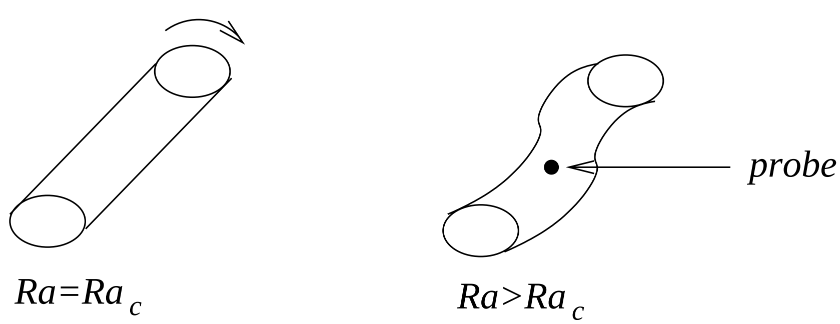



These *quantitative* results hold if a *qualitative* condition—the maximum of $f$ must be locally quadratic—holds.

At first glance this result may appear to pertain only to mathematical maps. However we have seen that more complicated systems can also behave as if they depend on only a few degrees of freedom. Due to dissipation, one may expect that a one-dimensional map is contained, so to speak, within them.

The first experimental verification of this idea was due to Libchaber, in a Rayleigh-Bénard system.

As the Rayleigh number increases beyond its critical value, a single convection roll develops an oscillatory wave:




A probe of temperature $X(t)$ is then oscillatory with frequency $f_1$ and period $1/f_1$.

Successive increases of Ra then yield a sequence of period doubling bifurcations at Rayleigh numbers

$$\mathrm{Ra}_1 < \mathrm{Ra}_2 < \mathrm{Ra}_3 < \dots$$

The experimental results are shown in

$\boxed{\text{BPV, Figure VIII.13a and VIII.13b}}$.

Identifying Ra with the control parameter $\mu$ in Feigenbaum's theory, Libchaber found

$$\delta \simeq 4.4$$

which is amazingly close to Feigenbaum's prediction, $\delta = 4.669\ldots$.

Such is the power of scaling and universality!