Problem set 8

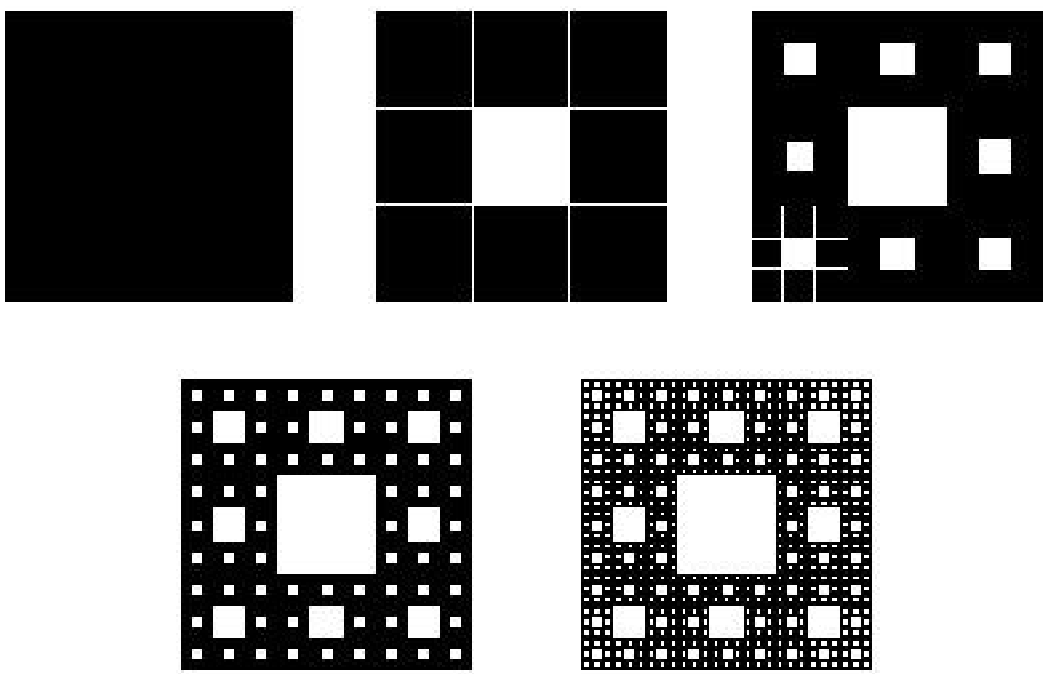
Suggested reading: Strogatz, Chapter 12; Baker and Gollub, Chapters 5,6.

We have provided a number of programs to help you do this problem set. They are described in the file ps8supp.pdf.

1. Fractal Dimensions

   (a) Construct a fractal that is similar to the Cantor set, but instead remove the middle 1/2 from each previous section. Show analytically that its fractal dimension is 1/2.

   (b) Calculate the fractal dimension of the Sierpinski carpet shown in the pictures.



2. The Hénon map is given by

$$\begin{aligned} X_{n+1} &= 1 + Y_n - \alpha X_n^2 \\ Y_{n+1} &= \beta X_n \end{aligned}$$

   (a) What are the fixed points? (Show analytically.)

   (b) Find (analytically) the value α_c such that for $\alpha > \alpha_c$ the fixed points are unstable.

   (c) Choose $\beta = 0.3$ and $\alpha = 1.4$. After some experimentation, initialize the iterates close to (but not exactly on) the unstable fixed point which lies close to the attractor. From your answers to a) and b) and some additional analysis, predict the slope of the attractor near the fixed points and then confirm your prediction from the numerical results.

   (d) Estimate the largest Lyaponov exponent for $\beta = 0.2$ and $\alpha = 1.3$. What is the smaller of the two Lyaponov exponents? How do these compare to crude predictions you are able to make in the vicinity of the fixed points? Can you qualitatively explain any differences?

3. The Lorenz model is given by

$$\begin{aligned} \dot{X} &= PY - PX \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ \end{aligned}$$

   where P is the Prandtl number, b is a constant, and r is the ratio of the Rayleigh number to the critical Rayleigh number. Take $r = 28$, $P = 10$ and $b = 8/3$. Using the time series of only one coordinate (X, Y, or Z) and embedding it in higher dimensions (as embodied in our program `fracdim` and described in ps8supp.pdf), calculate the dimension of the Lorenz attractor.