PROBLEM SET 6

In this problem set, you will derive the linear stability relations for a special case of Rayleigh-Bénard convection.

The equations describing thermal convection are

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}p + \theta\hat{z} + \nabla^2\vec{v} \tag{1}$$

$$\frac{\partial \theta}{\partial t} + \vec{v}\cdot\vec{\nabla}\theta = \nabla^2\theta + \mathrm{Ra}\ w \tag{2}$$

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{3}$$

where $\hat{z}$ is the vertical unit vector, $\vec{v}$ the velocity, w the vertical velocity, Pr the Prandtl number, Ra the Rayleigh number, and θ the deviation of the temperature from the value it would have in the absense of convection. By taking the curl of the first equation twice and neglecting nonlinear terms from both, we obtain the following set of linear equations for the vertical velocity valid for small values of w and θ:

$$\frac{1}{\mathrm{Pr}}\nabla^2\left(\frac{\partial w}{\partial t}\right) = \nabla_1^2\theta + \nabla^4 w \tag{4}$$

$$\frac{\partial \theta}{\partial t} = \nabla^2\theta + \mathrm{Ra}\ w. \tag{5}$$

Here $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, the horizontal Laplacian $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $\nabla^4 = (\nabla^2)^2$. The fluid is confined between the planes $z = 0$ and $z = 1$, so $w = 0$ and $\theta = 0$ at these boundaries. Convection experiments usually have rigid boundaries, but for mathematical simplicity we consider free boundaries, at which tangential stress vanishes. This yields the additional boundary condition $\frac{\partial^2 w}{\partial z^2} = 0$ at $z = 0$ and $z = 1$. To derive the conditions under which a purely conductive system is unstable, we consider small sinusoidal perturbations of the velocity and temperature fields:

$$w = A(t)\cos(k_x x + k_y y)\sin \pi z \tag{6}$$

$$\theta = B(t)\cos(k_x x + k_y y)\sin \pi z. \tag{7}$$

Here k_x and k_y are the horizontal wavenumbers (ie. $k_x = 2\pi/\lambda_x$, where λ_x is the wavelength of the perturbation in the x direction).

1. Verify that the sinusoidal perturbations, given in equations (6) and (7), satisfy the boundary conditions.

2. Obtain a set of linear ode's describing the evolution of $\dot{A}(t)$ and $\dot{B}(t)$ by replacing equations (6) and (7) into the linear evolution equations (4) and (5). Determine the fixed point in the plane of A and B, and show that it is unstable when

$$\mathrm{Ra} > \mathrm{Ra}^* = \frac{(k^2 + \pi^2)^3}{k^2},$$

where $k^2 = k_x^2 + k_y^2$. Sketch a graph of $\mathrm{Ra}^*(k)$. What physical state of the system does the fixed point correspond to?

3. What is the value of the minimum, called Ra_c, of the function $\mathrm{Ra}^*(k)$? At what value of k does it occur, and how does the wavelength corresponding to this critical wavenumber compare to the distance between the free boundaries of the fluid? For comparison, $\mathrm{Ra}_c = 1707$ for rigid-rigid boundaries and 1100 for rigid-free boundaries.