

Kinematic transformation of mechanical behavior

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Generalized coordinates are fundamental

If we assume that a linkage may accurately be described as a collection of linked rigid bodies, their generalized coordinates are a fundamental requirement for any model of mechanical behavior. However, to describe functional behavior it will often be necessary to express mechanical behavior as it appears in a different frame of reference, for example, the Cartesian coordinates of the end-point. Knowledge of the geometry relating the two frames is sufficient to transform mechanical behavior, but care is required.

Transformation to end-point coordinates

Express the kinematic equations relating end-point coordinates to generalized coordinates.

$$\mathbf{x} = \tilde{\mathbf{L}}(\boldsymbol{\theta})$$

The relations between incremental displacements, velocities, forces and momenta are obtained by differentiating and using power continuity.

$$d\mathbf{x} = \frac{\partial \tilde{\mathbf{L}}}{\partial \boldsymbol{\theta}} d\boldsymbol{\theta} = \tilde{\mathbf{J}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$\mathbf{v} = \tilde{\mathbf{J}}(\boldsymbol{\theta}) \boldsymbol{\omega}$$

$$\boldsymbol{\tau} = \tilde{\mathbf{J}}(\boldsymbol{\theta})' \mathbf{f}$$

$$\boldsymbol{\eta} = \tilde{\mathbf{J}}(\boldsymbol{\theta})' \mathbf{p}$$

Note that transformation of motion variables (displacement, velocity) is always well-defined *from* generalized coordinates to any other coordinates. Conversely, the transformation of force variables (force, momentum) is always well-defined *to* generalized coordinates from any other coordinates. In general, the inverse of these transformations may not be well-defined.

Inertia

Inertia relates momentum and velocity.

$$\boldsymbol{\eta} = \mathbf{I}(\boldsymbol{\theta}) \boldsymbol{\omega}$$

If every rigid body in the linkage has non-zero mass, the inertia tensor is positive-definite and its inverse exists. To define stored kinetic energy, the inverse of this relation is required, the causally-preferred form for an inertia.

$$\boldsymbol{\omega} = \mathbf{I}(\boldsymbol{\theta})^{-1} \boldsymbol{\eta}$$

Transformation to end-point coordinates is a straightforward matter of substitution.

$$\mathbf{v} = \mathbf{M}_x^{-1} \mathbf{p}$$

$$\mathbf{v} = \tilde{\mathbf{J}}(\boldsymbol{\theta}) \mathbf{I}(\boldsymbol{\theta})^{-1} \tilde{\mathbf{J}}(\boldsymbol{\theta})' \mathbf{p}$$

$$\mathbf{M}_x(\boldsymbol{\theta})^{-1} = \tilde{\mathbf{J}}(\boldsymbol{\theta})\mathbf{I}(\boldsymbol{\theta})^{-1}\tilde{\mathbf{J}}(\boldsymbol{\theta})^t$$

The inverse inertia at the end-point can always be defined. Due to linkage geometry it varies with end-point position and linkage configuration. In generalized coordinates, inertia (and hence inverse inertia) is positive definite. In end-point coordinates, inverse inertia is only positive *semi*-definite (strictly non-negative); in some configurations it may lose rank. In those cases the end-point inertia approaches infinity—force may be applied but no motion results. This is an argument for considering an inertial mechanism fundamentally to be an admittance, not an impedance.

Friction

Energy dissipation (friction) is characterized by a relation between force and velocity. In end-point coordinates a nonlinear form is as follows.

$$\mathbf{f} = \boldsymbol{\Pi}_x(\mathbf{v})$$

where $\boldsymbol{\Pi}_x(\cdot)$ denotes a function with $\mathbf{f}^t \mathbf{v} \geq 0$ ¹ and $\mathbf{f} = \mathbf{0}$ at $\mathbf{v} = \mathbf{0}$. The corresponding force-velocity relation in generalized coordinates is always well-defined and is obtained by substitution.

$$\boldsymbol{\tau} = \tilde{\mathbf{J}}(\boldsymbol{\theta})^t \boldsymbol{\Pi}_x(\tilde{\mathbf{J}}(\boldsymbol{\theta})\boldsymbol{\omega}) = \boldsymbol{\Pi}(\boldsymbol{\theta}, \boldsymbol{\omega})$$

Note that the linkage kinematics introduces a dependence on configuration.

Damping

Damping is defined as the gradient of force with respect to velocity. In end-point coordinates:

$$\mathbf{B}_x = \frac{\partial \boldsymbol{\Pi}_x}{\partial \mathbf{v}}$$

In generalized coordinates:

$$\mathbf{B}_\theta(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\omega}}$$

$$\mathbf{B}_\theta(\boldsymbol{\theta}) = \tilde{\mathbf{J}}(\boldsymbol{\theta})^t \mathbf{B}_x \tilde{\mathbf{J}}(\boldsymbol{\theta})$$

From this we see that it is always possible to find the generalized-coordinate damping corresponding to a specified end-point damping. However, this generalized-coordinate damping may not have full rank and its inverse may not be defined. For example, if the dimension of configuration space (the number of generalized coordinates) exceeds the dimension of end-point space (the number of end-point coordinates) the generalized-coordinate damping corresponding to *any* end-point damping has rank less than the dimension of configuration space. In the limit of very large damping, the end-point is effectively immobilized but the linkage, which has more degrees of freedom than the end-point, remains free to move².

¹ This assumes a sign convention with power positive into any passive element.

Generalized-coordinate damping in end-point coordinates

To transform a full-rank generalized-coordinate damping to end-point coordinates we express it in admittance form.

$$\boldsymbol{\omega} = \mathbf{B}_\theta^{-1} \boldsymbol{\tau}$$

The corresponding end-point damping is obtained by substitution.

$$\mathbf{v} = \tilde{\mathbf{J}}(\boldsymbol{\theta}) \mathbf{B}_\theta^{-1} \tilde{\mathbf{J}}(\boldsymbol{\theta})^t \mathbf{f}$$

$$\boxed{\mathbf{B}_x^{-1} = \tilde{\mathbf{J}}(\boldsymbol{\theta}) \mathbf{B}_\theta^{-1} \tilde{\mathbf{J}}(\boldsymbol{\theta})^t}$$

Again we see that even if the generalized-coordinate damping is independent of configuration, the end-point damping will, in general, vary with configuration.

Static impedance

Damping and inertia (two of an infinite set of admittance parameters) transform between coordinates in the same way. However, *static* impedance or admittance parameters transform differently.

A general form for static impedance (i.e., force-position behavior) in generalized coordinates is

$$\boldsymbol{\tau} = \boldsymbol{\Phi}(\boldsymbol{\theta}, \boldsymbol{\theta}_o)$$

where $\boldsymbol{\Phi}(\cdot)$ denotes a function and $\boldsymbol{\theta}_o$ denotes a zero of the function, a configuration at which the (generalized) force is zero. Causal arguments³ indicate this impedance form must always exist though it may not always have an inverse (admittance) form. Consider the differential of this function.

$$d\boldsymbol{\tau} = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\theta}} d\boldsymbol{\theta} + \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\theta}_o} d\boldsymbol{\theta}_o$$

In the following, we assume the zero configuration does not change: $d\boldsymbol{\theta}_o = 0$. Stiffness is defined as the gradient of force with respect to displacement.

$$\mathbf{K}_\theta(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\theta}}$$

Note that stiffness is (in general) a function of configuration. As the static impedance form always exists, stiffness may always be defined though it may vanish or lose rank at some configurations. When the stiffness has full rank, compliance is defined as the inverse of stiffness.

$$\mathbf{C}_\theta(\boldsymbol{\theta}) = \mathbf{K}_\theta^{-1}$$

Kinematic stiffness

Differentiate the force transformation to find the relation between stiffness in end-point and generalized coordinates.

³ Hogan, N. (1985) Biol. Cyb. 52:315-331.

$$d\boldsymbol{\tau} = \tilde{\mathbf{J}}(\boldsymbol{\theta})^t d\mathbf{f} + \frac{\partial \tilde{\mathbf{J}}(\boldsymbol{\theta})^t}{\partial \boldsymbol{\theta}} \mathbf{f} d\boldsymbol{\theta}$$

This reveals the fundamental difficulty: quite aside from any elastic behavior, any non-zero force acts through configuration-dependent moment arms⁴ to produce an apparent stiffness. Denote the apparent *kinematic stiffness* by Γ .

$$\Gamma(\boldsymbol{\theta}) = \frac{\partial \tilde{\mathbf{J}}(\boldsymbol{\theta})^t}{\partial \boldsymbol{\theta}} \mathbf{f}$$

Note that the kinematic stiffness vanishes for zero force. However, if non-zero it may be positive or negative (stabilizing or destabilizing) depending on the orientation of the force.

End-point stiffness in generalized coordinates

Static impedance in end-point coordinates may be expressed as follows

$$\mathbf{f} = \Phi_x(\mathbf{x}, \mathbf{x}_o)$$

where \mathbf{x}_o denotes a position at which force is zero. The corresponding static impedance in generalized coordinates is always well-defined and is obtained by substitution.

$$\boldsymbol{\tau} = \tilde{\mathbf{J}}^t \mathbf{f} = \tilde{\mathbf{J}}^t \Phi_x(\tilde{\mathbf{L}}(\boldsymbol{\theta}), \tilde{\mathbf{L}}(\boldsymbol{\theta}_o)) = \Phi(\boldsymbol{\theta}, \boldsymbol{\theta}_o)$$

End-point stiffness:

$$\mathbf{K}_x = \frac{\partial \Phi_x}{\partial \mathbf{x}}$$

Stiffness in generalized coordinates:

$$\mathbf{K}_\theta = \frac{\partial \Phi}{\partial \boldsymbol{\theta}} = \frac{\partial \tilde{\mathbf{J}}^t}{\partial \boldsymbol{\theta}} \mathbf{f} + \tilde{\mathbf{J}}^t \frac{\partial \Phi_x}{\partial \mathbf{x}} \frac{\partial \tilde{\mathbf{L}}}{\partial \boldsymbol{\theta}}$$

$$\boxed{\mathbf{K}_\theta = \Gamma + \tilde{\mathbf{J}}^t \mathbf{K}_x \tilde{\mathbf{J}}}$$

If the stiffness is evaluated at zero net end-point force, the kinematic stiffness vanishes.

$$\mathbf{K}_\theta \Big|_{\mathbf{f}=0} = \tilde{\mathbf{J}}^t \mathbf{K}_x \tilde{\mathbf{J}}$$

As with damping, we see that it is always possible to find the generalized-coordinate stiffness corresponding to a specified end-point stiffness. However, this generalized-coordinate stiffness may not have full rank and its inverse (generalized-coordinate compliance) may not be defined. If the dimension of configuration space exceeds the dimension of end-point space the generalized-coordinate stiffness corresponding to *any* end-point stiffness has rank less than the dimension of configuration space. In the limit of very large end-point stiffness, the end-point is effectively immobilized but the linkage, which has more degrees of freedom than the end-point, remains free to move.

⁴ These moment arms are defined by the columns of the Jacobian.

Generalized-coordinate compliance in end-point coordinates

To transform a full-rank generalized-coordinate stiffness to end-point coordinates we express it in compliance form.

$$d\theta = C_\theta d\tau = C_\theta (\tilde{\mathbf{J}}^t d\mathbf{f} + \Gamma d\theta)$$

$$(\mathbf{1} - C_\theta \Gamma) d\theta = C_\theta \tilde{\mathbf{J}}^t d\mathbf{f}$$

$$\mathbf{K}_\theta (\mathbf{1} - C_\theta \Gamma) d\theta = \mathbf{K}_\theta C_\theta \tilde{\mathbf{J}}^t d\mathbf{f}$$

$$(\mathbf{K}_\theta - \Gamma) d\theta = \tilde{\mathbf{J}}^t d\mathbf{f}$$

Assuming the inverse exists, this expression may be solved for $d\theta$ and $d\mathbf{x}$.

$$d\theta = (\mathbf{K}_\theta - \Gamma)^{-1} \tilde{\mathbf{J}}^t d\mathbf{f}$$

$$d\mathbf{x} = \tilde{\mathbf{J}} d\theta = \tilde{\mathbf{J}} (\mathbf{K}_\theta - \Gamma)^{-1} \tilde{\mathbf{J}}^t d\mathbf{f}$$

$$d\mathbf{x} = C_x d\mathbf{f}$$

$$\boxed{C_x = \tilde{\mathbf{J}} (\mathbf{K}_\theta - \Gamma)^{-1} \tilde{\mathbf{J}}^t}$$

If the compliance is evaluated at zero end-point force, the kinematic stiffness vanishes and the expression simplifies.

$$C_x|_{\mathbf{f}=0} = \tilde{\mathbf{J}} C_\theta \tilde{\mathbf{J}}^t$$

If the Jacobian may be inverted these expressions are equivalent to those found above.

$$\tilde{\mathbf{J}}^{-1} C_x \tilde{\mathbf{J}}^{-t} = (\mathbf{K}_\theta - \Gamma)^{-1} \text{ hence } \mathbf{K}_\theta = \Gamma + \tilde{\mathbf{J}}^t C_x \tilde{\mathbf{J}}$$

In general the inverse Jacobian may not exist but, provided the difference between generalized-coordinate stiffness and kinematic stiffness can be inverted, end-point compliance can always be defined. However, it may not always have full rank, and hence it may not always be possible to define end-point stiffness.

Remark

Note the profound influence of linkage kinematics on *all* components of impedance. For example, at singular configurations (at which the Jacobian loses rank) *all* of the admittance parameters (compliance, inverse damping, inverse inertia, etc.) lose rank and all corresponding impedance parameters (stiffness, damping, inertia, etc.) become undefined (i.e., approach infinity in at least one direction). Thus *pose* is one of the most important ways to modulate interactive behavior.