2.002 MECHANICS & MATERIALS II

STRESS AND STRAIN, CONTINUED

Tensor

In mathematics, a linear map $\mathbf{S}$ which assigns to each vector $\mathbf{a}$, another vector

$$\mathbf{b} = \mathbf{S}\mathbf{a},$$

is called a **tensor**.

$\mathbf{S}$ linear means

$$\mathbf{S}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{S}\mathbf{a} + \beta\mathbf{S}\mathbf{b}.$$

The components S_{ij} of a tensor $\mathbf{S}$ with respect to an orthonormal basis $\{\mathbf{e}_i | i = 1,3\}$ are defined by the scalar product of the vector $\left(\mathbf{S}\mathbf{e}_j\right)$ with $\mathbf{e}_i$:

$$\boxed{S_{ij} = \mathbf{e}_i \cdot \left(\mathbf{S}\mathbf{e}_j\right)}.$$

With this definition, $\mathbf{b} = \mathbf{S}\mathbf{a}$ is equivalent to

$$b_i = \sum_j S_{ij} a_j .$$

We shall write $[\mathbf{S}]$ for the matrix of the components S_{ij} of a tensor $\mathbf{S}$:

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} .$$

A tensor $\mathbf{S}$ is **symmetric** if

$$\mathbf{S} = \mathbf{S}^T, \quad \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix},$$

and **skew** if

$$\mathbf{S} = -\mathbf{S}^T, \quad \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = - \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}.$$

Note that the **transpose** of $\mathbf{S}$ is denoted by $\mathbf{S}^T$. The transpose of the matrix of components of $\mathbf{S}$ is obtained by interchanging the rows and columns of the matrix.

In our study of mechanics and materials, both the stress, $\boldsymbol{\sigma}$, and strain, ϵ, are **symmetric tensors** with components

$$\sigma_{ij} = \mathbf{e}_i \cdot \left(\boldsymbol{\sigma}\mathbf{e}_j\right), \quad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix},$$

and

$$\epsilon_{ij} = \mathbf{e}_i \cdot \left(\boldsymbol{\epsilon}\mathbf{e}_j\right), \quad [\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}.$$

Returning to our discussion of strain, the matrix of the components ϵ_{ij} of the infinitesimal strain tensor $\boldsymbol{\epsilon}$ is

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}.$$

Of the nine components listed above, only six are independent since the strain is symmetric:

$$\epsilon_{12} = \epsilon_{21}, \quad \epsilon_{13} = \epsilon_{31}, \quad \epsilon_{23} = \epsilon_{32}.$$

Written out in full, the components

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3},$$

are called the **normal strain components**, and the components

$$\epsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \epsilon_{21},$$
$$\epsilon_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \epsilon_{31},$$
$$\epsilon_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] = \epsilon_{32},$$

are called the **tensorial shear strain components**.

The **engineering shear strain components** are defined as

$$\gamma_{12} = \left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \gamma_{21},$$
$$\gamma_{13} = \left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \gamma_{31},$$
$$\gamma_{23} = \left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] = \gamma_{32}.$$

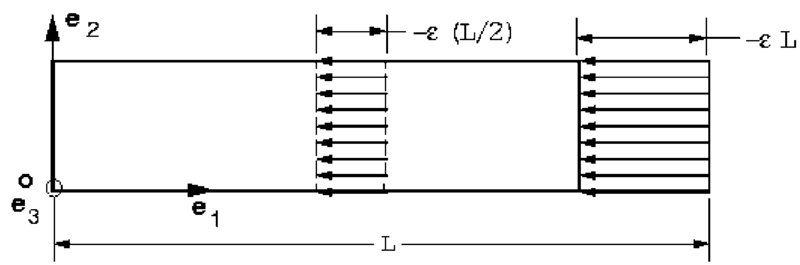
Note that the engineering shear strains are defined as twice the magnitude of the tensorial shear strains. Beware of this difference; it is often a source of error.

Some Simple States of Strain

Uniaxial Compression in the e_1 direction:

$$\mathbf{u} = -\epsilon\, x_1\, \mathrm{e}_1, \quad u_1 = -\epsilon x_1, \quad u_2 = u_3 = 0, \quad \epsilon = \text{const}$$

$$[\epsilon] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



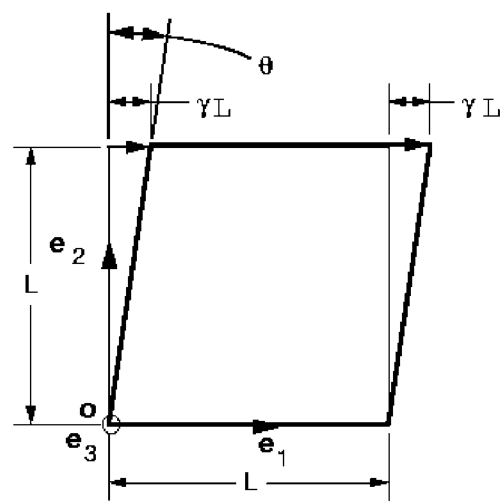
Bodies which experience constant strain are said to be **homogeneously deformed**.

Simple Shear with respect to $(\mathbf{e}_1, \mathbf{e}_2)$:

$$\mathbf{u} = \gamma \, x_2 \, \mathbf{e}_1, \quad u_1 = \gamma x_2, \quad u_2 = u_3 = 0, \quad \gamma > 0$$

$$[\epsilon] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_{12} = 2 \times \epsilon_{12} = \gamma = \tan\theta \approx \theta$$

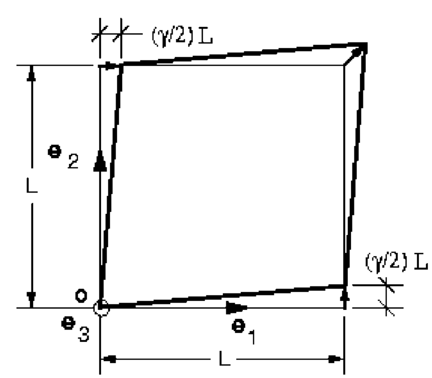


Pure Shear with respect to $(\mathrm{e}_1, \mathrm{e}_2)$:

$$\mathbf{u} = \frac{\gamma}{2} x_2 \, \mathbf{e}_1 + \frac{\gamma}{2} x_1 \, \mathbf{e}_2, \; u_1 = \frac{\gamma}{2} x_2, \; du_2 = \frac{\gamma}{2} x_1, \; u_3 = 0, \; \gamma > 0.$$

$$[\epsilon] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\gamma_{12} = 2 \times \epsilon_{12} = \gamma.$$

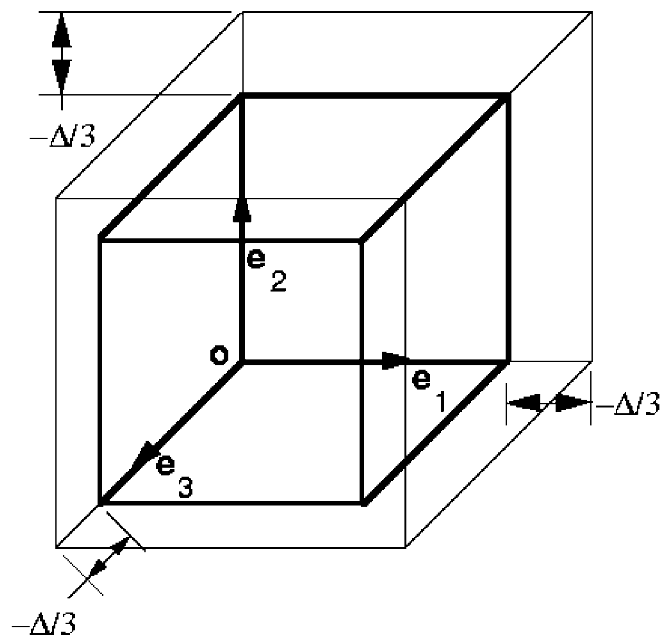


Uniform Compaction:

$$\mathbf{u} = \frac{-\Delta}{3} x_1 \mathbf{e}_1 + \frac{-\Delta}{3} x_2 \mathbf{e}_2 + \frac{-\Delta}{3} x_3 \mathbf{e}_3$$

$$u_1 = \frac{-\Delta}{3} x_1, \quad u_2 = \frac{-\Delta}{3} x_2, \quad u_3 = \frac{-\Delta}{3} x_3, \quad \Delta > 0$$

$$[\epsilon] = \begin{bmatrix} \frac{-\Delta}{3} & 0 & 0 \\ 0 & \frac{-\Delta}{3} & 0 \\ 0 & 0 & \frac{-\Delta}{3} \end{bmatrix}.$$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Note that in uniform compaction/dilatation the volume change per unit original volume is given by:

$$\frac{V - V_0}{V_0} = (1 + \epsilon_{11})(1 + \epsilon_{22})(1 + \epsilon_{33}) - 1,$$
$$= \epsilon_{11} + \epsilon_{22} + \epsilon_{33}, \qquad |\epsilon_{ij}| << 1$$

It may be shown, that this expression for the volume change always holds, whether or not shear strains are present and whether the normal strains are equal or not. That is, for infinitesimal deformations, the volume change per unit original volume, the dilatation, is always given by

$$\boxed{\sum_k \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \equiv \text{tr } \epsilon.}$$

Thus, the dilatation is given by the sum of the diagonal components, that is the **trace** of the strain tensor.

The components of the **strain deviator** tensor ϵ' are defined by

$$\epsilon'_{ij} = \epsilon_{ij} - \frac{1}{3}\left(\sum_k \epsilon_{kk}\right) \delta_{ij},$$

where

$$\delta_{ij} = \left\{ \begin{array}{lll} 1 & \text{if} & i = j, \\ 0 & \text{if} & i \neq j, \end{array} \right\}$$

is the **Kronecker delta**.

The tensor ϵ' is called the strain deviator because it measures the strain which deviates from the dilatational part of the strain.

Plane Strain

This corresponds to a displacement field in which u_3 is zero,and u_1 and u_2 are functions of only (x_1, x_2):

$$u_1 = \widehat{u}_1(x_1, x_2), \qquad u_2 = \widehat{u}_2(x_1, x_2), \qquad u_3 = 0.$$

This restriction on the displacement field in conjunction with the strain-displacement relations implies that in plane strain,

$$\epsilon = \widehat{\epsilon}(x_1, x_2)$$

with

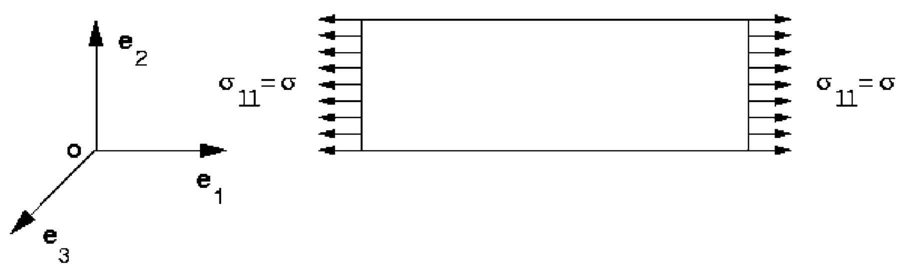
$$[\epsilon] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

SOME SIMPLE STATES OF STRESS:

Pure tension (or compression) with tensile stress σ in the e_1-direction:

$$\sigma_{11} = \sigma, \quad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$

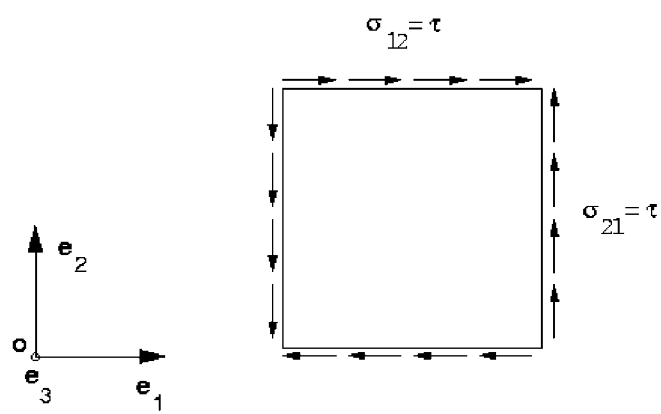
$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Pure shear stress τ relative to the direction pair $(\mathbf{e}_1, \mathbf{e}_2)$:

$$\sigma_{12} = \sigma_{21} = \tau, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

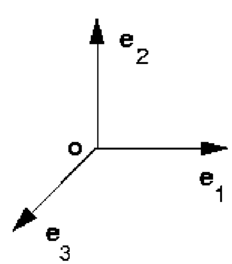


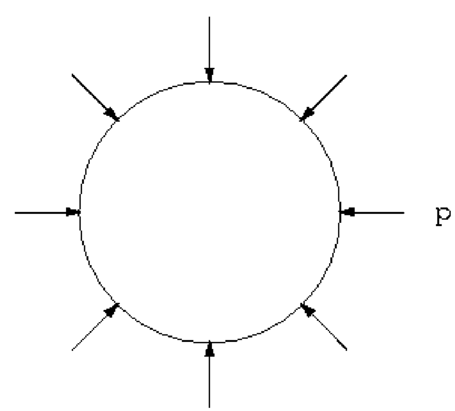
Hydrostatic pressure:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \text{and} \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$

$$[\boldsymbol{\sigma}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}, \quad \sigma_{ij} = -p\,\delta_{ij}.$$

This is the state of a stress in a fluid at rest. The scalar p is called the **pressure** of the fluid.





PLANE STRESS

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0,$$

with the remaining stress components functions of only (x_1, x_2):

$$\sigma_{11} = \hat{\sigma}_{11}(x_1, x_2), \quad \sigma_{22} = \hat{\sigma}_{22}(x_1, x_2), \quad \sigma_{12} = \hat{\sigma}_{12}(x_1, x_2).$$

This is the approximate state of stress in a thin sheet lying parallel to the x_3-plane and subject to forces acting in its plane.

STRESS DEVIATOR

The components of the **stress deviator** tensor $\boldsymbol{\sigma}'$ are defined by

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\left(\sum_k \sigma_{kk}\right)\delta_{ij},$$

where

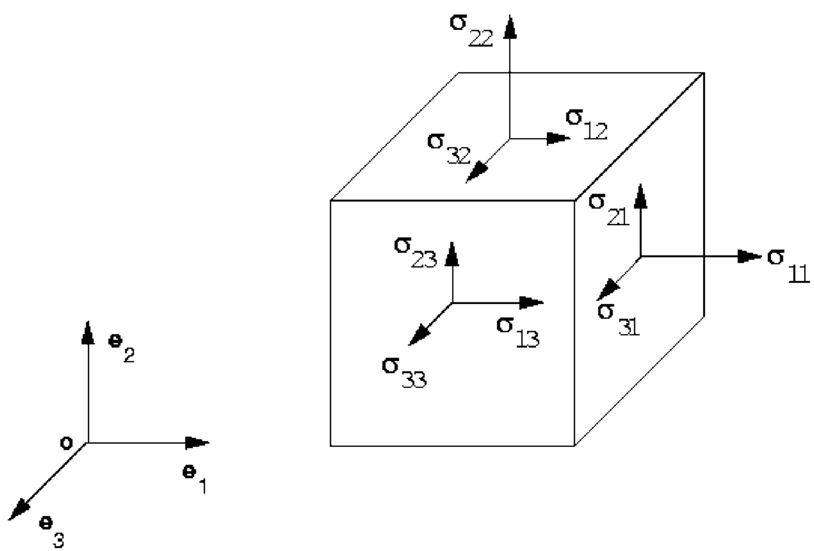
$$\delta_{ij} = \left\{ \begin{array}{lll} 1 & \text{if} & i = j, \\ 0 & \text{if} & i \neq j, \end{array} \right\}$$

is the **Kronecker delta**.

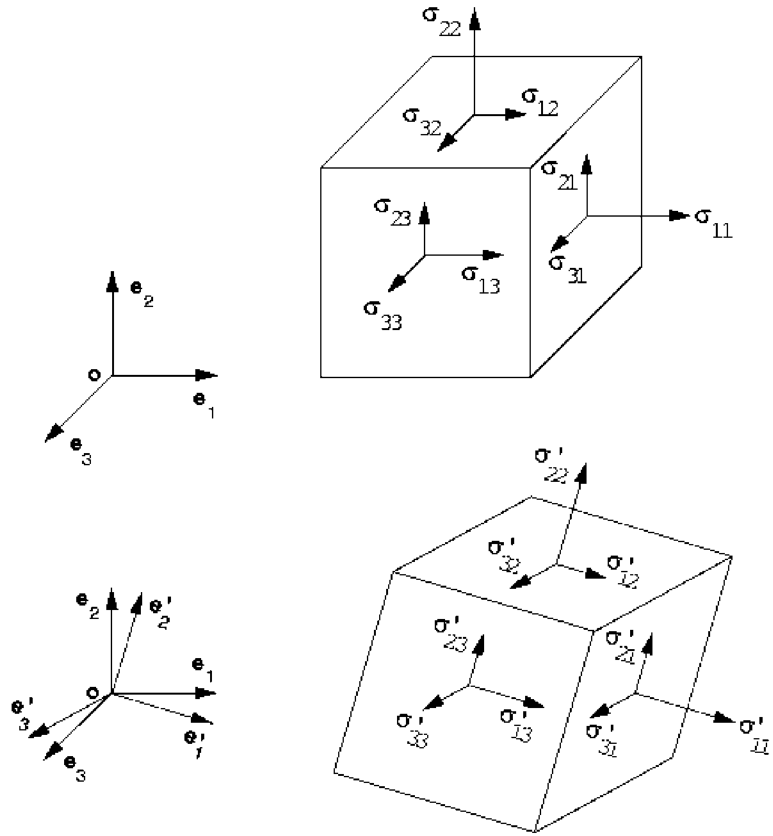
The tensor $\boldsymbol{\sigma}'$ has the property that $\mathrm{tr}\,\boldsymbol{\sigma}' = \sum_k \sigma'_{kk}$, is zero. A tensor whose trace is zero is called a **deviatoric tensor**, and $\boldsymbol{\sigma}'$ is called the **deviator** of $\boldsymbol{\sigma}$.

Change of stress components upon a transformation of coordinates

Recall, that the component $\sigma_{ij} = \mathrm{e}_i \cdot \boldsymbol{\sigma} \mathrm{e}_j$ is the stress component in the i-direction associated with the j-oriented face. Each of these components has the dimensions of force per unit area.



Up till this point, we have considered the components of stress σ_{ij} with respect to a rectangular cartesian coordinate system with origin $\mathbf{o}$ and base vectors $\{\mathbf{e}_i\}$. Let us now consider a second rectangular cartesian coordinate system with the same origin, but a different orthonormal basis $\left\{\mathbf{e}'_i\right\}$,



Transformation of stresss components under a rotation of the coordinate system.

Let

$$Q_{ij} = \mathbf{e}_i{}' \cdot \mathbf{e}_j$$

denote the cosine of the angle between $\mathbf{e}_i{}'$ and $\mathbf{e}_j$, so that

$$\mathbf{e}_i{}' = \sum_j Q_{ij}\mathbf{e}_j.$$

Then, the components of $\boldsymbol{\sigma}$ with respect to the basis $\left\{\mathbf{e}_i'\right\}$ are

$$\sigma_{ij}' = \mathbf{e}_i{}' \cdot \boldsymbol{\sigma}\mathbf{e}_j' = \left(\sum_k Q_{ik}\mathbf{e}_k\right) \cdot \boldsymbol{\sigma} \left(\sum_l Q_{jl}\mathbf{e}_l\right) = \sum_{k,l} Q_{ik}Q_{jl}\left(\mathbf{e}_k \cdot \boldsymbol{\sigma}\mathbf{e}_l\right)$$

or*

$$\boxed{\sigma_{ij}' = \sum_{k,l} Q_{ik}Q_{jl}\sigma_{kl}.}$$

*This equation is often used as the defining equation of a second rank tensor.

This transformation rule may be written in matrix notation as

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}^T$$

Thus, the a stress tensor $\boldsymbol{\sigma}$ will have different component and matrix representations with respect to the different bases $\{\mathrm{e}_i\}$ and $\left\{\mathrm{e}'_i\right\}$.

Principal Stresses and Principal Directions of Stress

The components $\sigma_{ij} = \mathrm{e}_i \cdot \boldsymbol{\sigma} \mathrm{e}_j$ are the stress components with respect to a basis $\{\mathrm{e}_i | i = 1, 2, 3\}$. Since $\boldsymbol{\sigma}$ is symmetric, another orthonormal basis $\left\{\mathrm{e}'_i\right\}$ can be found such that with $Q_{ij} = \mathrm{e}_i{}' \cdot \mathrm{e}_j$:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}^T$$

The particular basis $\left\{\mathrm{e}'_i\right\}$ with respect to which the stress matrix is diagonal are called the **principal directions of stress**, and the corresponding stress components are called the **principal stresses**. Physically, each of the principal stresses is a normal stress acting on a plane determined by the corresponding principal direction of stress, and there is no shear component of the traction vector on this plane.

Recall that the traction vector $\mathbf{t}$ on a plane defined by a unit normal $\mathbf{n}$ is given by

$$t_i = \sum_{j=1}^{3} \sigma_{ij} n_j.$$

If

$$\sum_{j=1}^{3} \sigma_{ij} n_j = \sigma n_i,$$

holds, that is when $\mathbf{t}$ and $\mathbf{n}$ have the same direction, then σ is called a **principal stress**, and $\mathbf{n}$ is the associated **principal direction** of stress. The principal stresses and the principal directions of the stress are solutions of the **eigenvalue problem**

$$\sum_{j=1}^{3} \left(\sigma_{ij} - \sigma \delta_{ij}\right) n_j = 0 \qquad (i = 1, 2, 3).$$

This eigenvalue problem may be stated in matrix notation as

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The condition for this eigenvalue problem to have a non-trivial solution for n_j is

$$\det \begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} = -\sigma^3+I_1\sigma^2-I_2\sigma+I_3=0,$$

where

$$I_1=\sum_k \sigma_{kk}, \quad I_2=\frac{1}{2}\left[I_1^2-\sum_{i,j}\sigma_{ij}\sigma_{ij}\right], \quad I_3=\det\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

Since the principal stresses are determined by I_1, I_2, I_3, and can have no dependence on the coordinate system with respect to which we refer the components of stress, I_1, I_2, and I_3 must be independent of that choice, and are therefore called the **stress invariants**.

Solutions to the cubic equation

$$-\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

give the three roots $(\sigma_1, \sigma_2, \sigma_3)$, knowing which one can determine the principal directions $\left(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right)$ from the eigenvalue equation.

Symmetry of the stress tensor has the important consequence that there exist **three mutually perpendicular** directions such that there are no shearing forces on area elements perpendicular to these directions.

The **principal directions of stress** are denoted by $\left(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right)$, and the **principal stresses** are denoted by $(\sigma_1, \sigma_2, \sigma_3)$, and are usually ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

If σ_1, σ_2, and σ_3 are distinct, then the (normalized) principal directions $\left(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right)$ are unique and mutually orthogonal. If these three orthogonal vectors are taken as base vectors at $\mathbf{x}$, then referred to this basis the matrix of stress components is a diagonal matrix with elements σ_1, σ_2 and σ_3.

If $\sigma_1 = \sigma_2 \neq \sigma_3$, then the principal direction $\mathbf{n}^{(3)}$ is uniquely determined and $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are **any two** mutually orthogonal unit vectors lying in the plane perpendicular to $\mathbf{n}^{(3)}$.

If $\sigma_1 = \sigma_2 = \sigma_3$, then **any three** mutually orthogonal unit vectors are the principal directions.

The discussion of principal components and principal directions for the symmetric stress tensor $\boldsymbol{\sigma}$ holds for **any** symmetric tensor $\mathbf{A}$.

That is, any symmetric tensor $\mathbf{A}$ possesses three **eigenvalues** (a_1, a_2, a_3) and an associated orthonormal set of **eigenvectors** $\left(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right)$.

In particular, because the infinitesimal strain tensor $\boldsymbol{\epsilon}$ is a symmetric, it possesses three eigenvalues $(\epsilon_1, \epsilon_2, \epsilon_3)$ and an orthonormal set of eigenvectors associated with these eigenvalues.

Problem

(Reading Assignment:
Sections 2.4 through 2.6 of Bickford)

1. In plane stress ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$) show that if the $\mathbf{e}'_1$ and $\mathbf{e}'_2$ axes are obtained by rotating the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes in a counter-clockwise direction through an angle θ about the $\mathbf{e}_3$-axis, then

$$\begin{aligned}
\sigma'_{11} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\theta + \sigma_{12} \sin 2\theta \\
\sigma'_{22} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\theta - \sigma_{12} \sin 2\theta \\
\sigma'_{12} &= -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\theta + \sigma_{12} \cos 2\theta.
\end{aligned}$$

2. From these equations deduce that

$$\frac{\partial \sigma'_{11}}{\partial \theta} = 2\sigma'_{12}, \quad \frac{\partial \sigma'_{22}}{\partial \theta} = -2\sigma'_{12},$$

and that

$$\sigma'_{12} = 0 \quad \text{when} \quad \theta = \theta^* \equiv \frac{1}{2}\tan^{-1}\left\{\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right\}.$$

For plane stress, the directions $(\mathrm{e}'_1, \mathrm{e}'_2)$ corresponding to θ^* are the in-plane principal axes of stress, and the corresponding values of stress $\sigma'_{11}, \sigma'_{22}$ are the principal stress components. Let us denote the maximum (most positive) of these principal stresses by σ_1 and the minimum by σ_2. Show that

$$\left.\begin{array}{c}\sigma_1 \\ \sigma_2\end{array}\right\} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}.$$

3. Further show that σ'_{12} has an extremum when $\theta = \theta^* \pm 45^\circ$, and that the maximum value of σ'_{12} is

$$(\sigma'_{12})_{\mathrm{max}} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}.$$

Solution

1. In plane stress

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If e_1' and e_2' axes are obtained by rotating the e_1 and e_2 axes in a counter-clockwise direction through an angle θ about the e_3-axis, then from $Q_{ij} = \mathrm{e}_i' \cdot \mathrm{e}_j$ we have

$$\left[Q_{ij}\right] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & 0 \\ \sigma'_{21} & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$\sigma'_{11} = \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + \sigma_{12}\, 2\sin\theta\cos\theta,$$
$$\sigma'_{22} = \sigma_{11} \sin^2\theta + \sigma_{22} \cos^2\theta - \sigma_{12}\, 2\sin\theta\cos\theta,$$
$$\sigma'_{12} = (-\sigma_{11} + \sigma_{22}) \sin\theta\cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right).$$

Since

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta),$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta),$$
$$\sin 2\theta = 2\sin\theta\cos\theta,$$

we may rewrite the preceeding equation as

$$\begin{aligned}
\sigma'_{11} &= \frac{1}{2}(\sigma_{11}+\sigma_{22}) + \frac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\theta + \sigma_{12}\sin 2\theta \\
\sigma'_{22} &= \frac{1}{2}(\sigma_{11}+\sigma_{22}) - \frac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\theta - \sigma_{12}\sin 2\theta \\
\sigma'_{12} &= -\frac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\theta + \sigma_{12}\cos 2\theta.
\end{aligned}$$

2. Hence,

$$\frac{\partial \sigma'_{11}}{\partial \theta} = -(\sigma_{11}-\sigma_{22})\sin 2\theta + 2\sigma_{12}\cos 2\theta = 2\sigma'_{12},$$

$$\frac{\partial \sigma'_{22}}{\partial \theta} = (\sigma_{11}-\sigma_{22})\sin 2\theta - 2\sigma_{12}\cos 2\theta = -2\sigma'_{12},$$

$$\sigma'_{12} = 0 \text{ when } \theta = \theta^* \equiv \frac{1}{2}\tan^{-1}\left\{\frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}}\right\}.$$

Since $\sigma'_{12} = 0$ when $\tan 2\theta^* = \dfrac{2\sigma_{12}}{\sigma_{11} - \sigma_{12}}$, we have

$$\sin 2\theta^* = \frac{\sigma_{12}}{\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}, \quad \cos 2\theta^* = \frac{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}{\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}$$

$$\begin{aligned}
\sigma_1 &= \sigma'_{11}(\theta^*), \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} + \frac{(\sigma_{11} - \sigma_{22})}{2} \cos 2\theta^* + \sigma_{12} \sin 2\theta^* \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} + \frac{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}{\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}, \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\sigma_2 &= \sigma'_{22}(\theta^*), \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} - \frac{(\sigma_{11} - \sigma_{22})}{2}\cos 2\theta^* - \sigma_{12}\sin 2\theta^* \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} - \frac{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}{\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}, \\
&= \frac{(\sigma_{11} + \sigma_{22})}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}.
\end{aligned}
$$

3. Since

$$\frac{\partial \sigma'_{12}}{\partial \theta} = -\frac{(\sigma_{11} - \sigma_{22})}{2} 2\cos 2\theta - \sigma_{12} 2\sin 2\theta,$$

$\dfrac{\partial \sigma'_{12}}{\partial \theta} = 0$ when θ has the value $\tilde{\theta}$ given by

$$\tan 2\tilde{\theta} = -\frac{(\sigma_{11} - \sigma_{22})}{2\sigma_{12}}.$$

Recall that $\tan 2\theta^* = \dfrac{2\sigma_{12}}{(\sigma_{11} - \sigma_{22})}$. Thus

$$\tan 2\tilde{\theta} = -\cot 2\theta^*.$$

Using the trignometric identity $\tan(\alpha \pm (\pi/2)) = -\cot \alpha$, we have

$$\tilde{\theta} = \theta^* \pm (\pi/4).$$

Thus σ'_{12} has an extremum when $\theta = \theta^* \pm (\pi/4)$.

Finally, since

$$\cos 2\tilde{\theta} = \frac{\sigma_{12}}{\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}, \quad \sin 2\tilde{\theta} = \frac{-\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)}{\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}},$$

we have

$$\begin{aligned}\sigma'_{12}|_{\max} = \sigma_{12}(\tilde{\theta}) &= -\frac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\tilde{\theta} + \sigma_{12}\cos 2\tilde{\theta}, \\ &= \frac{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}{\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}},\end{aligned}$$

or

$$\sigma'_{12}|_{\max} = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2} = \frac{\sigma_1 - \sigma_2}{2}.$$