2.20 - Marine Hydrodynamics
Lecture 7

Chapter 3 – Ideal Fluid Flow

The structure of Lecture 7 has as follows: In paragraph 3.0 we introduce the concept of inviscid fluid and formulate the governing equations and boundary conditions for an ideal fluid flow. In paragraph 3.1 we introduce the concept of circulation and state Kelvin's theorem (a conservation law for angular momentum). In paragraph 3.2 we introduce the concept of vorticity.

$$\boxed{\textbf{Ideal Fluid Flow} \equiv \begin{cases} \textbf{Inviscid Fluid} & \nu = 0 \\ \quad + & \\ \textbf{Incompressible Flow (§ 1.1)} & \dfrac{D\rho}{Dt} = 0 \text{ or } \nabla \cdot \vec{v} = 0 \end{cases}}$$

3.0 Governing Equations and Boundary Conditions for Ideal Flow

- Inviscid Fluid, Ideal Flow

  Recall Reynolds number is a qualitative measure of the importance of viscous forces compared to inertia forces,

$$R_e = \frac{UL}{\nu} = \frac{\text{inertia forces}}{\text{viscous forces}}$$

For many marine hydrodynamics problems studied in 13.021 the characteristic lengths and velocities are L $\geq$ 1m and U $\geq$ 1m/s respectively. The kinematic viscosity in water is $\nu_{water} = 10^{-6}\text{m}^2/\text{s}$ leading thus to typical Reynolds numbers with respect to U and L in the order of

$$\begin{aligned} R_e &= \frac{UL}{\nu} \geq 10^6 >>> 1 \Rightarrow \\ \frac{1}{R_e} &\sim \frac{\text{viscous forces}}{\text{inertia forces}} \simeq 0 \end{aligned}$$

This means that viscous effects are << compared to inertial effects - or confined within very small regions. In other words, for many marine hydrodynamics problems, viscous effects can be neglected for the bulk of the flow.

Neglecting viscous effects is equivalent to setting the kinematic viscosity $\nu = 0$, but

$$\nu = 0 \Leftrightarrow \textbf{ inviscid fluid}$$

Therefore, for the typical marine hydrodynamics problems we assume

$$\textbf{incompressible} \text{ flow} + \textbf{inviscid} \text{ fluid} \equiv \textbf{ideal fluid flow}$$

which turns out to be a good approximation for many problems.

- Governing Equations for Ideal Fluid Flow

  - Continuity Equation:

$$\boxed{\nabla \cdot \vec{v} = 0}$$

  - Momentum (Navier-Stokes ⇒ Euler) equations:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p - g\hat{j}}$$

    By neglecting the viscous stress term ($\nu \nabla^2 \vec{v}$) the Navier-Stokes equations reduce to the Euler equations. (Careful not to confuse this with the Euler equation in §1.6).

    The N-S equations are second order PDE's with respect to space (2^{nd} order in ∇^2), thus: (a) require 2 kinematic boundary conditions, and (b) produce smooth solutions in the velocity field.

    The Euler equations are first order PDE's, thus: (a) require 1 kinematic boundary condition, and (b) may allow discontinuities in the velocity field.

- Boundary Conditions for Euler equations (Ideal Flow):

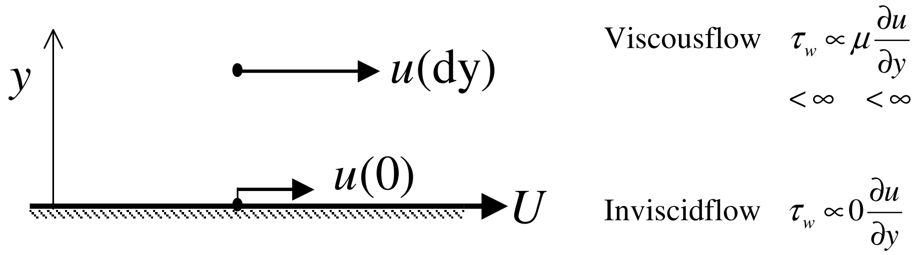
  - KBC:

$$\boxed{\vec{v} \cdot \hat{n} = \underbrace{\vec{u} \cdot \hat{n} = U_n}_{\text{given}} \quad \leftarrow \text{‘no flux’ + free (to) slip}}$$

    Note: ‘No slip’ condition $\vec{v} \cdot \hat{t} = \vec{U} \cdot \hat{t}$ does not apply.

    The ‘no slip’ condition is required to ensure that the velocity gradients are finite and therefore the viscous stresses $\hat{\tau}_{ij}$ are finite.

    But since $\nu = 0$ the viscous stresses are identically zero ($\hat{\tau}_{ij} \propto \mu = \rho\nu = 0$) and the velocity gradients can be infinite. Or else the velocity field need not be continuous.



– DBC:

$$\boxed{p = \ldots \text{ Pressure given on the boundary}}$$

Similarly to the argument for the KBC, viscous stresses $\hat{\tau}_{ij}$ cannot be specified on any boundary since $\nu = 0$.

- Summary of consequences neglecting viscous effects this far:
  – Neglecting viscous effects is equivalent to setting the kinematic viscosity equal to zero:

$$\nu = 0$$

Setting $\nu = 0 \Longleftrightarrow$ **inviscid fluid**

  – Setting $\nu = 0$ the viscous term in the Navier-Stokes equations drops out and we obtain the Euler equations.

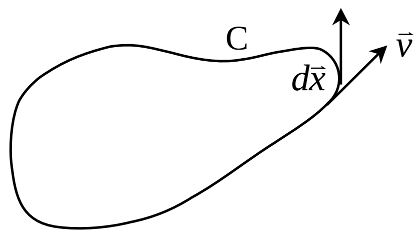
The Euler equations are $1st$ order PDE's in space, thus (a) require only one boundary condition for the velocity and (b) may allow for velocity jumps.

  – Setting $\nu = 0$ all the viscous stresses $\hat{\tau}_{ij} \propto \mu = \rho\nu$ are identically 0. This may allow for infinite velocity gradients.

This affects (a) the KBC, allowing free slip, and (b) the DBC, where no viscous stresses can be specified on any boundary.

3.1 Circulation – Kelvin's Theorem

3.1.1 $\Gamma \equiv$ **Instantaneous** circulation around any arbitrary **closed** contour C.



$$\Gamma = \int_C \underbrace{\vec{v} \cdot d\vec{x}}_{\substack{\text{tangential} \\ \text{velocity}}}$$

The circulation Γ is an Eulerian idea and is **instantaneous**, a 'snapshot'.

3.1.2 Kelvin's Theorem (KT) :

For <u>ideal</u> fluid under <u>conservative</u> body forces,

$$\frac{d\Gamma}{dt} = 0 \text{ following any } \textbf{material contour } C,$$

i.e., Γ remains constant under for Ideal Fluid under Conservative Forces (IFCF).
This is a statement of conservation of angular momentum.
(Mathematical Proof: cf JNN pp 103)

Kinematics of a small deformable body:

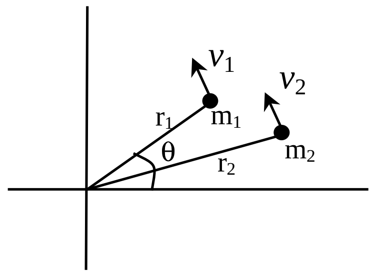
(a) Uniform translation $\rightarrow$ Linear momentum
(b) Rigid body rotation $\rightarrow$ Angular momentum
(c) Pure strain$\rightarrow$ No linear or angular momentum involved (no change in volume
(d) Volume dilatation

For Ideal Fluid under Conservative body Forces:

(a) Linear momentum $\rightarrow$ Can change
(b) Angular momentum $\rightarrow$ **By K.T., cannot change**
(c) Pure strain$\rightarrow$ Can change
(d) Volume dilatation $\rightarrow$ Not allowed (incompressible fluid)

Kelvin's Theorem is a statement of conservation of angular momentum under IFCF.

Example 1: Angular momentum of point mass.



Angular momentum of point mass:

$$\left|\vec{\mathcal{L}}\right| = |\vec{r} \times (m\vec{v})| = mvr = mr^2\dot{\theta}$$

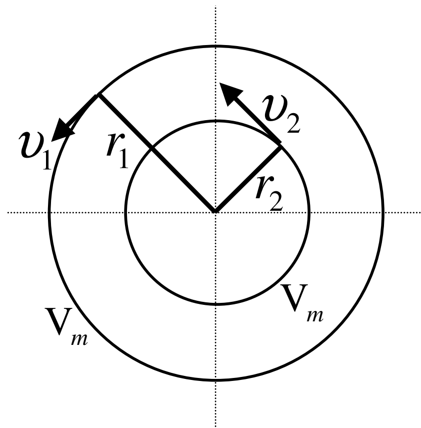
Conservation of angular momentum:

$$\begin{aligned}
\left|\vec{\mathcal{L}}\right|_1 &= \left|\vec{\mathcal{L}}\right|_2 \\
m_1 v_1 r_1 &= m_2 v_2 r_2 \overset{m_1=m_2}{\Longrightarrow} \\
v_1 r_1 &= v_2 r_2 \text{ or} \\
r_1^2\dot{\theta}_1 &= r_2^2\dot{\theta}_2
\end{aligned}$$

Conservation of angular momentum does **not** imply constant angular velocity:

$$\boxed{\text{Angular Momentum } \not\Rightarrow \text{ angular velocity } \vec{\omega}}$$

Example 2: Conservation of circulation around a shrinking circular material volume V_m.



$$\Gamma_1 = \int_0^{2\pi} d\theta r_1 v_1 = \int_0^{2\pi} d\theta r_2 v_2 = \Gamma_2$$

Example 3: Conservation of circulation around a shrinking arbitrary material volume V_m, C_m.

$$\Gamma_1 = \int_{C_1} \vec{v}_1 \cdot d\vec{x} = \int_{C_2} \vec{v}_2 \cdot d\vec{x} = \Gamma_2$$

3.2 Vorticity

3.2.1 **Definition of Vorticity**

$$\vec{\omega} = \nabla \times \vec{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Relationship of vorticity to circulation - Apply Stokes' Theorem:

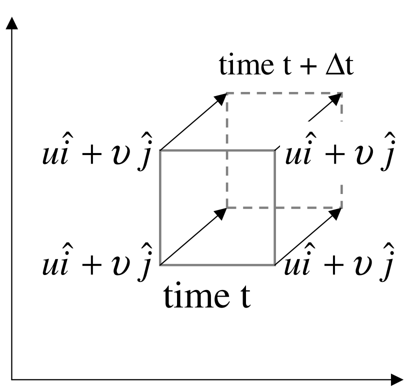
$$\Gamma = \oint_C \vec{v} \cdot d\vec{x} = \iint_{\text{any S `covering' C}} (\nabla \times \vec{v}) \cdot \hat{n} dS = \iint_{\text{any S `covering' C}} \vec{\omega} \cdot \hat{n} dS \equiv \textbf{F}\text{lux of vorticity out of S}$$

3.2.2 **What is Vorticity?**

For example, special case: 2D flow - $w = 0$; $\frac{\partial}{\partial z} = 0$; $\omega_y = \omega_x = 0$ and

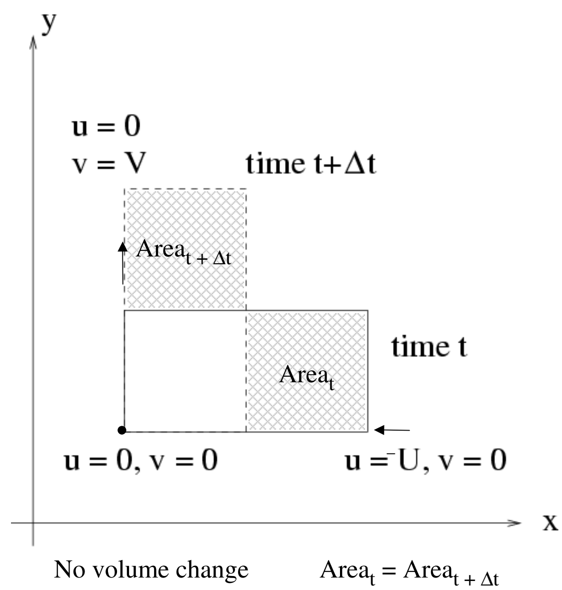
$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

(a) Translation: $u = \text{constant}$, $v = \text{constant}$



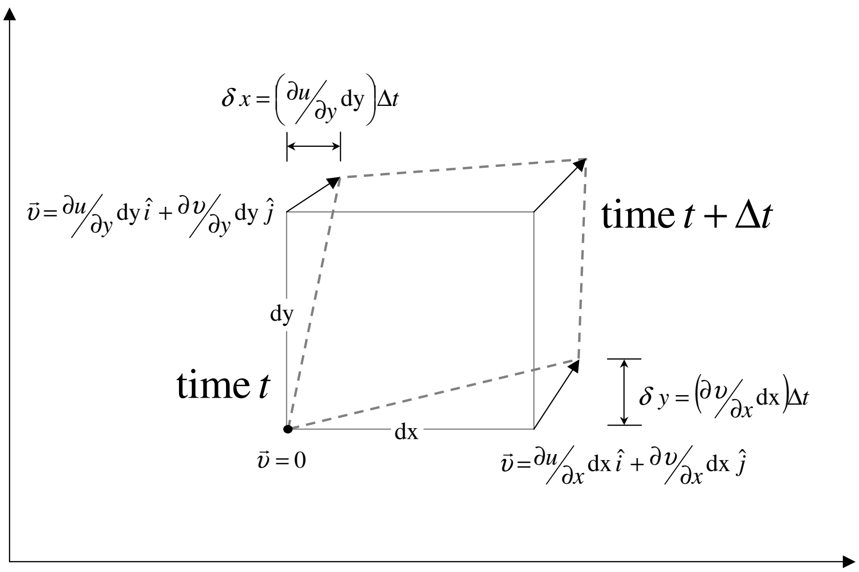
$$\frac{\partial v}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \Rightarrow \omega_z = 0 \rightarrow \text{no vorticity}$$

(b) Pure Strain (no volume change):



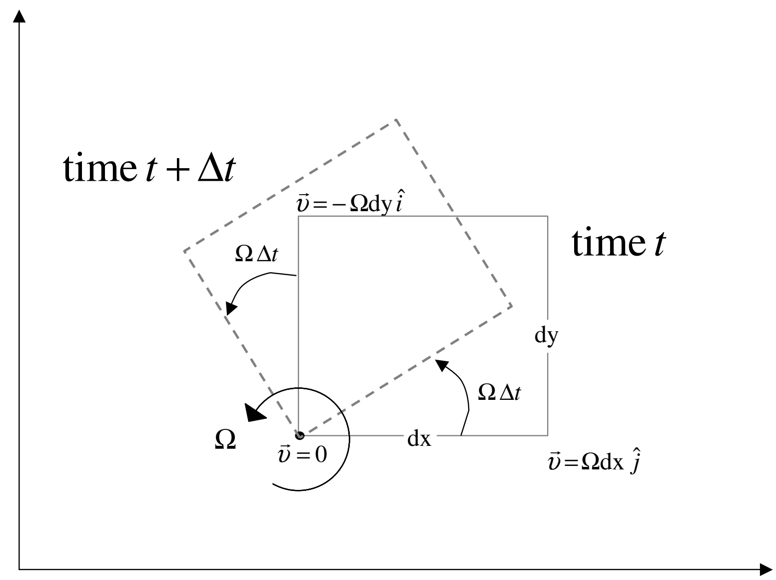
$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}; \quad \text{u} = \text{-v}; \quad \frac{\partial u}{\partial y} = 0; \quad \frac{\partial v}{\partial x} = 0 \Rightarrow \omega_z = 0$$

(c) Angular deformation



$$\vec{\omega} = 0 \text{ only if } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \rightarrow \delta x = \delta y (\text{ for dx = dy})$$

(d) Pure rotation with angular velocity Ω



$$\frac{\partial v}{\partial x} = \Omega; \quad \frac{\partial u}{\partial y} = -\Omega; \quad \omega_z = 2\Omega$$

i.e. vorticity $\propto$ 2(angular velocity).

3.2.3 Irrotational Flow

A flow is irrotational if the vorticity is zero everywhere or if the circulation is zero along *any* arbitrary closed contour:

$$\boxed{\vec{\omega} \equiv 0 \text{ everywhere } \Leftrightarrow \Gamma \equiv 0 \text{ for any } C}$$

Further on, if at $t = t_o$, the flow is irrotational, i.e., $\Gamma \equiv 0$ for all C, then Kelvin's theorem states that under IFCF, $\Gamma \equiv 0$ for all C for all time t:

$$\boxed{\textbf{once irrotational, always irrotational}}$$

(Special case of Kelvin's theorem)