13.002

Introduction to Numerical Methods for Engineers

Solution to Problem Set 2

1.

$$y_n = \int_0^1 \frac{x^n}{x+5} dx,\ n = 0, 1, ...\infty$$

$$y_n + 5y_{n-1} = \int_0^1 \frac{x^n + 5x^{n-1}}{x+5} dx = \int_0^1 \frac{x^{n-1}(x+5)}{x+5} dx = \int_0^1 x^{n-1} dx = \frac{1}{n}$$

2.

$$y_0 = \int_0^1 \frac{dx}{x+5} = [ln(x+5)]_0^1 = ln6 - ln5 \approx 0.182$$

3. The the definite integral in part 1 can be represented by the area under the function $f(x) = \frac{x^n}{x+5}$. In the range $x = 0$ to $x = 1$ the area is positive for any n. Therefore both y_n and y_{n-1} are positive. From the recurrence

$$y_n + 5y_{n-1} = \frac{1}{n} \qquad as\ n \to \infty\ y_n + 5y_{n-1} \to 0$$

Since both of the terms on the left side are non-negative $y_n \to 0$ and $y_{n-1} \to 0$ as $n \to \infty$

4.

$$y_0 \approx 0.182 \quad y_1 \approx 0.090 \quad y_2 \approx 0.050 \quad y_3 \approx 0.0830 \quad y_4 \approx -0.165$$

It is unexpected that $y_3 > y_2$ and y_4 is negative. The reason for the incorrect result is the round-off error in y_0 whose magnitude is multiplied by -5 in the calculation of y_1. The error propagates making the recursion unstable.

5. The list of y_n values and the graph show the numerical instability for forward recursion. As n increases y_n becomes increasingly negative.

6. The problem can be solved using the recursion formula in the other direction,

$$y_{50} = 0\,, \qquad y_{n-1} = \frac{1}{5n} - \frac{y_n}{5}$$

The list of y_n for backward recursion shows that the values of y_n converge for much smaller n than 50 and the value of y_0 is 0.182, as expected. Also, all of them are positive. The backward recurrence works well because the error is now divided by -5 in each step.

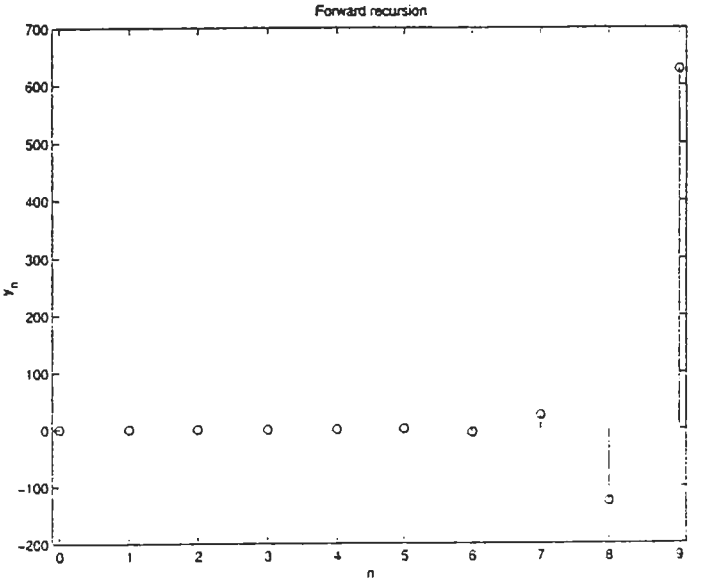
Forward Recursion

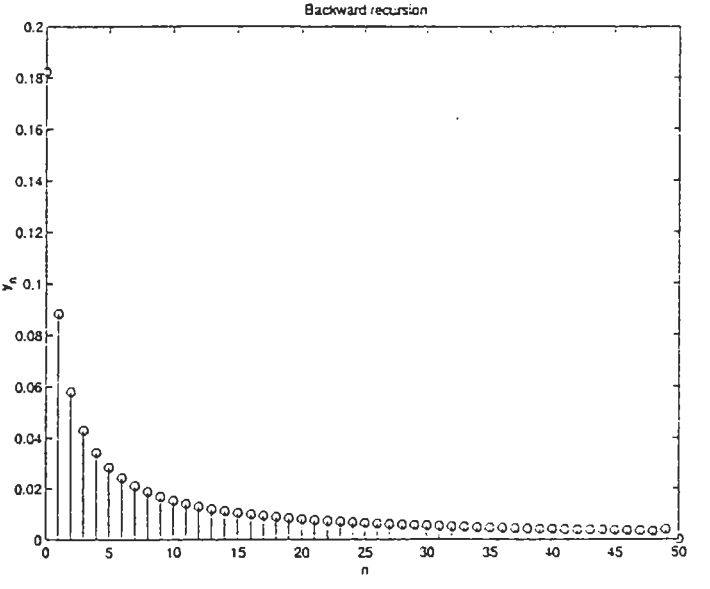
n	y(n)
0	0.1820
1.0000	0.0900
2.0000	0.0500
3.0000	0.0833
4.0000	-0.1667
5.0000	1.0333
6.0000	-5.0000
7.0000	25.1429
8.0000	-125.5893

Backward Recursion

n	y(n)
0	0.1823
1.0000	0.0884
2.0000	0.0580
3.0000	0.0431
4.0000	0.0343
5.0000	0.0285
6.0000	0.0243
7.0000	0.0214
8.0000	0.0182
9.0000	0.0200
10.0000	0

```
% Forward recursion
clear
n=8
    .182
     k=1:n
y=1/k-5*y
end;


% Forward recursion (vector form)
clear
n=9
y=zeros(1,n)
y(1)=0.182
for k=1:n
y(k+1)=1/k-5*y(k)
end;


x=[0:n];

%Ploting
figure(1)
stem(x,y)
xlabel('n')
ylabel('y_n')
axis([-0.1 9.1 -200 700])
title('Forward recursion')

    ackward recursion (vector form)
ear
n=10
y=zeros(1,n+1)
y(n+1)=0
for k=n:-1:1
y(k)=1/(5*k)-.2*y(k+1)
end

x=[0:n]

%Ploting
figure(2)
stem(x,y)
xlabel('n')
ylabel('y_n')
title('Backward recursion')
```


Forward recursion
700
600
500
400
300
200
100
0
-100
-200
y_n
0
1
2
3
4
5
6
7
8
9
n


Backward recursion
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
y_n
0
5
10
15
20
25
30
35
40
45
50
n