Session 22

2.032
11/29 – 1

Linearized eq of motion for holonomic systems

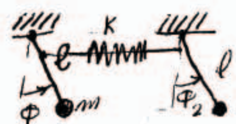
$\underline{x} = q - q_0, \quad \underline{\underline{M}}\,\ddot{\underline{x}} + (\underline{\underline{G}} + \underline{\underline{C}})\dot{\underline{x}} + (\underline{\underline{K}} + \underline{\underline{B}})\underline{x} = \underline{0}$

$-\frac{\partial V}{\partial q}\Big|_{q_0} = Q\big|_{q_0}$

In the Conservative & natural Case
(potential) $\qquad \underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{0} \qquad (1)$

Simplest derivation from quadratic lagrangian

$L = \frac{1}{2}\dot{\underline{x}}^T \underline{\underline{M}}\,\dot{\underline{x}} - \frac{1}{2}\underline{x}^T \underline{\underline{K}}\,\underline{x} + O(\varepsilon)$

Example  Spring is unstretched at $\varphi_1 = \varphi_2 = 0$
$g\downarrow$ Assume. Unstretched length $= \underline{0}$

$T = \frac{1}{2}m\ell^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$; Equilibrium. $\varphi_{10} = \varphi_{20} = 0 \Rightarrow \underline{x} = \varphi - \varphi_0 = \varphi$

$= \frac{1}{2}m\ell^2(\dot{x}_1^2 + \dot{x}_2^2)$

$$V = mgl(1-\cos\varphi_1) + mgl(1-\cos\varphi_2) + \frac{1}{2}k\left[(h\cos\varphi_1 - h\cos\varphi_2)^2 + (h\sin\varphi_1 - h\sin\varphi_2)^2\right]$$

$$\Rightarrow L = \frac{1}{2}ml^2(\dot{x}_1^2+\dot{x}_2^2) - mgl\left[1-(1-\frac{\varphi_1^2}{2}+\cdots)+1-(1-\frac{\varphi_2^2}{2}+\cdots)\right] - \frac{1}{2}kh^2\left[\underbrace{(1-\frac{\varphi_1^2}{2}+\cdots-1+\frac{\varphi_2^2}{2}+\cdots)^2}_{O(4)}+(\varphi_1-\varphi_2+\cdots)^2\right]$$

$$= \frac{1}{2}ml^2(\dot{x}_1^2+\dot{x}_2^2) - \frac{1}{2}mgl(x_1^2+x_2^2) - \frac{1}{2}kh^2(x_1-x_2)^2 + O(3)$$

Sym, pos. def. Sym

$$\Rightarrow \underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}; \quad \underline{\underline{K}} = \begin{pmatrix} mgl+kh^2 & -kh^2 \\ -kh^2 & mgl+kh^2 \end{pmatrix} \to \text{Stiffness}$$

(mass matrix)

if we expand about any other equilibrium point, we'll get different matrices

Since the equilibrium is stable $\to$ $\underline{K}$ has to be positive definite and it is

Systematic Solution of (1):

Trial Solution $\tilde{\underline{x}}(t) = \underline{x}_0 e^{\lambda t}$; general Solution $x(t) = \sum c_i \tilde{x}_i(t)$

$\underline{x}_0 \neq \underline{0}$ From initial Conditions

plug into (1):

(2) $(\lambda^2 \underline{\underline{M}} + \underline{\underline{K}})\underbrace{\underline{x}_0}_{a} = \underline{0}$, let $\underline{x}_0 = \underline{a}$

Note: if a is a Solution, so is $c\underline{a}$ $\Rightarrow$ mode shapes are never unique in magnitude

- left multiply (2) by $\underline{a}^T$ $\Rightarrow$ $\boxed{\lambda^2 = -\frac{\underline{a}^T \underline{\underline{K}}\, \underline{a}}{\underline{a}^T \underline{\underline{M}}\, \underline{a}}}$

Case (a): $\underline{\underline{K}} = \left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0}$ is positive definite

$\Rightarrow \lambda^2 < 0 \Rightarrow \lambda = \pm i\omega \Rightarrow$ $\tilde{x}_1(t) = \underline{a}e^{i\omega t}$, $\tilde{x}_2(t) = \underline{a}e^{-i\omega t}$

$\tilde{\underline{x}}(t)$: normal mode

$\underline{a}$: mode shape

ω : natural frequency

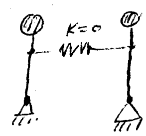
Case (b) $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2}\Big|_{q_0}$ is negative definite $\Rightarrow$ maximum for V at q_0

$\Rightarrow \lambda^2 > 0 \Rightarrow \lambda = \pm\nu \Rightarrow \tilde{x}_1(t) = \underline{a}e^{\nu t}$

$\tilde{x}_2(t) = \underline{a}\,e^{-\nu t}$

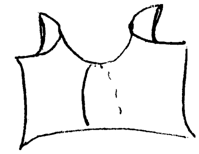
$\Rightarrow \underline{x} = \underline{0}$ is unstable

example:



Case (c): $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2}\Big|_{q_0}$ is indefinite $\rightarrow V$ has a saddle type nature

(has a point of inflection)



Instability

Example: Coupled pendulum & spring $K \neq 0$

How do we find λ & $\underline{a}$?

Consider the case of pos. def. $\underline{\underline{K}} \Rightarrow (-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\underline{a} = \underline{0}$

$\Rightarrow -\omega^2\underline{\underline{M}} + \underline{\underline{K}}$ is a

Singular matrix $\Rightarrow$
$$\begin{vmatrix} -\omega^2 m_{11} + k_{11} & \cdots & -\omega^2 m_{1n} + k_{1n} \\ \vdots & & \\ -\omega^2 m_{n1} + k_{n1} & \cdots & -\omega^2 m_{nn} + k_{nn} \end{vmatrix} = 0$$

$\cdot\; n^{th}$ order polynomial for ω^2 (characteristic equation)

$a_1(\omega^2)^n + a_2(\omega^2)^{n-1} + \cdots + a_{n+1} = 0$

always has n-roots (all roots are real here - since $\underline{\underline{M}}, \underline{\underline{K}}$ are pos. def.)

$\Rightarrow \omega_1^2 \ldots \omega_n^2$

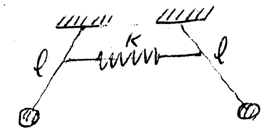
$\Rightarrow \omega_1, \ldots, \omega_n$

n-natural frequencies

Finding the mode shapes: $(-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\,\underline{a} = \underline{0} \Rightarrow \underline{a}_j = - -$

terminology $\underline{\underline{\Phi}} = [\underline{a}_1, \ldots, \underline{a}_n]$ modal matrix (is not unique, but the directions are unique)

Example: Reconsider pendulum-spring system



$$\underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \qquad K = \begin{pmatrix} mgl + kh^2 & -kh^2 \\ -kh^2 & mgl + kh^2 \end{pmatrix}$$

Let: $ml^2 = 1$ $[kg\,m^2]$ $\Rightarrow$ Char. eq. $\begin{pmatrix} -\omega^2+3 & -1 \\ -1 & -\omega^2+3 \end{pmatrix} = \omega^4 - 6\omega^2 + 8 = 0$

$mgl = 2$ $[N\,m]$

$kh^2 = 1$ $[N\,m]$

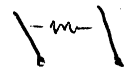
$\Rightarrow \omega_1 = \sqrt{2}$ $[\frac{1}{s}]$

$\omega_2 = 2$ $[\frac{1}{s}]$

Mode shapes: (1) $\omega_1^2 = 2$ $\begin{pmatrix} -2+3 & -1 \\ -1 & -2+3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{0}$

$\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $\omega_2^2 = 4$ $\begin{pmatrix} -4+3 & -1 \\ -1 & -4+3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{0} \longrightarrow \vec{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

a_1 

a_2