Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 6[1]

Problem 6.1

For the following statement, verify whether it is true or false (give a proof if true, give a counterexample if false):

Assume that

(a) n, m are positive integers;

(b) $f: \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ and $g: \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^m$ are continuously differentiable functions;

(c) the ODE

$$\dot{y}(t) = g(\bar{x}, y(t))$$

has a globally asymptotically stable equilibrium for every $\bar{x} \in \mathbf{R}^n$;

(d) functions $x_0: [0,1] \mapsto \mathbf{R}^n$ and $y_0: [0,1] \mapsto \mathbf{R}^m$ are continuously differentiable and satisfy

$$\dot{x}_0(t) = f(x_0(t), y_0(t)), \quad g(x_0(t), y_0(t)) = 0 \quad \forall\, t \in [0,1].$$

Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the differential equation

$$\dot{x}(t) = f(x(t), y(t)), \quad \dot{y}(t) = \epsilon^{-1} g(x(t), y(t)),$$

has a solution $x_\epsilon: [0,1] \mapsto \mathbf{R}^n$, $y_\epsilon: [0,1] \mapsto \mathbf{R}^m$ such that $x_e(0) = x_0(0)$, $y_\epsilon(0) = y_0(0)$, and $x_\epsilon(t)$ converges to $x_0(t)$ as $\epsilon \to 0$ for all $t \in [0,1]$.

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Problem 6.2

Find all values of parameter $k \in \mathbf{R}$ for which solutions of the ODE system

$$\begin{aligned} \dot{x}_1 &= -x_1 + k(x_1^3 - 3x_1x_2^2) \\ \dot{x}_2 &= -x_2 + k(3x_1^2x_2 - x_2^3) \end{aligned}$$

with almost all initial conditions converge to zero as $t \to \infty$. **Hint:** use weighted volume monotonicity with density function

$$\rho(x) = |x|^\gamma.$$

Also, for those who remember complex analysis, it is useful to pay attention to the fact that

$$(x_1^3 - 3x_1x_2^2) + j(3x_1^2x_2 - x_2^3) = (x_1 + jx_2)^3.$$

Problem 6.3

Equations for steering a two-wheeled vehicle (with one wheel used for steering and the other wheel fixed) on a planar surface are given by

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)u_1, \\ \dot{x}_2 &= \sin(x_3)u_1, \\ \dot{x}_3 &= x_4u_1, \\ \dot{x}_4 &= u_2, \end{aligned}$$

where x_1, x_2 are Decart coordinates of the fixed wheel, x_3 is the angle between the vehicle's axis and the x_1 axis, x_4 is a parameter characterizing the steering angle, u_1, u_2 are functions of time with values restricted to the interval $[-1, 1]$.

(a) Find all states reachable (time finite but not fixed) from a given initial state $\bar{x}_0 \in \mathbf{R}^4$ by using appropriate control.

(b) Design and test in computer simulation an algorithm for moving in finite time the state from a given initial position $\bar{x}_0 \in \mathbf{R}^4$ to an arbitrary reachable state $\bar{x}_1 \in \mathbf{R}^4$.