2.003J/1.053J Dynamics and Control I, Spring 2007
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4/23/2007

Lecture 18

Lagrangian Dynamics: Equilibrium Analysis - Cart with Pendulum and Spring and Spinning Hoop with Sliding Mass Examples

Example: Cart with Pendulum and Spring (Continued)

Recall:

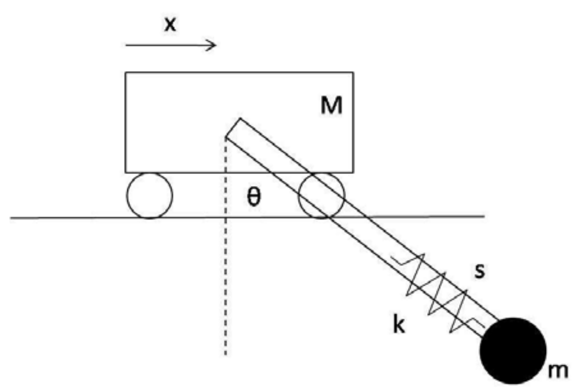

Figure 1: Cart with pendulum and spring. Figure by MIT OCW.

Equations of Motion

$$(M+m)\ddot{x} + m(\ddot{s}\sin\theta + 2\dot{s}\dot{\theta}\cos\theta + s\ddot{\theta}\cos\theta - s\dot{\theta}^2\sin\theta) = 0 \qquad (1)$$

$$s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \qquad (2)$$

$$m\ddot{s} + m\ddot{x}\sin\theta - ms\dot{\theta}^2 - mg\cos\theta + k(s-l) = 0 \qquad (3)$$

Equilibria

$$\ddot{s} = \dot{s} = \ddot{\theta} = \dot{\theta} = \ddot{x} = \dot{x} = 0$$

Set all variables to 0 except for position variables.

Configuration:

$\theta_0 = 0$, $s_0 = l + \frac{mg}{k}$ (Stable)
$\theta_0 = \pi$, $s_0 = l - \frac{mg}{k}$ (Stable or unstable? Expect it to be unstable based on physical intuition.)
x for both can be any value.

Linearize Equations (1), (2), and (3) about $\theta_0 = \pi$ and $s_o = l - mg/k$

$\theta = \theta_0 + \phi$, $s = s_0 + \epsilon$
$\dot{\theta} = \dot{\phi}$, $\ddot{\phi} = \ddot{\theta}$, $\dot{s} = \dot{\epsilon}$, $\ddot{s} = \ddot{\epsilon}$

Taylor Series

$$\cos(\theta_0 + \phi) = \cos\theta_0 + \frac{d}{d\theta}\cos\theta\bigg|_{\theta_0}\phi + \ldots \approx -1 + 0 \text{ for } \theta = \pi$$

$$\sin(\theta_0 + \phi) = \sin\theta_0 + \frac{d}{d\theta}\sin\theta\bigg|_{\theta_0}\phi = 0 - \phi \text{ for } \theta = \pi$$

Linearization

$$(M+m)\ddot{x} + m[\dot{\epsilon}(-\phi) + 2\dot{\epsilon}\dot{\phi}(-1) + (s_0+\epsilon)\ddot{\phi}(-1) - (s_0+\epsilon)\dot{\phi}^2(-\phi)] = 0$$

$$\boxed{(M+m)\ddot{x} - ms_0\ddot{\phi} = 0} \qquad (1\text{L}_\phi)$$

$$(s_0+\epsilon)\ddot{\phi} + 2\dot{\epsilon}\dot{\phi} + \ddot{x}(-1) + g(-\phi) = 0$$

$$\boxed{s_0\ddot{\phi} - \ddot{x} - g\phi = 0} \qquad (2\text{L}_\phi)$$

$$m\ddot{\epsilon} + m\ddot{x}(-\phi) - m(s_0+\epsilon)\dot{\phi}^2 - mg(-1) + k(s_0+\epsilon+l) = 0$$

$$s_0 = l - \frac{mg}{k}$$

$$\boxed{m\ddot{\epsilon} + k\epsilon = 0} \tag{3L$_\phi$}$$

Solution and Analysis

From ($1L_\phi$):

$$\ddot{x} = \frac{ms_0}{(M+m)}\ddot{\phi}$$

Substitute in ($2L_\phi$).

$$s_0\ddot{\phi} - \frac{ms_0}{(M+m)}\ddot{\phi} - g\phi = 0$$

$$\ddot{\phi} - \frac{g(M+m)}{Ms_0}\phi = 0$$

Solutions are of form $\phi = \phi_0 e^{\lambda t}$. $\ddot{\phi} = \lambda^2\phi$.

$$\lambda^2 - g\frac{(M+m)}{Ms_0} = 0$$

Finally:

$$\lambda = \pm\sqrt{\frac{g(M+m)}{Ms_0}}$$

Predicts exponential growth of θ disturbances in time, therefore unstable.

Equation ($3L_\phi$) still predicts that small stretches of the spring lead to oscillations.

Spinning Hoop with Sliding Mass

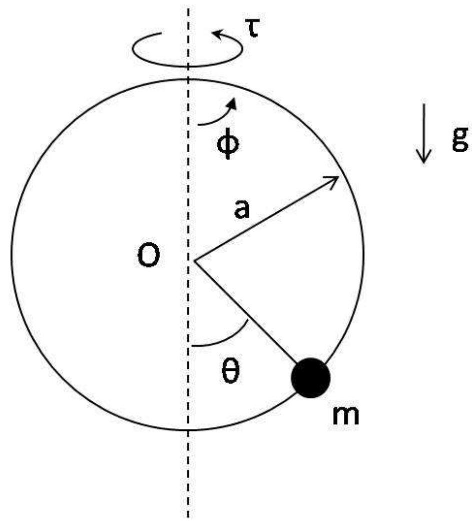

Figure 2: Spinning hoop with sliding mass. The hoop of radius, a rotates. The mass, m slides around the hoop. Figure by MIT OCW.

Massless ring - Frictionless
Rotating about the vertical axis
Center O with Radius a
m slides on hoop - 2 degrees of freedom. If m were a free particle, 3 degrees of freedom.
Torque, τ about z axis.

Generalized Coordinates and Generalized Forces

Two generalized coordinates: θ, ϕ

$$\Xi_\theta = 0$$

What is the work done with a small change θ? None. Only gravity.

$$\Xi_\phi = \tau$$

External torque applied.

Kinematics

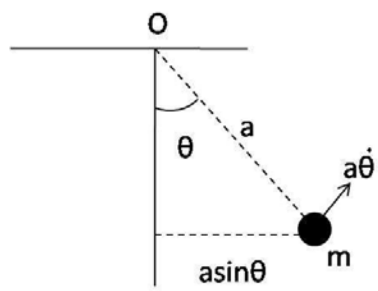

Figure 3: Kinematic diagram of sliding mass on hoop. Figure by MIT OCW.

$\perp$ components sliding on hoop, rotating into page.

Lagrangian

$$L = T - V$$

$$T = \frac{1}{2}mv^2_{\text{particle}} = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2)$$

$$V = -mga\cos\theta$$

$$L = T - V = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2) - mga\cos\theta$$

Equations of Motion

θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \Xi_\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}$$

$$\frac{\partial L}{\partial\theta} = ma^2\sin\theta\cos\theta\dot{\phi}^2 - mga\sin\theta$$

$$\boxed{ma^2\ddot{\theta} - ma^2\sin\theta\cos\theta\dot{\phi}^2 + mga\sin\theta = 0} \tag{4}$$

ϕ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \Xi_\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi}$$
$$\frac{\partial L}{\partial \phi} = 0$$
$$\Xi_\phi = \tau$$

$$\boxed{\frac{d}{dt}(ma^2 \sin^2 \theta \dot{\phi}) = \tau}$$

Modification: Add Controller

Imagine a controller that keeps $\dot{\phi}$ a constant.

Assume $\dot{\phi} = \text{constant} = \Omega$.

$$\tau = \frac{d}{dt}(ma^2 \sin \theta \Omega)$$

Controller measures θ and $\dot{\theta}$, then sets τ so that Ω is constant.

Assume that this equation is always satisfied by controller.

Equation (4) becomes:

$$\boxed{\ddot{\theta} - \sin \theta \cos \theta \Omega^2 + \frac{g}{a} \sin \theta = 0}$$

Equilibrium Points

Equilibria - must use original nonlinear equations to determine equilibrium points.

$$\dot{\theta} = \ddot{\theta} = 0$$
$$-\sin \theta \cos \theta \Omega^2 + \frac{g}{a} \sin \theta = 0$$

Physical intuition tells us some equilibria should fall at $\theta = 0$ or $\theta = \pi$.

$$\boxed{\sin \theta (\frac{g}{a} - \cos \theta \Omega^2) = 0} \qquad (5)$$

$$\theta_0 = 0, \pi \qquad (\sin \theta = 0)$$

Three equilibrium positions from the equation.

$\cos\theta = \frac{g}{a\Omega^2}$, $\theta_e = \arccos\frac{g}{a\Omega^2}$
Balance between gravity and centripetal force (normal force from hoop)
Note that the solution $\theta_e = \arccos\frac{g}{a\Omega^2}$ only exists, provided $\frac{g}{a\Omega^2} < 1$ or θ_e equilibrium only exists when rotation is fast enough ($\Omega^2 \geq \frac{g}{a}$).

Stability

Stability around $\theta_e = \arccos(g/a\Omega^2)$

Look at $\theta_e = \arccos\frac{g}{a\Omega^2}$
$\theta = \theta_e + \epsilon$, $\dot{\theta} = \dot{\epsilon}$, $\ddot{\theta} = \ddot{\epsilon}$

$$\ddot{\epsilon} - \sin(\theta_e + \epsilon)\cos(\theta_e + \epsilon)\Omega^2 + \frac{g}{a}\sin(\theta_e + \epsilon) = 0$$

Use angle addition formulas to expand (an alternative to Taylor series expansion):

$$\ddot{\epsilon} - [(\sin\theta_e\cos\epsilon + \cos\theta_e\sin\epsilon)(\cos\theta_e\cos\epsilon - \sin\theta_e\sin\epsilon)]\Omega^2 + \frac{g}{a}(\sin\theta_e\cos\epsilon + \cos\theta_e\sin\epsilon) = 0$$

$$\ddot{\epsilon} - (\sin\theta_e\cos\theta_e\cos^2\epsilon - \sin^2\theta_e\cos\epsilon\sin\epsilon + \cos^2\theta_e\sin\epsilon\cos\epsilon - \sin^2\epsilon\sin\theta_e\cos\theta_e)\Omega^2 + \frac{g}{a}(\sin\theta_e\cos\theta_e + \cos\theta_e\epsilon) = 0 \quad (6)$$

We approximate:
$\cos\epsilon \approx 1$
$\sin\epsilon \approx \epsilon$

$$\ddot{\epsilon} - \Omega^2\sin\theta_e\cos\theta_e + (\sin^2\theta_e - \cos^2\theta)\Omega^2\cdot\epsilon + \frac{g}{a}\sin\theta_e + \frac{g}{a}\epsilon\cos\theta_e = 0$$

Terms with no ϵ (no perturbation variable) are a restatement of the equilibrium configuration you already found.

$$\frac{g}{a}\sin\theta_e - \Omega^2\sin\theta_e\cos\theta_e \to \sin\theta_e\left(\frac{g}{a} - \Omega^2\cos\theta_e\right)$$

So those terms cancel out by the equilibrium condition shown in (5).

$$\ddot{\epsilon} - \Omega^2(\cos^2\theta_e - \sin^2\theta_e)\epsilon + \frac{g}{a}\epsilon\cos\theta_e = 0$$

But $\cos\theta_e = \frac{g}{a\Omega^2}$, so

$$\boxed{\ddot{\epsilon} + \Omega^2 \sin^2 \theta_e \epsilon = 0}$$

Stable, because this is a positive sign. sin or cos solutions. Oscillations. Stable.

Next time: Equilibrium points $\theta = 0, \pi$.