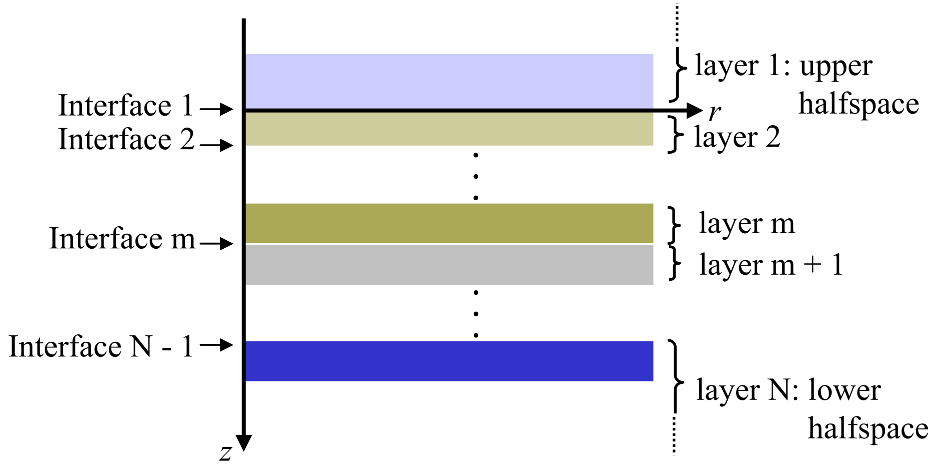
Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Wavenumber Integration

- Range-independent – Integral Transform solution
- Exact depth-dependent solution
- Numerical issues:
  - Numerical stability of depth solution
  - Evaluation of inverse transforms
- Example: Pekeris waveguide
  - Relation to Normal Mode solution



Horiztonally stratified environment

Wavenumber Integration Techniques

Integral Transform Solution

Helmholtz Equation for Displacement Potentials

$$\left[\nabla^2 + k_m^2(z)\right] \psi_m(r,z) = f_s(z,\omega)\,\frac{\delta(r)}{2\pi r}\,,$$

Medium wavenumber: $\quad k_m(z) = \dfrac{\omega}{c(z)}$

Hankel Transform Pair

$$f(r,z) = \int_0^\infty f(k_r,z)\, J_0(k_r r)\, k_r\, dk_r \,,$$
$$f(k_r,z) = \int_0^\infty f(r,z)\, J_0(k_r r)\, r\, dr \,,$$

Integral Transform Solution

Depth-separated Wave Equation

$$\left[\frac{d^2}{dz^2} - \left[k_r^2 - k_m^2(z)\right]\right] \psi_m(k_r,z) = \frac{f_s(z)}{2\pi} \,,$$

Superposition Principle

$$\psi_m(k_r,z) = \underbrace{\hat{\psi}_m(k_r,z)}_{\text{Source}} + \underbrace{A_m^+(k_r)\, \psi_m^+(k_r,z) + A_m^-(k_r)\, \psi_m^-(k_r,z)}_{\text{Homogeneous Solution}} \,,$$

Homogeneous Fluid Layers

$$c = \sqrt{\frac{K}{\rho}}$$

$$k_m(z) = k_m = \omega/c$$

Depth Solutions

$$\phi^+(k,z) = e^{ik_z z} \qquad \text{Downward Propagating}$$

$$\phi^-(k,z) = e^{-ik_z z} \qquad \text{Upward Propagating}$$

$$k_z = \sqrt{k_m^2 - k_r^2} \qquad \text{Vertical wavenumber}$$

Layers without Sources

$$\phi(r,z) = \int_0^\infty \left[\boxed{A^-} e^{-ik_z z} + \boxed{A^+} e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r \,.$$

2 Unknowns

Layers without Sources

$$\phi(r,z) = \int_0^\infty \left[A^- e^{-ik_z z} + A^+ e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r \,.$$

Interface Condition Parameters

Vertical Particle Displacements

$$\begin{aligned} w(r,z) &= \frac{\partial \phi}{\partial z} \\ &= \int_0^\infty \left[-ik_z A^- e^{-ik_z z} + ik_z A^+ e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r \,, \end{aligned}$$

Vertical Normal Stress

$$\begin{aligned} \sigma_{zz}(r,z) &= -p(r,z) \\ &= K\,\nabla^2 \phi(r,z) \\ &= -\rho\omega^2\, \phi(r,z) \\ &= -\rho\omega^2 \int_0^\infty \left[A^- e^{-ik_z z} + A^+ e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r \,. \end{aligned}$$

Homogeneous Fluid Layers

Simple Point Source

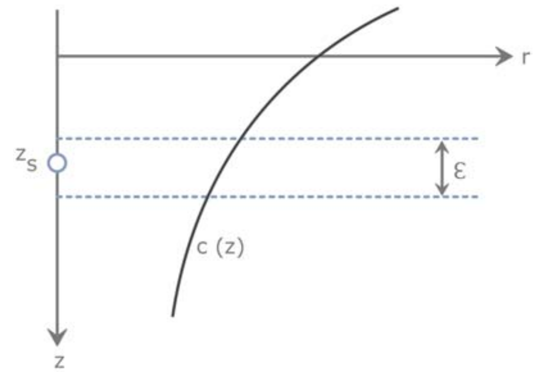
$$f_s(z,\omega) = S_\omega\, \delta(z - z_s)$$

$$\hat{\phi}(k_r, z) = \frac{S_\omega}{4\pi} \frac{e^{ik_z|z-z_s|}}{ik_z},$$

Displacements and Stresses

$$\hat{w}(r,z) = \frac{S_\omega}{4\pi} \int_0^\infty \mathrm{sign}(z - z_s)\, e^{ik_z|z-z_s|}\, J_0(k_r r)\, k_r\, dk_r$$

$$\hat{\sigma}_{zz}(r,z) = -\frac{S_\omega\, \rho\omega^2}{4\pi} \int_0^\infty \frac{e^{ik_z|z-z_s|}}{ik_z} J_0(k_r r)\, k_r d\, k_r$$



Point source in n^2-linear fluid medium

Gradient Fluid Layers

$$k_m^2(z) = \omega^2(az+b) \,,$$

Depth-separated Wave Equation

$$\left[\frac{d^2}{dz^2} - \left[k_r^2 - \omega^2(az+b)\right]\right]\phi(k_r,z) = 0 \,.$$

Depth-Solutions

$$\begin{aligned}
\phi^+(k_r,z) &= \mathrm{Ai}\,(\zeta) \,, \\
\phi^-(k_r,z) &= \mathrm{Ai}\,(\zeta) - i\,\mathrm{Bi}\,(\zeta) \,, \\
\zeta &= (\omega^2 a)^{-2/3}\left[k_r^2 - \omega^2(az+b)\right] \,.
\end{aligned}$$

See Fig 10.6 and 10.7, “Bessel Functions of Fractional Order,” in Abramowitz, M. and Stegun, I. *Handbook of Mathematical Functions*. Mineola NY: Dover, 1965

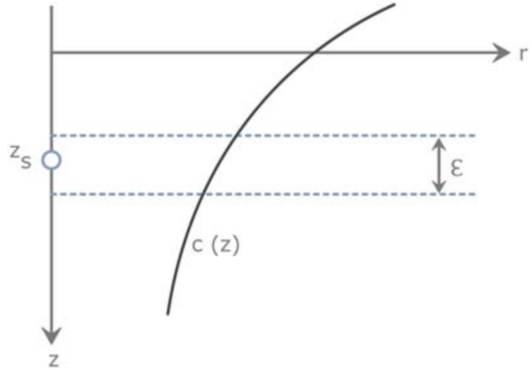
Depth-Solutions

$$\begin{aligned}\phi^{+}(k_r, z) &= \mathrm{Ai}\,(\zeta)\,,\\ \phi^{-}(k_r, z) &= \mathrm{Ai}\,(\zeta) - i\,\mathrm{Bi}\,(\zeta)\,,\\ \zeta &= (\omega^2 a)^{-2/3}\left[k_r^2 - \omega^2(az+b)\right]\,.\end{aligned}$$

Homogeneous Solution

$$\phi(r,z) = \int_0^\infty \left\{\boxed{A^{+}}\mathrm{Ai}\,(\zeta) + \boxed{A^{-}}\left[\mathrm{Ai}\,(\zeta) - i\,\mathrm{Bi}\,(\zeta)\right]\right\} J_0(k_r r)\, k_r\, dk_r\,,$$

2 Unknowns



Gradient Fluid Layers

Displacements and Stresses

$$
\begin{aligned}
w(r,z) &= -(\omega^2 a)^{1/3} \int_0^\infty \left\{ A^+ \mathrm{Ai}'(\zeta) + A^- \left[\mathrm{Ai}'(\zeta) - i\,\mathrm{Bi}'(\zeta)\right]\right\} J_0(k_r r)\, k_r\, dk_r \\
\sigma_{zz}(r,z) &= -p(r,z) \\
&= -\omega^2 \int_0^\infty \left\{ A^+ \mathrm{Ai}\,(\zeta) + A^- \left[\mathrm{Ai}\,(\zeta) - i\,\mathrm{Bi}\,(\zeta)\right]\right\} J_0(k_r r)\, k_r\, dk_r
\end{aligned}
$$

Source Contribution

$$
\widehat{\phi}(r,z) = -\frac{S_\omega}{4\pi} \int_0^\infty J_0(k_r r)\, k_r\, dk_r
$$

$$
\times \begin{cases}
\dfrac{2\,(\omega^2 a)^{-1/3} [\mathrm{Ai}\,(\zeta_s) - i\,\mathrm{Bi}\,(\zeta_s)]\mathrm{Ai}\,(\zeta)}{\mathrm{Ai}'(\zeta_s)[\mathrm{Ai}\,(\zeta_s) - i\,\mathrm{Bi}\,(\zeta_s)] - \mathrm{Ai}\,(\zeta_s)[\mathrm{Ai}'(\zeta_s) - i\,\mathrm{Bi}'(\zeta_s)]}, & a(z - z_s) \le 0 \\
\dfrac{2\,(\omega^2 a)^{-1/3} \mathrm{Ai}\,(\zeta_s)[\mathrm{Ai}\,(\zeta) - i\,\mathrm{Bi}\,(\zeta)]}{\mathrm{Ai}'(\zeta_s)[\mathrm{Ai}\,(\zeta_s) - i\,\mathrm{Bi}\,(\zeta_s)] - \mathrm{Ai}\,(\zeta_s)[\mathrm{Ai}'(\zeta_S) - i\,\mathrm{Bi}'(\zeta_s)]}, & a(z - z_s) \ge 0\,.
\end{cases}
$$

Homogeneous Elastic Layers

Scalar Displacement Potentials

$$
\begin{aligned}
u(r,z) &= \frac{\partial}{\partial r}\phi(r,z) + \frac{\partial^2}{\partial r \partial z}\psi(r,z)\,, \\
w(r,z) &= \frac{\partial}{\partial z}\phi(r,z) - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\psi(r,z)\,,
\end{aligned}
$$

$$
\begin{aligned}
\left[\nabla^2 + k_m^2\right]\phi(r,z,t) &= 0\,, \qquad k_m = \omega/c_p \qquad \text{Compressional waves} \\
\left[\nabla^2 + \kappa_m^2\right]\psi(r,z,t) &= 0\,, \qquad \kappa_m = \omega/c_s \qquad \text{Shear waves}
\end{aligned}
$$

Wave Speeds

$$
\begin{aligned}
c_p &= \sqrt{\frac{\lambda + 2\mu}{\rho}}\,, \\
c_s &= \sqrt{\frac{\mu}{\rho}}\,.
\end{aligned}
$$

Integral Representations

$$
\begin{aligned}
\phi(r,z) &= \int_0^\infty \left[\boxed{A^-} e^{-ik_z z} + \boxed{A^+} e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r\,, \\
\psi(r,z) &= \int_0^\infty k_r^{-1} \left[\boxed{B^-} e^{-i\kappa_z z} + \boxed{B^+} e^{i\kappa_z z}\right] J_0(k_r r)\, k_r\, dk_r\,,
\end{aligned}
$$

4 Unknowns

Vertical Wavenumbers

$$
\begin{aligned}
k_z &= \sqrt{k_m^2 - k_r^2}\,, \\
\kappa_z &= \sqrt{\kappa_m^2 - k_r^2}\,.
\end{aligned}
$$

Boundary Conditions

Table 4.1 Boundary conditions (= : continuous; 0 : vanishing; - : not involved).

Type	Field parameter				Number of Boundary Conditions
	w	u	σ_{zz}	σ_{rz}	
Fluid–vacuum	-	-	0	-	1
Fluid–fluid	=	-	=	-	2
Fluid–solid	=	-	=	0	3
Solid–vacuum	-	-	0	0	2
Solid–solid	=	=	=	=	4

Global Equations and Unknowns

Wavefield Unknowns		Boundary Conditions
0	Vacuum	
		2
4	Elastic Ice Cover	
		3
2	Fluid Water Column	
		2
2	Fluid Sediment Layer	
		3
4	Elastic Sediment Layer	
		4
2	Elastic Halfspace	
14 unknowns		14 equations