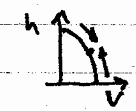
2.26 Lecture 4

1D duct flow - common variations

i) changes in cross-sectional area  (lect. 2)

ii) wall friction $(\rightarrow (M=1))$  (lect. 3)

iii) heating or cooling (today)

For starters, implement these one at a time.

<u>Friction</u>

Recall from incompressible flow:

Darcy friction factor: $\dfrac{f}{4} = \dfrac{\tau_w}{\frac{1}{2}\rho u^2}$ (check factor of 4 in white)

$f = f(Re, \underset{\substack{\uparrow \\ \text{roughness}}}{\varepsilon})$ (Moody chart)

e.g. for laminar flow, $f = \dfrac{64}{Re}$

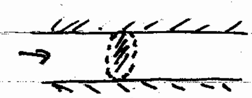
use <u>hydraulic diameter</u>: $D_H = \dfrac{4A}{P}$

For compressible flows:

Fanning friction factor: $f = \dfrac{\tau_w}{\frac{1}{2}\rho u^2}$

$f = f(M, Re, \varepsilon)$

Find f as a function of Mach #:



$\leftarrow$ Force/length $= \tau_w \pi D$ on perimeter

$$\frac{\text{Force}}{\text{length}\cdot\text{unit cross-section area}} = \frac{\tau_w \cancel{\pi} \cancel{D}}{(\cancel{\pi} D^{\cancel{2}}/4)} = \frac{2\frac{1}{2} f \rho u^2}{D}$$

put this in the momentum equation:

(mom.) $$\rho u \underbrace{\frac{du}{dx}}_{\frac{1}{2}\frac{d(u^2)}{dx}} = -\frac{2}{\rho}\frac{dP}{dx} - \frac{4}{D} f \rho u^2 \Rightarrow \frac{1}{u^2}\frac{du^2}{dx} + \frac{4}{D} f = -\frac{2}{\rho u^2}\frac{dP}{dx}$$

(energy) $$h_0 = h + \frac{u^2}{2} = \text{const.}$$

(mass) $$\rho u = J = \text{const.}$$

For a <u>perfect gas</u>

$$\underbrace{\frac{h_0}{c_p}}_{T_0} = \underbrace{\frac{h}{c_p}}_{T} + \frac{u^2}{2c_p} \Rightarrow c_0^2 = c^2 + \underbrace{\frac{\gamma R}{2c_p}}_{\frac{\gamma(c_p-c_v)}{2c_p}} u^2 = c_0^2 = c^2 + \frac{\gamma-1}{2}u^2$$

$$\Downarrow$$

$$\frac{dc^2}{dx} = -\frac{\gamma-1}{2}\frac{du^2}{dx}$$

$$M^2 = u^2/c^2$$

$$\frac{dM^2}{dx} = \frac{c^2\, du^2/dx - u^2\, dc^2/dx}{c^4} \Rightarrow \frac{1}{M^2}\frac{dM^2}{dx} = \frac{1}{u^2}\frac{du^2}{dx} - \frac{1}{c^2}\frac{dc^2}{dx}$$

$$\frac{1}{M^2}\frac{dM^2}{dx} = \frac{du^2}{dx}\left(\frac{1}{u^2} + \frac{\gamma-1}{2}\frac{1}{c^2}\right)$$

$$= \underbrace{\frac{1}{u^2}\frac{du^2}{dx}}_{\text{use this in cons of mom.}}\left(1 + \frac{\gamma-1}{2}M^2\right)$$

$$P = \rho R T = \rho\frac{c^2}{\gamma}$$

$$\frac{dP}{dx} = \frac{1}{\gamma}\left[\rho\frac{dc^2}{dx} + c^2\frac{d\rho}{dx}\right] = \frac{1}{\gamma}\left[-\rho\frac{\gamma-1}{2}\frac{du^2}{dx} + c^2\frac{d\rho}{dx}\right]$$

$$J = \rho u \Rightarrow \rho\frac{du}{dx} = -u\frac{d\rho}{dx}$$

$$= \frac{1}{\gamma}\left[-\rho\frac{\gamma-1}{2}\frac{du^2}{dx} + c^2\left(\cancel{\frac{u}{\rho}}\right)\left(\frac{-\rho}{u}\right)\frac{du}{dx}\right]$$

$$= \frac{1}{\gamma}\left[-\rho\frac{\gamma-1}{2}\frac{du^2}{dx} - \frac{c^2\rho}{u^2}\frac{1}{2}\frac{du^2}{dx}\right]$$

$$\underbrace{\frac{2\,dP}{\rho u^2\,dx}}_{\text{use this in cons. of mom.}} = \frac{1}{\gamma}\left[-\frac{(\gamma-1)}{u^2} - \frac{c^2}{u^4}\right]\frac{du^2}{dx} = \frac{1}{\gamma}\left[-(\gamma-1) - \frac{1}{M^2}\right]\frac{1}{u^2}\frac{du^2}{dx}$$

$$\frac{1}{M^2}\frac{dM^2}{dx}\frac{1}{(1+\frac{\gamma-1}{2}M^2)} + \frac{4f}{D} = \frac{1}{\gamma}\left[(\gamma-1) + \frac{1}{M^2}\right]\frac{1}{M^2}\frac{dM^2}{dx}\frac{1}{(1+\frac{\gamma-1}{2}M^2)}$$

$$\frac{4f}{D} = \frac{dM^2}{dx}\frac{1}{M^4(1+\frac{\gamma-1}{2}M^2)}\left\{\frac{\cancel{\gamma}-M^2}{\gamma} + \frac{M^2}{\gamma M^2} - \cancel{1}\right\}$$

$$\boxed{\frac{4f}{D} = \frac{1-M^2}{\gamma M^4(1+\frac{\gamma-1}{2}M^2)}\frac{dM^2}{dx}}$$

Note: for $f>0$

$M<1 \Rightarrow \frac{dM^2}{dx} > 0 \; (\therefore M \to 1)$

$M>1 \Rightarrow \frac{dM^2}{dx} < 0 \; (\therefore M \to 1)$

just as we saw from the Fanno lines!

Finally, recall that if $M \to 1$ the flow can no longer accelerate (decelerate) → "frictionally choked."

Find this length by integrating $x: 0 \to L_{max}$, $M: M \to 1$

$$\boxed{\frac{4\bar{f}L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2(1+\frac{\gamma-1}{2}M^2)}}$$

where $\bar{f} \equiv \frac{1}{L}\int_0^L f\,dx$

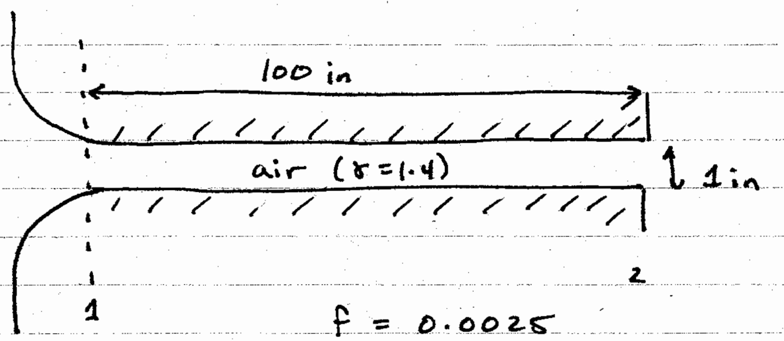
Example

reservoir

$P_r = 10$ atm

$T_r : 300$ k

choked



$P_a = 1$ atm

Find: Velocity @ 2 (exit velocity)

P and M @ 1

Compare mass flow w. that of a short converging nozzle w. same res. conditions

$\frac{P}{P_o} = \frac{1\text{atm}}{10\text{atm}} = 0.1$ (Note from table 5.1 $P_*/P_o = 0.5283$ ∴ Flow is super ~~subsonic~~)

$\Rightarrow$ choked ~~chock~~ ∴ at throat:

$M = 1 \qquad \frac{c}{c_o} = 0.9129 \qquad \frac{\rho}{\rho_o} = 0.6339$

$M = \frac{u}{c} \Rightarrow u = Mc = Mc_o(0.9129) = 0.9129\, c_o$

$\dot{m} = \rho u A = (0.6339)\rho_o (0.9129) c_o (1 \text{in})$

$$\boxed{\dot{m} = 0.5787\, \rho_o c_o A}$$

For the pipe:

"chokked" $\Rightarrow L = L_{max} \Rightarrow 4fL_{max}/D = 1.00 \Rightarrow \boxed{M_1 = 0.51}$

(show plot)

$M = 0.51 \Rightarrow P/P_o = 0.8374 \Rightarrow \boxed{P_1 = 8.374 \text{ atm}}$

$\dot{m} = \rho u A$ $\qquad \frac{\rho}{\rho_0} = 0.8809 \qquad \frac{c}{c_0} = 0.9750$

$= 0.8809\, \rho_0 (0.51)(0.9750) c_0 A$ $\quad u/c = M \Rightarrow u = Mc = M \frac{c}{c_0} c_0$

$$\boxed{\dot{m} = 0.438\, \rho_0 c_0 A}$$

(5) exit: $c^2 + \frac{\gamma - 1}{2} u^2 = c_0^2 \qquad M = 1 \Rightarrow c^2 = u^2$

$$\Rightarrow \frac{\gamma+1}{2} u^2 = c_0^2$$

$$u^2 = \frac{2}{\gamma+1} T_0 \gamma R = \frac{2}{2.4}(300\cancel{K})\, 1.4 \left(287 \frac{m^2}{s^2 \cancel{K}}\right)$$

$$\boxed{u = 316.9 \ m/s}$$

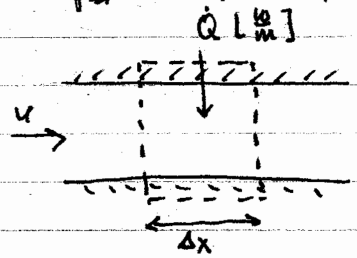
Note cons of energy is the <u>same</u> for the exit velocity in the nozzle case. But $\dot{m}$ is different b.c. ρ is different.

Check if flow is really choked:

$$\frac{P_2}{P_0} = \frac{\rho_2 \cancel{R_2} T_2}{\rho_0 \cancel{R_0} T_0} = \frac{\rho_2 c_2^2}{\rho_0 c_0^2} \qquad \dot{m} = \rho_2 u_2 A = \rho_0 c_0 \cancel{A}\, 0.5787$$

Frictionless flow w. heat added

steady state, 1D, const. cross-section



(mass) $\dot{m} = \rho u A = \text{const}$

$J = \rho u = \text{const}$

(mom) $\rho u \frac{du}{dx} = -\frac{dP}{dx} \quad \Rightarrow \quad \boxed{P + \rho u^2 = \text{const}}$

$$\frac{d}{dt}\int \cancel{\ } \, dV = -\int P \cdot n \, dS - \int \rho \bar{u} \, \bar{u} \cdot \hat{n} \, dS$$

(energy) $$\frac{d}{dt}\int \cancel{(\ )} \, dV = -\int \rho \left(h + \tfrac{1}{2}u^2\right) \bar{u} \cdot \hat{n} \, dS + \dot{Q} \Delta x$$

neglect visc. + P.E.

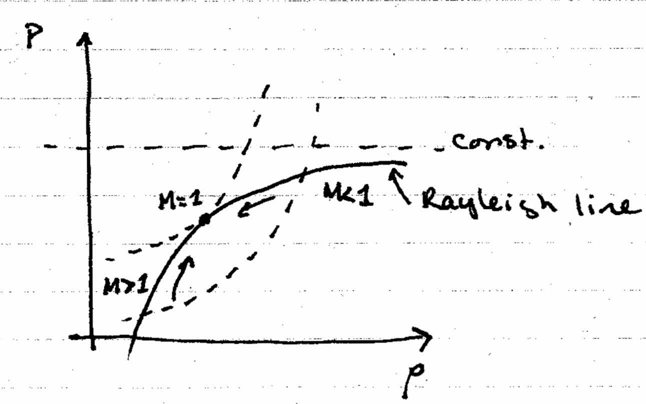
$$0 = \frac{-\rho u A \left(h + \tfrac{1}{2}u^2\right)\big|_x + \rho u A \left(h + \tfrac{1}{2}u^2\right)_{x+\Delta x}}{\Delta x} + \dot{Q} \cancel{\Delta x}$$

$$\boxed{\dot{m} \frac{d}{dx}\left(h + \tfrac{1}{2}u^2\right) = \dot{Q}}$$

From e.g. condensation, evaporation, combustion

Combine mass + mom:

$$\boxed{P = \text{const} - \frac{J^2}{\rho}}$$



Lines of const. entropy

$$\cancel{\Delta} s^{0} = c_v \ln\left(\frac{P}{P_0}\right) - c_p \ln\left(\frac{\rho}{\rho_0}\right)$$

$$\left(\frac{P}{P_0}\right)^{c_v} = \left(\frac{\rho}{\rho_0}\right)^{c_p}$$

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}.$$

Tangent @ $\left(\frac{\partial P}{\partial \rho}\right)_{Rayleigh} = \underbrace{\left(\frac{\partial P}{\partial \rho}\right)_s}_{c^2} \Rightarrow \boxed{M = 1}$ again!

$$\frac{J^2}{\rho^2} = u^2$$

Heating:

$M < 1$ $\quad \frac{dM}{dx} > 0 \quad M \to 1$

$M > 1$ $\quad \frac{dM}{dx} < 0 \quad M \to 1$ (opposite for cooling)

Too much heat $\Rightarrow$ shock $\Rightarrow$ maximum $\dot{Q}_L$ (similar to max L we saw w. friction)

Need to find out how other properties change as a function of T_0 ("temp that the stream would assume if it was adiabatically decelerated to zero velocity.")

$$\frac{dh_0}{dx} = \frac{\dot{Q}_L}{\dot{m}} \Rightarrow h_{02} - h_{01} = \int_{x_1}^{x_2} \frac{\dot{Q}}{\dot{m}}\, dx \equiv \dot{q}$$

$$= c_p (T_{02} - T_{01})$$

Find relations of ratios btwn stream properties @ pt. 1 and pt. 2 (pg 195-196 in handout)