2.25 ADVANCED FLUID MECHANICS

QUIZ 1 **FALL 2004** October 14, 2004

Problem 1

A) This is a straightforward simple application of simple buoyancy ideas.

i) Force balance gives $$F_R + mg = \rho g\left(\pi R^2 h + \frac{2}{3}\pi R^3\right)$$

or $$F_R = \rho g\left(\pi R^2 h + \frac{2}{3}\pi R^3\right) - mg$$

for specific values given (in cgs units) $$F_R = (1)(1000)\left(\pi \cdot 1^2 \cdot 5 + \frac{2}{3}\pi \cdot 1^3\right) - 15(1000)$$

$$F_R = \left(\frac{2\pi}{3}\right)(1000) = 2749 \text{ dynes}$$

ii) The 'float' is neutrally buoyant when $F_R \to 0$ then

$$mg = \rho g V_{new} = \rho g\left(\frac{2}{3}\pi R^3 + \pi R^2 h_{new}\right) \quad \Rightarrow \quad V_{new} = \frac{m}{\rho} = 15 cm^3$$

If compression is slow then we can assume it is isothermal so $P_1V_1 = P_2V_2$ with $P_1 \cong 1$ atm.

$$\left(\frac{P_2}{P_1}\right) = \frac{V_1}{V_2} = \frac{17\pi/3}{15} = \frac{17\pi}{45}$$

The new height of the membrane is $\frac{2}{3}\pi \cdot 1^3 + \pi \cdot 1^2 \cdot h_2 = 15 \;\Rightarrow h_2 = 4.107 cm$

iii) As the pressure increases beyond this value, volume decreases further => the float valve sinks. This results in further compression of air and so it sinks further. Final resting place (labeled point '3') is on bottom of tank.

$$P_3 = P_2 + \rho g L = 1.5P_a + \rho g L \; = 1.6P_a = 1.6\times 10^5 \, N/m^2$$

Volume of air at this level is given by expression for adiabatic compression $P_3V_3^{\gamma} = P_1V_1^{\gamma}$

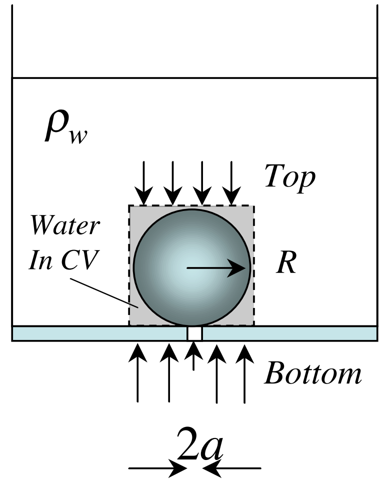
$$V_3 = V_1\left(P_a/P_3\right)^{1/\gamma} \qquad \Rightarrow V_3 = \left(\frac{17\pi}{3}\right)\left(\frac{1}{1.6}\right)^{1/1.4} = 12.72 cm^3$$

final height is $\frac{2}{3}\pi \cdot 1^3 + \pi \cdot 1^2 \cdot h_3 = 12.72 cm^3 \qquad \Rightarrow \qquad h_3 = 3.38 cm$

B) Here we must be more careful as the buoyant force concept only applies to fully submerged fraction.

For sphere to remain sitting on bottom of tank requires

$$P_T A_T + W_{sphere} + W_{water} \geq P_B \pi\left(R^2 - a^2\right) + P_a \pi a^2$$



substitute $P_T = P_a + \rho_w g(L-2R)$, $\quad P_B = P_a + \rho_w gL$

$$W_{water} = \rho_w g\left[\pi R^2(2R) - \frac{4}{3}\pi R^3\right] \qquad W_{sphere} = \frac{4}{3}\pi R^3 \rho_s g = \frac{4}{3}\pi R^3 \rho_w g X$$

P_a terms cancel throughout as expected and we find:

$$\rho_w g(L-2R)\pi R^2 + X\rho_w g\frac{4}{3}\pi R^3 + \rho_w g\left(\frac{2}{3}\pi R^3\right) \geq \rho_w gL\pi\left(R^2 - a^2\right)$$

Canceling $\rho_w g\pi R^2$ throughout gives

$$(L-2R) + \frac{4}{3}XR + \frac{2}{3}R \geq L\left(1 - a^2/R^2\right)$$

Or

$$\boxed{X \geq 1 - \frac{3}{4}\left(\frac{L}{R}\right)\left(\frac{a^2}{R^2}\right)}$$

Alternately; consider a buoyancy force arising from a displaced volume consisting of the sphere *minus a "core"* of volume $V_{core} = \pi a\ V_{core} = \pi a^2(2R)$. We thus get a force balance of:

$$p_T\pi a^2 + \tfrac{4}{3}\pi R^3 X\rho_w g \geq \left[\tfrac{4}{3}\pi R^3 - 2R\pi a^2\right]\rho_w g + p_a\pi a^2 \quad \Rightarrow \text{ same result!}$$

ii) For given values $\quad X \geq 1 - \frac{3}{4}\left(\frac{0.8}{0.02}\right)\left(\frac{0.002}{0.02}\right)^2 = 0.70$

As the height *decreases*, the required sphere density *increases* (otherwise the buoyant force of the displaced water upwards overwhelms the force pushing down from above).

iii) After the sphere has popped off we have a tank what is open to atmosphere and it will drain at velocity $V_{surface} = dL/dt$.

Apply steady Bernoulli equation : $\quad P_a + \frac{1}{2}\rho\left(\frac{dL}{dt}\right)^2 + \rho gL = P_a + \frac{1}{2}\rho V_2^{\,2} + 0$

Conservation of mass gives $A_1V_1 = A_2V_2 \quad \Rightarrow \quad V_2 = V_1\frac{A_{Tank}}{A_{Hole}} = \frac{R_T^{\,2}}{a^2}\left(\frac{dL}{dt}\right)$

Combining gives $\quad 2gL = \left(\frac{dL}{dt}\right)^2\left\{\frac{R_T^{\,4}}{a^4} - 1\right\}$

Note that if the tank is an infinite reservoir then we have $V_1 \to 0$ and $V_2 \cong \sqrt{2gL}$. However this is not necessarily true in this case (and as the level drops and V_2 decreases it will become a progressively less good approximation). However for a finite size tank we need to solve: $\quad \frac{dL}{dt} = -\frac{\sqrt{2gL}}{\left(R_T^{\,4}/a^4 - 1\right)^{1/2}}$

The solution is thus: $\quad \left[\frac{1}{2}\sqrt{L}\right]_{L_0}^{L} = -\frac{\sqrt{2g}}{\left(R_T^{\,4}/a^4 - 1\right)^{1/2}}t \quad$ or $\quad \boxed{L = \left\{L_0^{\,1/2} - \frac{\sqrt{g/2}}{\left(R_T^{\,4}/a^4 - 1\right)^{1/2}}t\right\}^2}$

the drainage time is $\quad \boxed{t_{drain} = \left(\frac{R_T^{\,4}}{a^4} - 1\right)^{1/2}\sqrt{\frac{2L_0}{g}}}$

PROBLEM 2

A small tube of length $2L$ is submerged in a pool of liquid (density ρ, pressure P_a). The tube is open at both ends, and filled with liquid. However, it is made of piezoelectric material, and its cross-sectional area A (which is uniform over the tube's length) can be controlled by the application of an electric voltage.
Suppose that, by the application of a suitable voltage, the tube's area is reduced in time according to a specified function $A(t)$, which is monotonically decreasing. Assuming that the flow inside the tube is incompressible and inviscid, obtain, in terms of the given quantities and the function $A(t)$ and its derivatives, expressions for

(a) the flow speed u at a station x in the tube, and

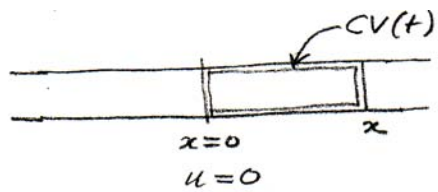
(b) the pressure at the tube's centerpoint, $x = 0$

(c) Are your results in (a) and (b) valid for increasing as well as decreasing $A(t)$? .

SOLUTION TO PROBLEM 2

(a) simplest method for u:

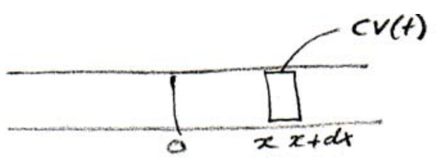
$$\rho \frac{d}{dt}(Ax) + \rho u A = 0$$



$$u = -\frac{1}{A}\frac{dA}{dt} \cdot x$$

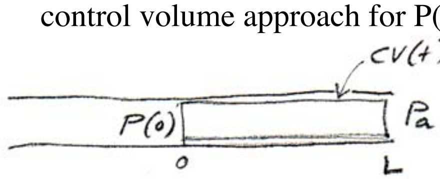
another method for u:

$$\rho \frac{\partial}{\partial t}(A dx) + \rho \frac{\partial}{0x}(uA) dx = 0$$



$$\boxed{u = -\frac{1}{A}\frac{dA}{dt} \cdot x}$$

control volume approach for P(0):



$$\frac{d}{dt}\int_{cv(t)} \rho v_x dV + \int \rho v_x (\vec{v} - \vec{v}_{cs}) \cdot d\vec{A} = (P_a)_{cv}$$

$$\frac{d}{dt}\int_0^L \rho u(x,t) A dx + \rho u^2(L) A = \left[P(0) - P_a\right] A$$

Plug in for $u(x,t)$ from (a), get:

$$\boxed{P(0,t)-P_a=\frac{\rho L^2}{A^2}\left(\frac{dA}{dt}\right)^2-\frac{\rho L^2}{2A}\frac{d^2A}{dt^2}}$$

Euler equation approach for *P(0):*

$$\frac{\partial P}{\partial x}=-\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)$$

$$=-\rho\left[-\frac{d}{dt}\left(\frac{1}{A}\frac{dA}{dt}\right)x+\left(\frac{1}{A}\frac{dA}{dt}\right)^2 x\right] \qquad \underline{\text{using (a)}}$$

Integrate from $x=0$ to $x=L$, get same result.

Unsteady Bernoulli equation

$$\rho\int_0^L \frac{\partial u}{\partial t}\cdot dx+\frac{\rho u^2(L)}{2}+P_a=P(0)$$

$u=-\frac{1}{A}\frac{dA}{dt}x$ get <u>same thing</u>

$P(L)\cong P_a-\frac{1}{2}\rho u^2(L)$ as <u>first approximation</u>.

Answer in (a) OK, <u>answer in (b) must be modified to account for different $P(L)$</u>, as indicated above.