The Generalized Boundary Layer Equations

Gareth H. McKinley (MIT-HML),
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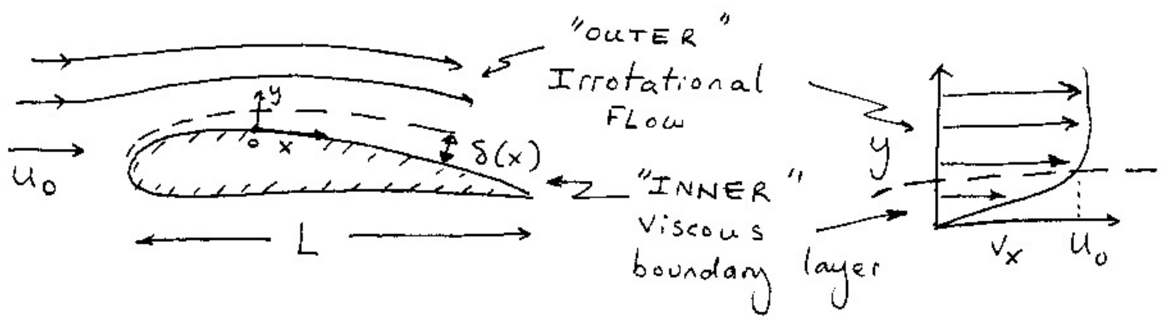
We have seen that, in general, high Reynolds number flow past a slender body such as an airfoil can be considered as an irrotational "outer" flow (that can be determined in principle at least from potential flow theory) and a thin "inner" flow regime in which viscous effects are important and lead to the generation of vorticity. This thin inner region is often referred to as a *viscous boundary layer* and denoted generically $\delta(x)$. Here we shall consider the "inner flow" region in detail and wish to see what simplifications to the equations of motion are possible due to the thinness of the boundary layer.

We consider a 2D boundary layer next to a solid wall on which the no-slip boundary condition is to be applied. We shall use a "boundary layer" coordinate system in which x is along the surface and y is normal to the surface. A similar analysis can be performed in three-dimensions but requires an explicit assumption of the lateral scale of the three-dimensional flow. The following analysis is also applicable to boundary layers on axisymmetric objects provided that the boundary layer is thin enough that local curvature terms are always negligible (i.e. $\delta(x)/R(x) \ll 1$).



The boundary layer thickness grows with x but is small with respect to the length scale of the surface $\delta(x)/L << 1$. We proceed as follows:

i) An estimate of the thickness where <u>both</u> inertial and viscous forces balance each other is (as we have seen before)

$$\left.\begin{aligned} \left(\rho v_x \frac{\partial v_x}{\partial x}\right) &\sim \mu \frac{\partial^2 v_x}{\partial y^2} \\ \rho U_0 \frac{U_0}{L} &\sim \mu \frac{U_0}{\delta^2} \end{aligned}\right\} \qquad \delta \sim \sqrt{\frac{\mu L}{U_\infty}} \qquad \text{or} \qquad \frac{\delta}{L} \sim \sqrt{\frac{1}{Re_L}}$$

ii) Next consider the continuity equation: $\dfrac{\partial v_x}{\partial x}+\dfrac{\partial v_y}{\partial y}=0$

We express this in dimensionless form with scaled variables $x^* = x/L$, $y^* = y/\delta$ and $u = v_x/U_0$, $v = v_y/V$ (where V is, at the moment, unknown):

$$\frac{U_\infty}{L}\left(\frac{\partial u}{\partial x^*}\right)+\frac{V}{\delta}\left(\frac{\partial v}{\partial y^*}\right)=0$$

If we have picked the characteristic y-component V correctly (so that the scaled partial derivatives, $\partial u/\partial x^*$ and $\partial v/\partial y^*$ are both 0(1)) then we expect:

$$V \sim \left(\frac{\delta}{L}\right)U_0$$

The variations in the *y*-component of the velocity are thus much smaller than the variations in v_x (at least within the boundary layer). This is analogous to our approach in Lubrication Theory so far.

iii) We next consider the magnitude of pressure variations within the boundary layer. The first order solution to this flow problem at *Re* >>1 is the inviscid flow given by Euler's Equation (or Bernoulli's equation) and we thus expect that $P_c \sim \rho U_0{}^2$.
The pressure field obtained from Bernoulli's equation gives an excellent description of the actual pressure field measured experimentally for slender bodies. This should be expected since introducing viscosity μ into the problem affects the tangential shear stresses only [you should be able to show that along a solid surface $\mu \partial v_i/\partial x_i = 0$ for $i = j$].

The full nondimensional equations of motion in the 2D boundary layer are thus:

Scalings: $x^* = x/L$ $\quad u = v_x/U_0$ $\quad p^* = p\big/\rho U_0^2$

$y^* = y/\delta$ $\quad v = v_y\big/(\delta U_\infty/L)$ $\quad \delta = L\big/\sqrt{\text{Re}} = \sqrt{\nu L/U_\infty}$

x-component:
$$u\frac{\partial u}{\partial x^*}+v\frac{\partial u}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{Re}\frac{\partial^2 u}{\partial x^{*2}}+\frac{\partial^2 u}{\partial y^{*2}} \qquad (1)$$

y-component:
$$\frac{1}{Re}\left(u\frac{\partial v}{\partial x^*}+v\frac{\partial v}{\partial y^*}\right)=-\frac{\partial p^*}{\partial y^*}+\frac{1}{Re^2}\frac{\partial^2 v}{\partial x^{*2}}+\frac{1}{Re}\frac{\partial^2 v}{\partial y^{*2}} \qquad (2)$$

continuity:
$$\frac{\partial u}{\partial x^*}+\frac{\partial v}{\partial y^*}=0 \qquad (3)$$

For high Reynolds number flows these equations can be simplified; we ignore all items $O(1/Re)$, $O\left(1/Re^2\right)$. The resulting equations when expressed in *dimensional* terms are:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2} \qquad (4)$$

$$0 = -\frac{\partial p}{\partial y} \qquad (5)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (6)$$

Note the difference between these equations and the lubrication analysis (for quasi-steady, locally fully-developed flows) we have discussed previously. In the flow direction, eq. (4) shows that *inertial, viscous* and *pressure* forces are all important. In the lubrication scaling only pressure and viscous forces remain.

Finally a note about pressure fields.

Equation (5) tells us that to leading order there is no variation in the pressure across the thickness of the boundary layer. The pressure field inside the boundary layer can thus be equated with the pressure field in the inviscid flow far from the solid surface.

This outer pressure field is said to be "imposed" on the inner flow and can be determined by solving the inviscid Euler equations (or by using Bernouilli…) *a priori* (for a given p_∞, U_0).

The boundary conditions for the equations are given by (in dimensionless terms):

$u(x^*,0) = 0$	No slip
$v(x^*,0) = 0$	No Penetration
$u(x^*,\infty) = U(x^*)$	Match Potential flow as $y^* \to \infty$
$u(0,y^*) = U_0(y^*)$	Initial Condition

Note that because of the nonlinear terms, such as $u_x \, \partial u_x/\partial x$ etc., these equations are not *elliptic* (like the lubrication equations) but rather are *PARABOLIC* convection-diffusion equations and thus require an initial condition. Here x plays the role of a time-like variable describing *diffusion* in the y-direction as convection in the x-direction increases.

The fundamental concept of the *boundary layer equations* given by (1–3) or (4–6) is that we introduce a "stretching" of the 'small' variable (in this case the length scale y) so that terms in the equations become of order $O(1)$. At a position x we know $\delta(x) \sim \sqrt{\nu x/U_0}$ and we "stretch" y by rescaling it (i.e. by dividing by a small parameter δ) to become $y^* = y/\delta$ where $0 \le y^* < 1$:

$$\text{Hence} \quad y^* = \sqrt{\frac{U_0}{\nu}}\, x^{-1/2} y \qquad \text{or} \qquad y^* = \sqrt{\mathrm{Re}_x}\,(y/x)$$

where Re_x is a local Reynolds number $\mathrm{Re}_x = U_0 x/\nu$

This rescaling of the equation is at the cornerstone of analytic techniques for many transport phenomena. For further information see L.G. Leal "Laminar Flow And Convective Processes" 1992 (Butterworths), or W. M. Deen "Analysis of Transport Phenomena" 2000 (Oxford).