6.895 Theory of Parallel Systems	Due: Thursday, November 27th

Problems 12–15

Problem 12. Denote the butterfly network running from major cycles to minor cycles by W, let its "transpose" W^{T} be the butterfly network running from minor cycles to major cycles. Thus, the Beneš network is essentially the network WW^{T}. We've seen that WW^{T} can route any permutation (offline) by node-disjoint paths.

Problem 12.a. Show that the network $WWWW$, formed by 4 butterfly networks connected in tandem, can route any permutation through node-disjoint paths. (*Hint:* Show that WW can implement the ***bit-reverse*** permutation.)

Problem 12.b. (*Harder.*) Show that the network WWW can route any permutation by node-disjoint paths.

Problem 12.c. (*Open research problem.*) Can the network WW route any permutation by node-disjoint paths?

Problem 13. Shuffling

Problem 13.a. Show that r perfect out-shuffles return an $N = 2^d$-card deck to its original order if and only if $2^r \equiv 1 \pmod{N-1}$.

Problem 13.b. Show that 8 perfect out-shuffles are necessary and sufficient to restore a 52-card deck to its original order.

Problem 13.c. How many perfect out-shuffles are necessary and sufficient to restore a 54-card deck to its original order?

Problem 14. De Bruijn

Problem 14.a. Construct a 32-bit de Bruijn sequence.

Problem 14.b. Prove that the bisection width of an N-node de Bruijn network is $O(N/\lg N)$. If you wish, you may use the fact that the bisection width of an N-node shuffle-exchange network is $O(N/\lg N)$.

Problem 15. Kautz networks

A ***(d, k) Kautz network*** is defined as follows:

- Each node is a word $\langle x_1, x_2, \ldots, x_k \rangle$ on the alphabet $\{0, 1, \ldots, d\}$, where $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, k-1$.
- For each node $\langle x_1, x_2, \ldots, x_k \rangle$ and all $z \in \{0, 1, \ldots, d\} - \{x_k\}$, an edge exists from $\langle x_1, x_2, \ldots, x_k \rangle$ to $\langle x_2, \ldots, x_k, z \rangle$.

Problem 15.a. Draw a $(2, 3)$ Kautz network.

Problem 15.b. Prove that every node in a (d, k) Kautz network has out-degree d and in-degree d. Prove (careful: two cases) that the diameter is k.

Problem 15.c. How many nodes N does a (d, k) Kautz network contain in terms of d and k? (Give a precise answer, not a bound.) Prove it.

Problem 15.d. Conjecture a good upper bound on the bisection width of a (d, k) Kautz network. *Optional:* Prove your bound.

Problem 15.e. Some architects have recently become enamored with the idea of interconnecting the processors of a parallel computer with a (d, k) Kautz network, typically with $d \geq 4$. List salient pros and cons.