# 6.003: Signals and Systems

## CT Fourier Transform

*April 8, 2010*

# CT Fourier Transform

Representing signals by their frequency content.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{("analysis" equation)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{("synthesis" equation)}$$

- generalizes Fourier series to represent aperiodic signals.
- equals Laplace transform $X(s)|_{s=\jmath\omega}$ if ROC includes $j\omega$ axis.
  $\rightarrow$ inherits properties of Laplace transform.
- complex-valued function of **real** domain $\omega$.
- simple "inverse" relation
  $\rightarrow$ more general than table-lookup method for inverse Laplace.
  $\rightarrow$ "duality."
- **filtering**.
- **applications in physics**.

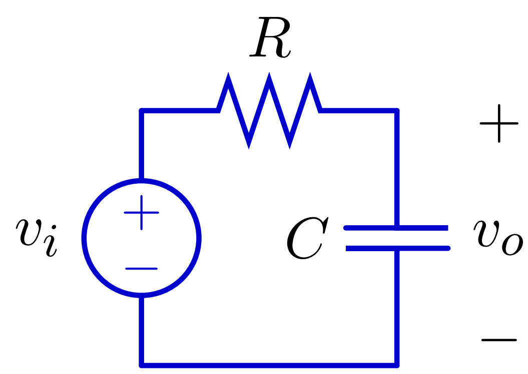
# Filtering

Notion of a filter.

LTI systems

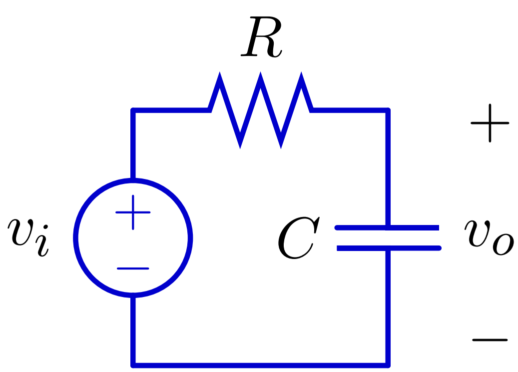
- cannot create new frequencies.
- can only scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit



# Lowpass Filter

Calculate the frequency response of an RC circuit.



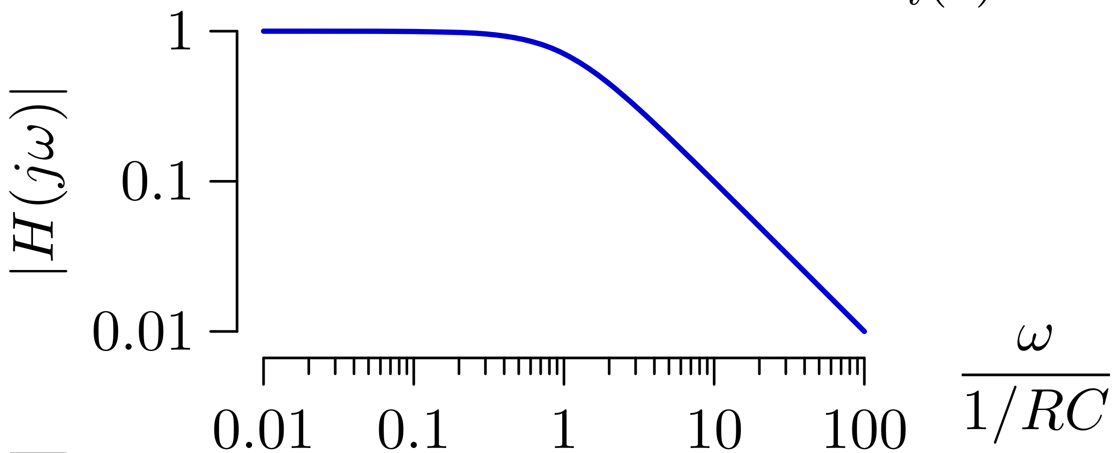

KVL: $v_i(t) = Ri(t) + v_o(t)$

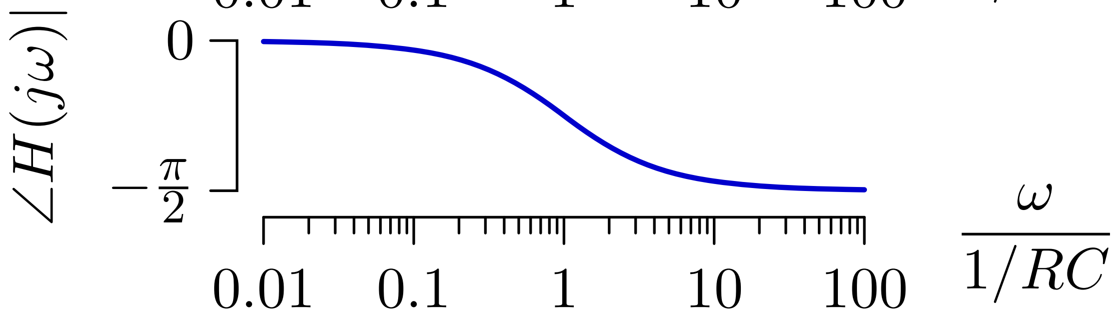
C: $i(t) = C\dot{v}_o(t)$

Solving: $v_i(t) = RC\dot{v}_o(t) + v_o(t)$

$$V_i(s) = (1 + sRC)V_o(s)$$

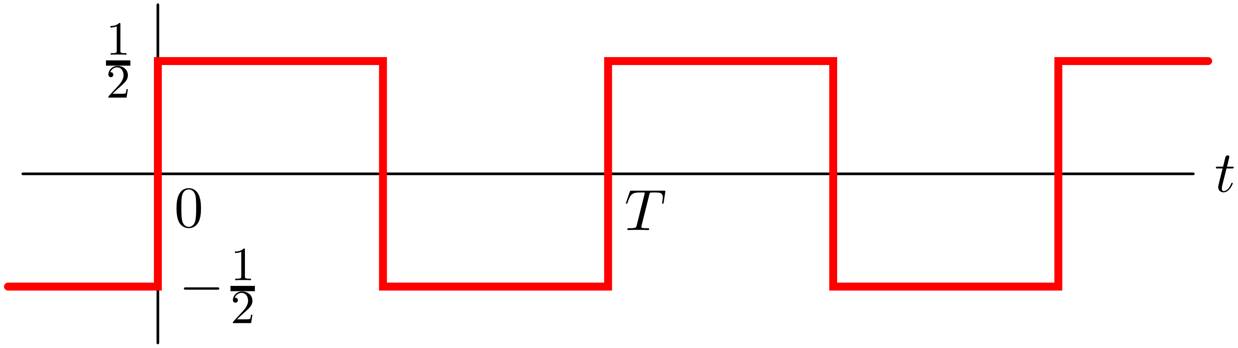
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$



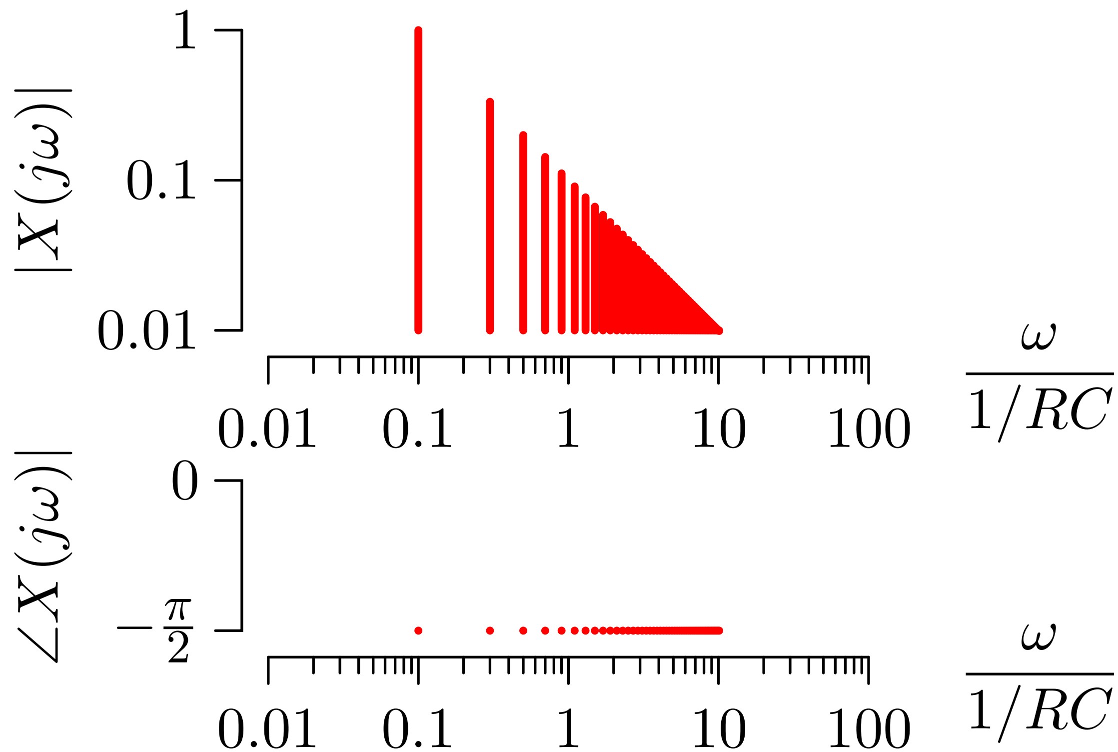



# Lowpass Filtering

Let the input be a square wave.

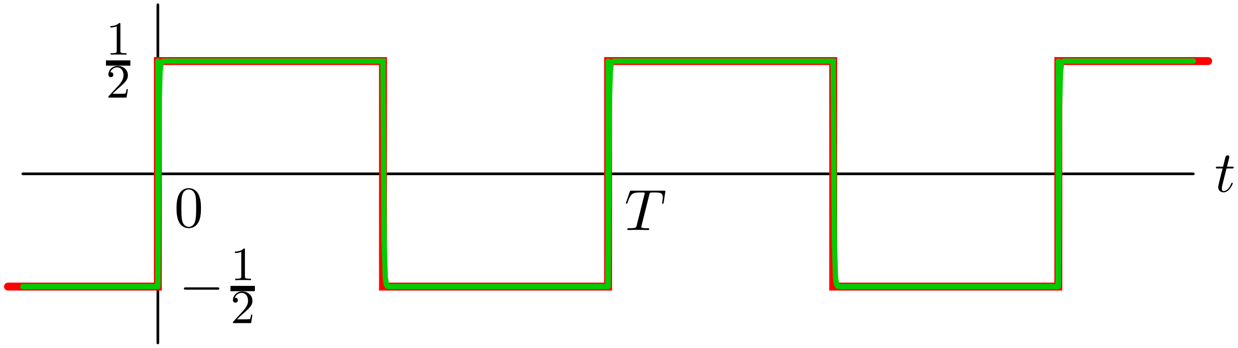



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

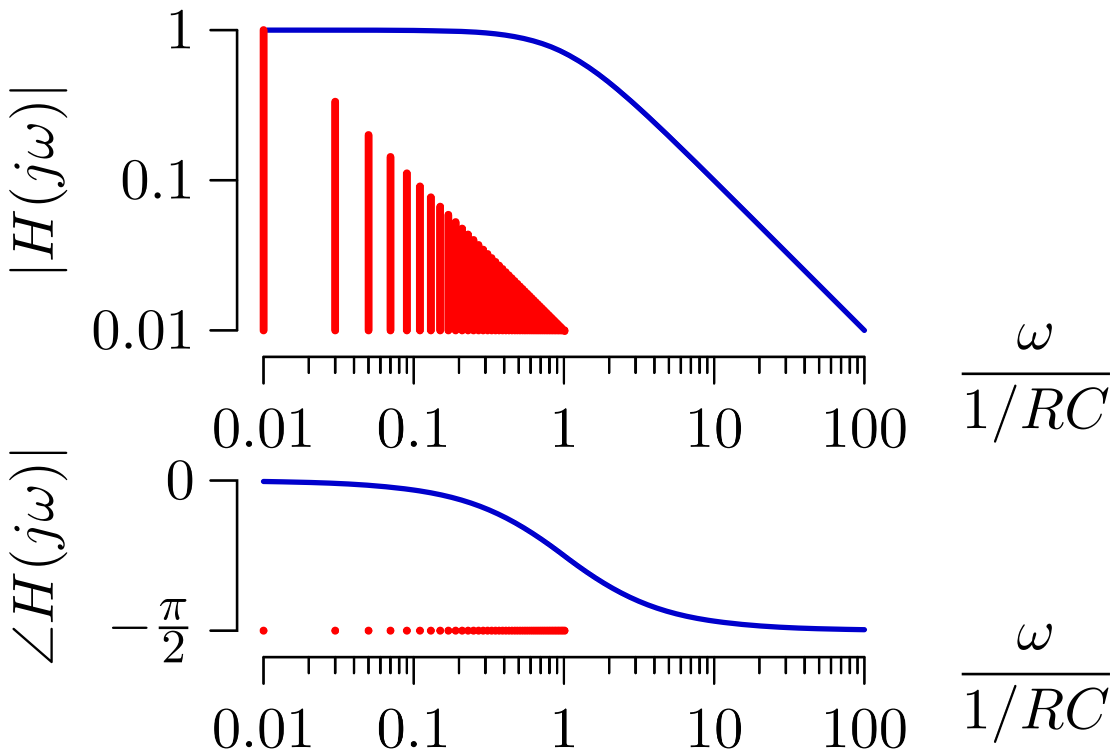


# Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

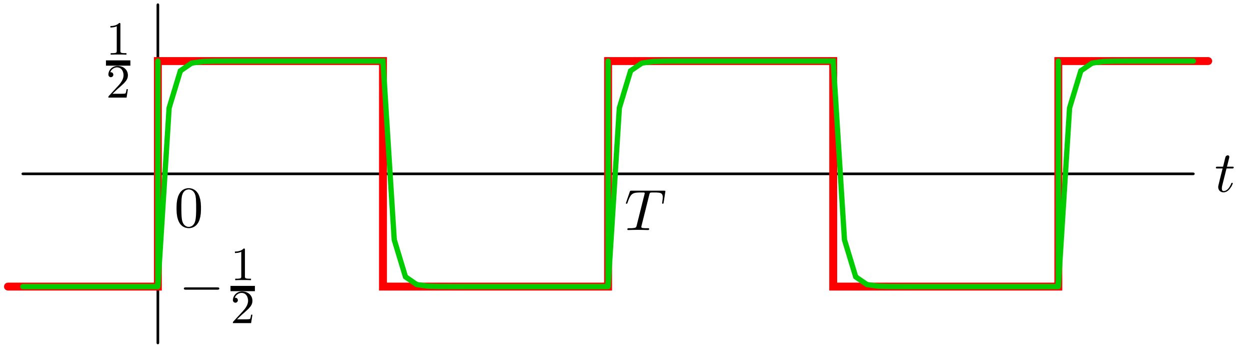



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

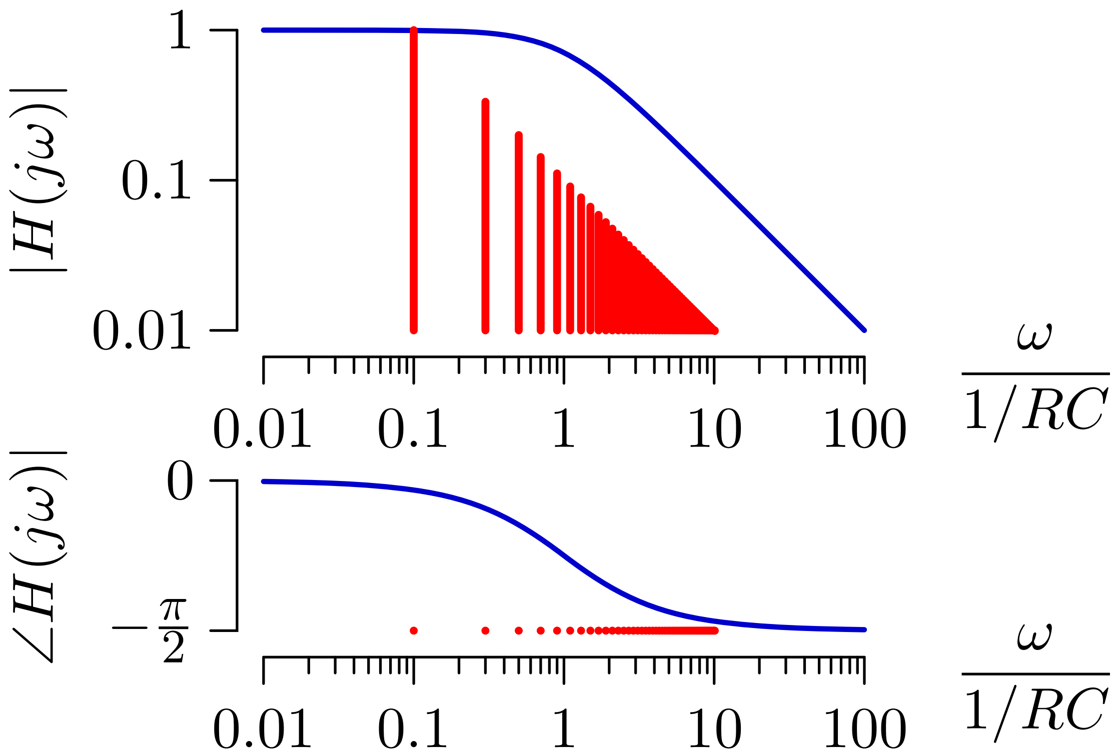


# Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

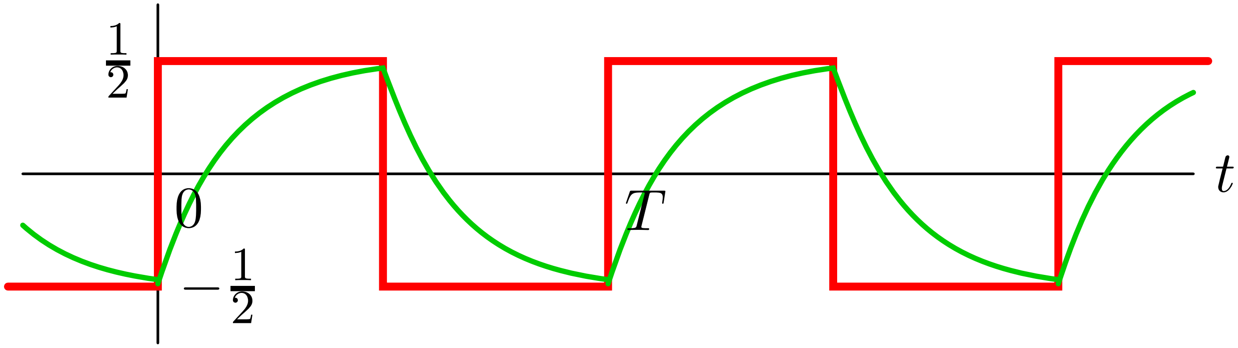



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

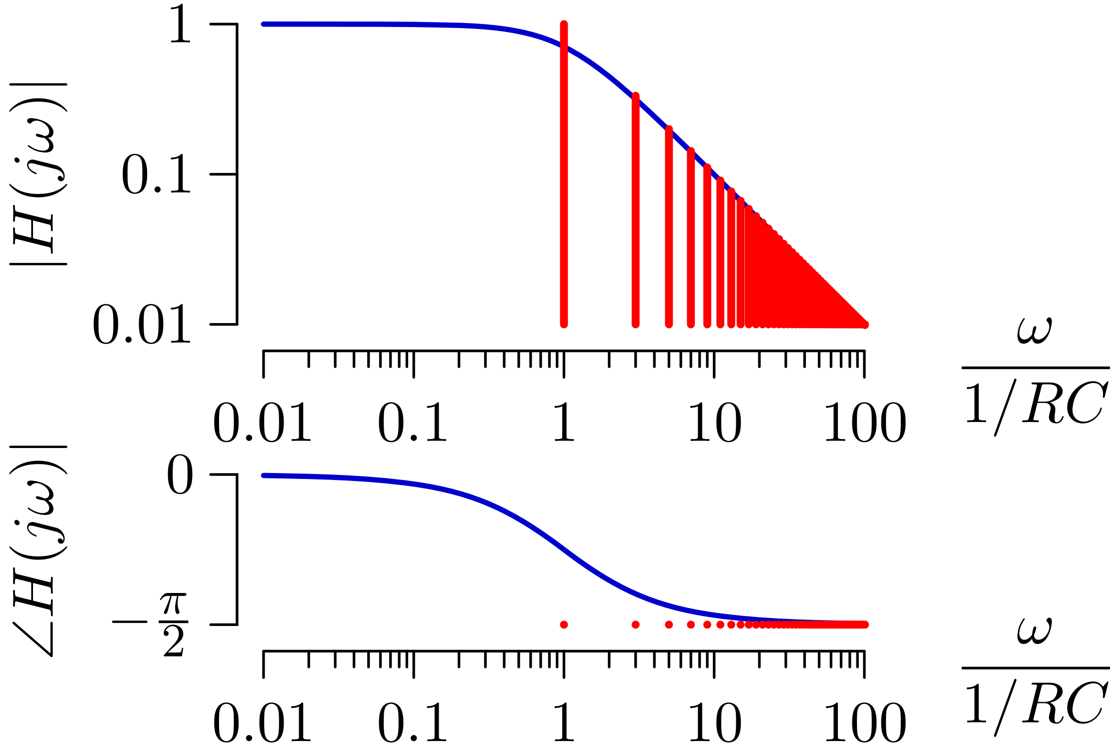


# Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

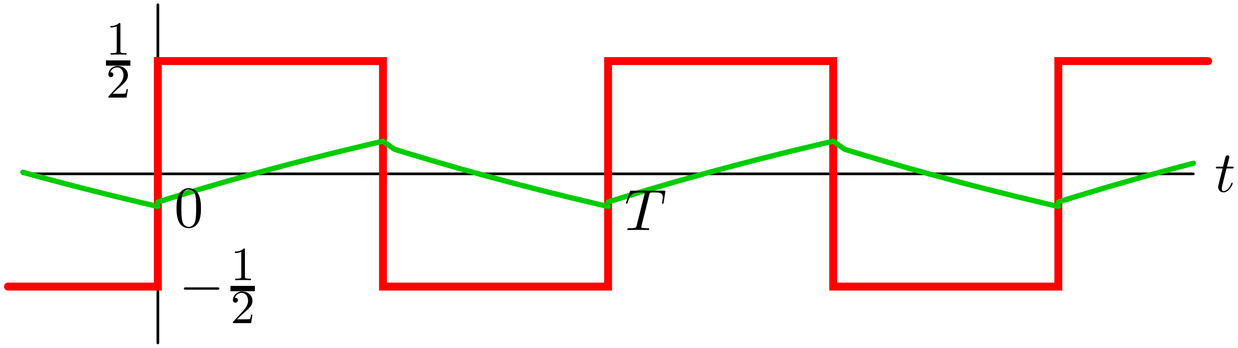



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

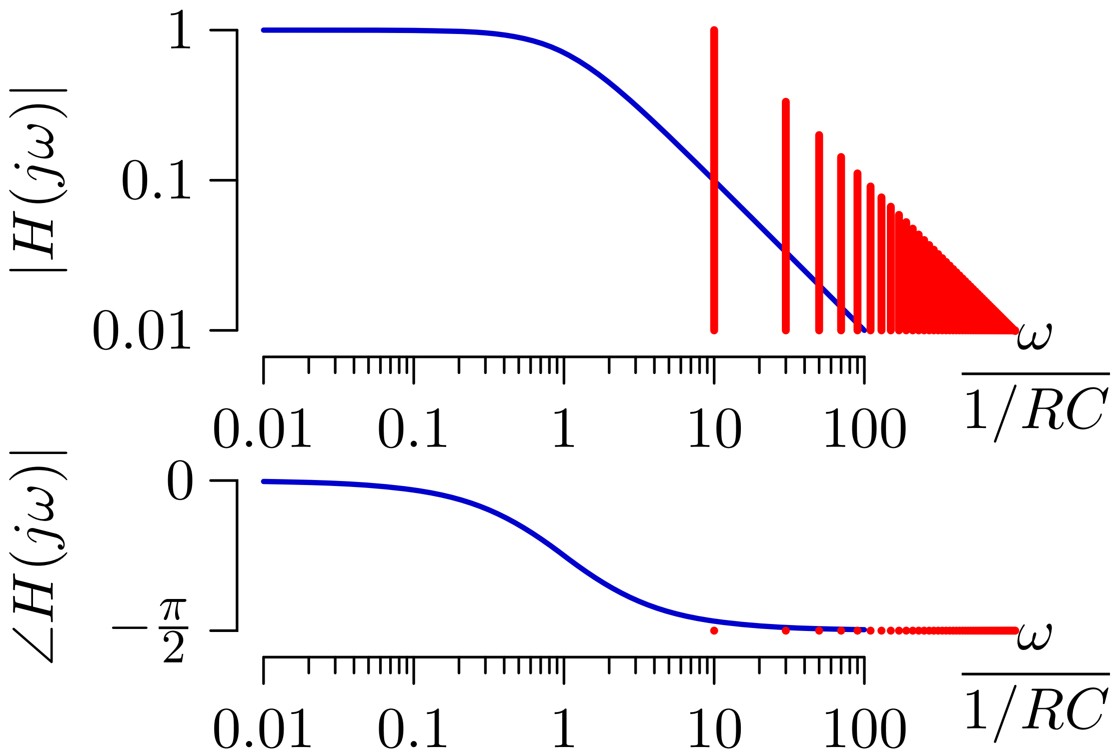


# Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.

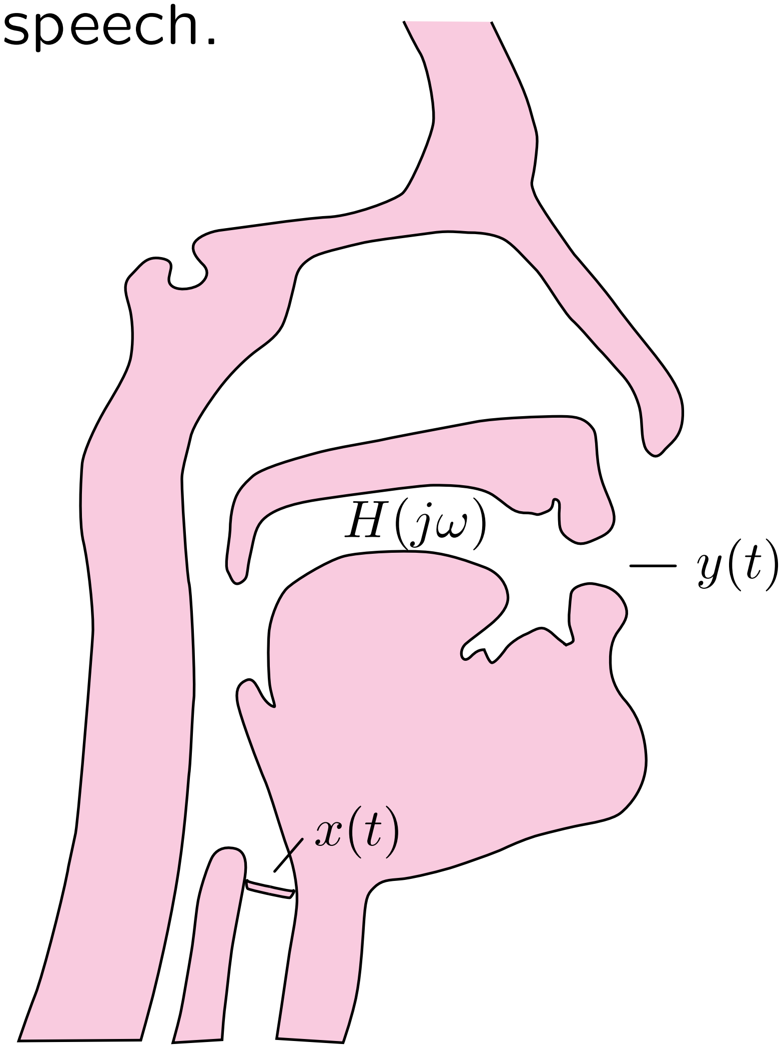



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



# Source-Filter Model of Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.




# Filtering

LTI systems "filter" signals based on their frequency content.

Fourier transforms represent signals as sums of complex exponentials.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}$$

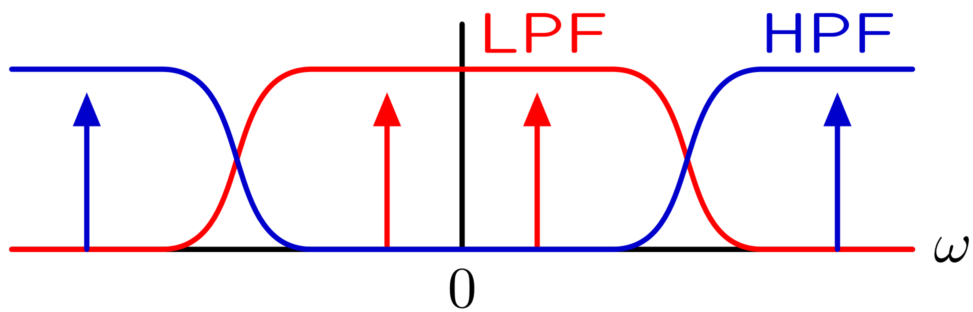
LTI systems "filter" signals by adjusting the amplitudes and phases of each frequency component.

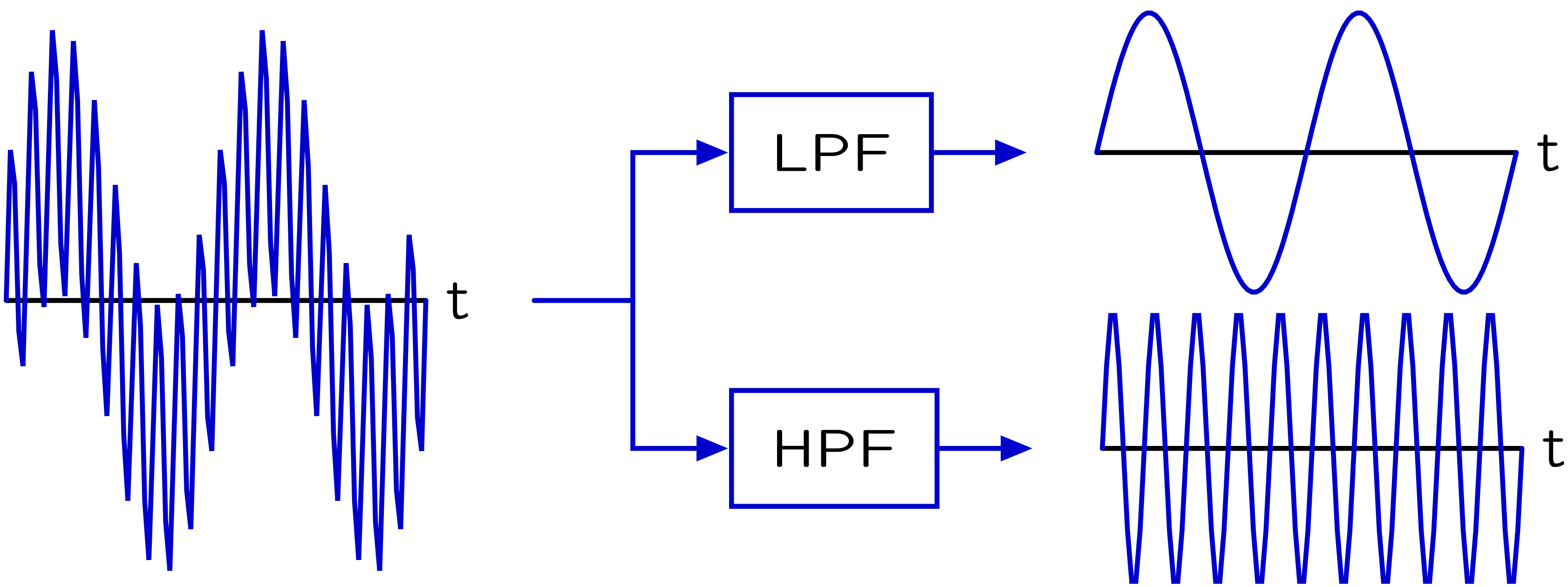
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \;\rightarrow\; y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

# Filtering

Systems can be designed to selectively pass certain frequency bands.

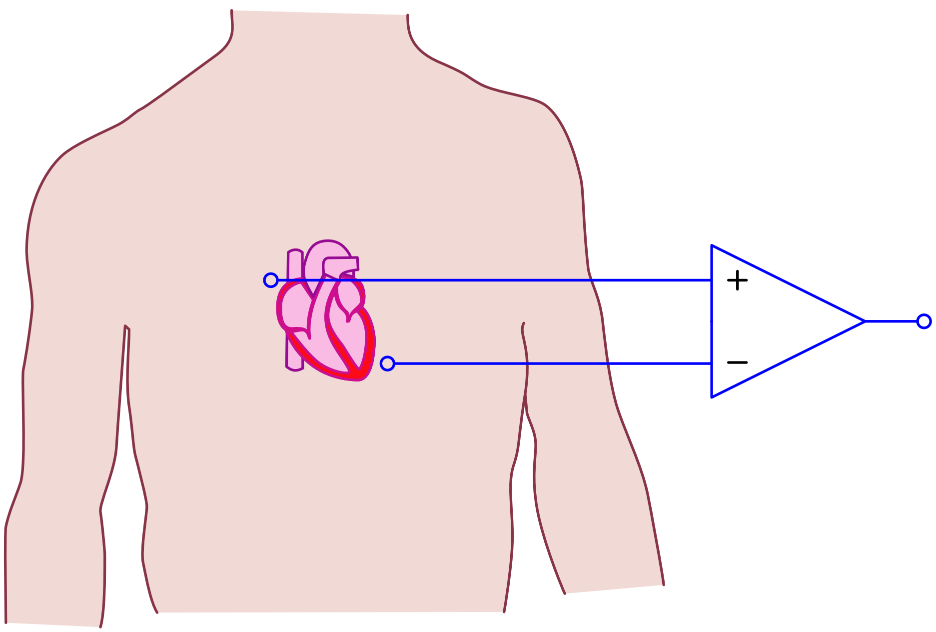
Examples: low-pass filter (LPF) and high-pass filter (HPF).

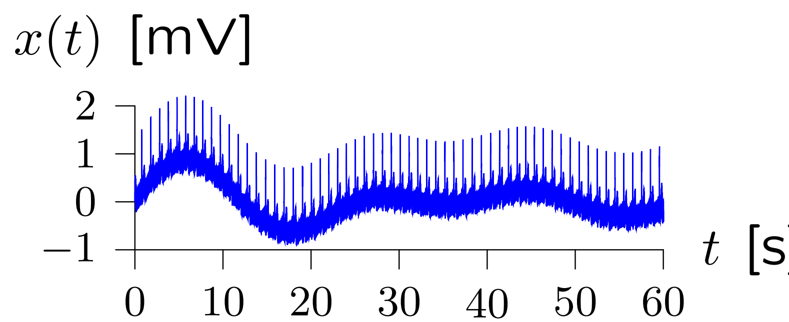


# Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.



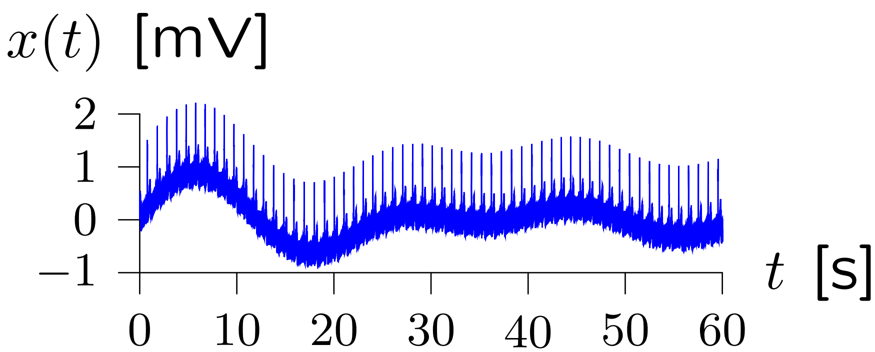



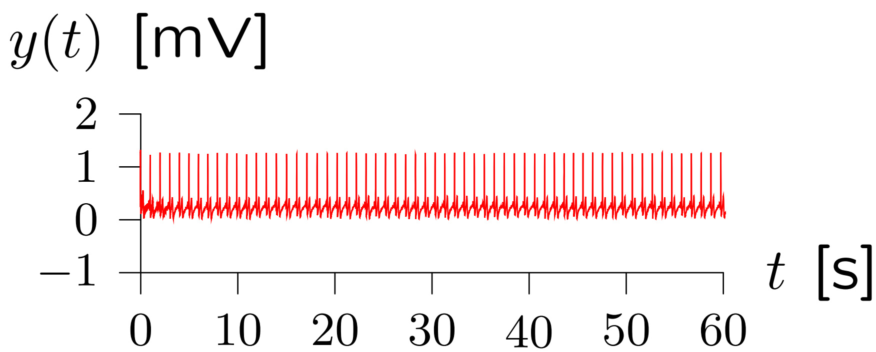

ECG and analysis by T. F. Weiss

# Filtering Example: Electrocardiogram

In addition to picking up electrical responses of the heart, electrodes on the skin also pick up a variety of other electrical signals that we regard as "noise."

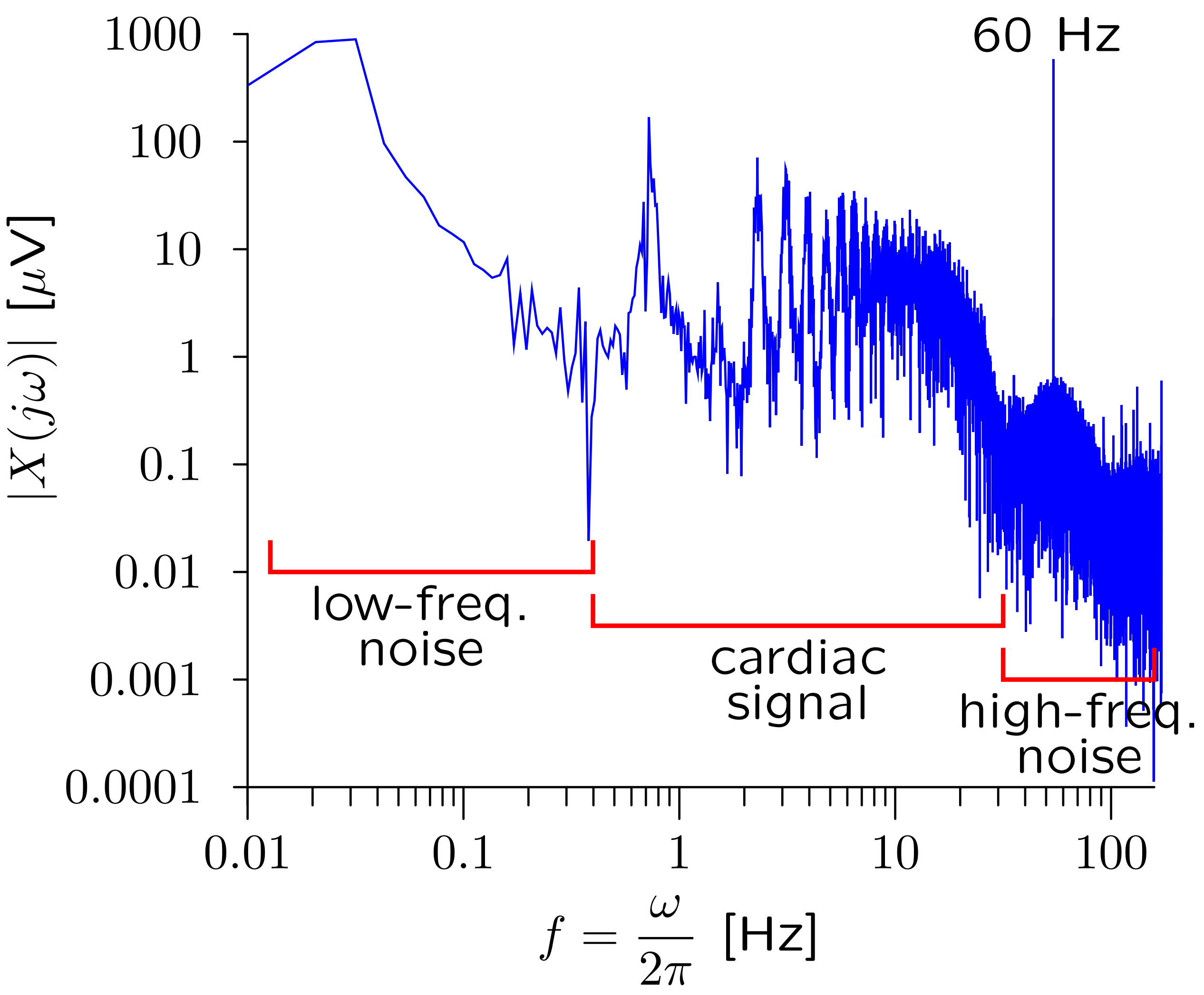
We wish to design a filter to eliminate the noise.

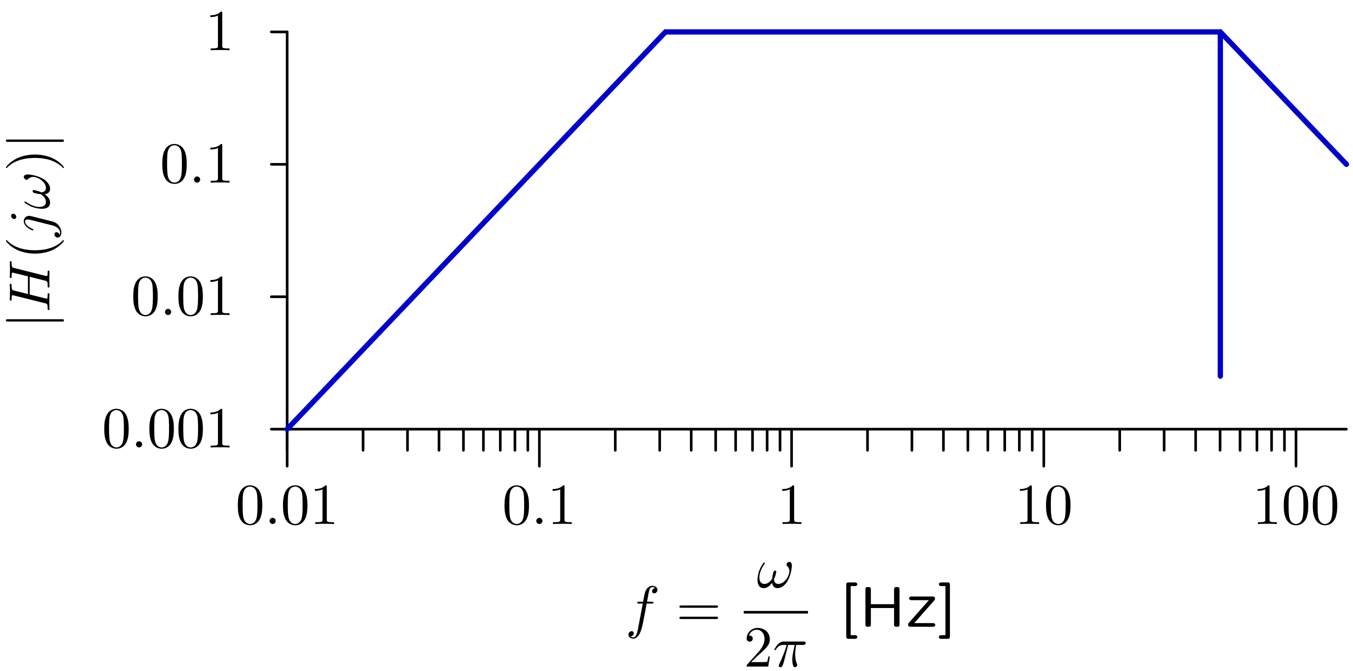
# Filtering Example: Electrocardiogram

We can identify the “noise” by breaking the electrocardiogram into frequency components using the Fourier transform.



# Filtering Example: Electrocardiogram

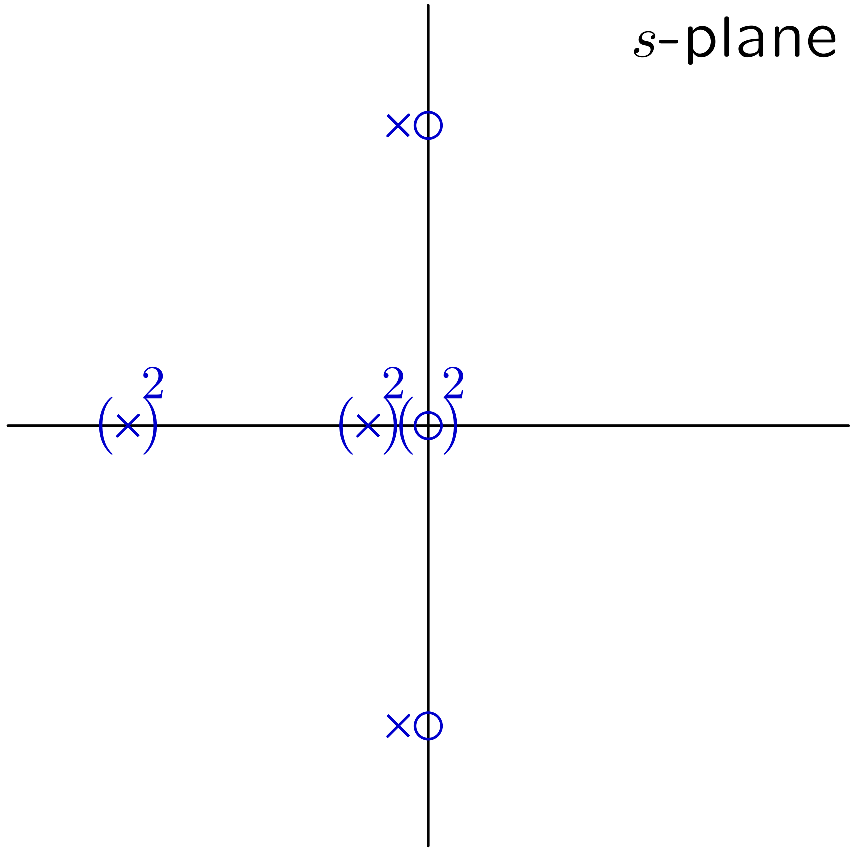
Filter design: low-pass flter + high-pass filter + notch.



# Electrocardiogram: Check Yourself

Which poles and zeros are associated with

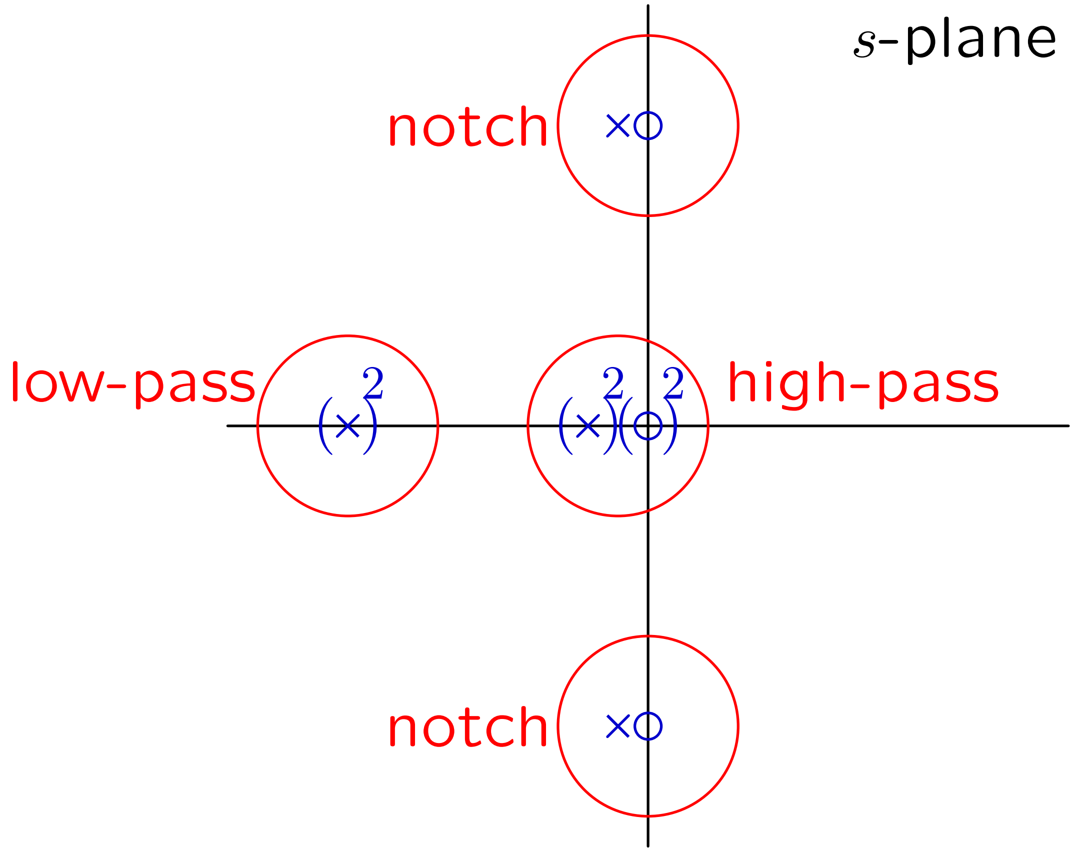
- the high-pass filter?
- the low-pass filter?
- the notch filter?



# Electrocardiogram: Check Yourself

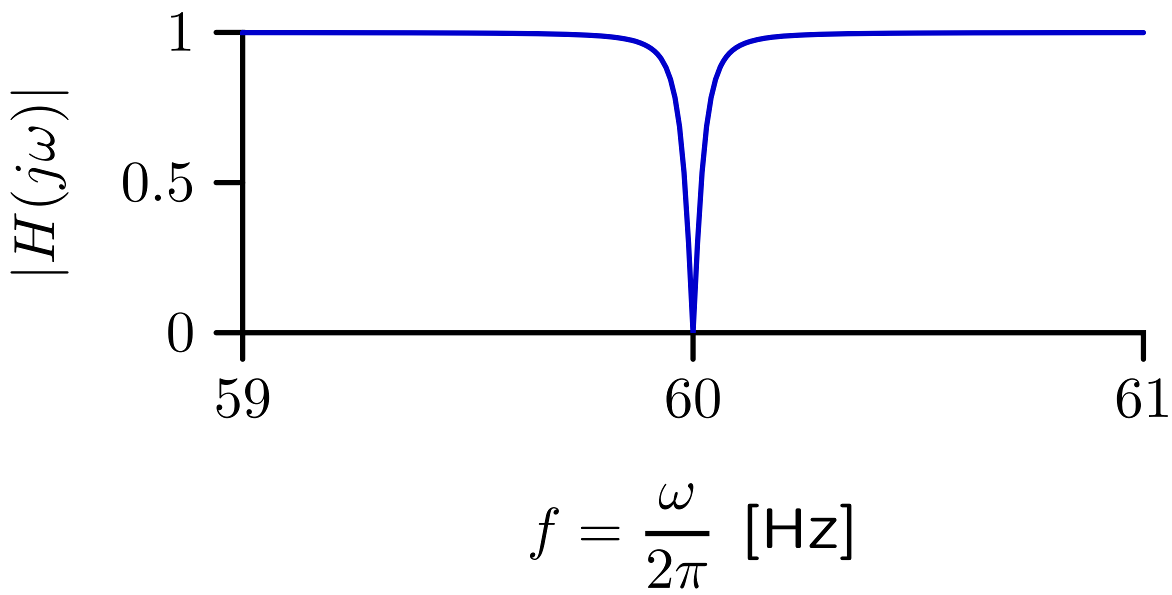
Which poles and zeros are associated with

- the high-pass filter?
- the low-pass filter?
- the notch filter?



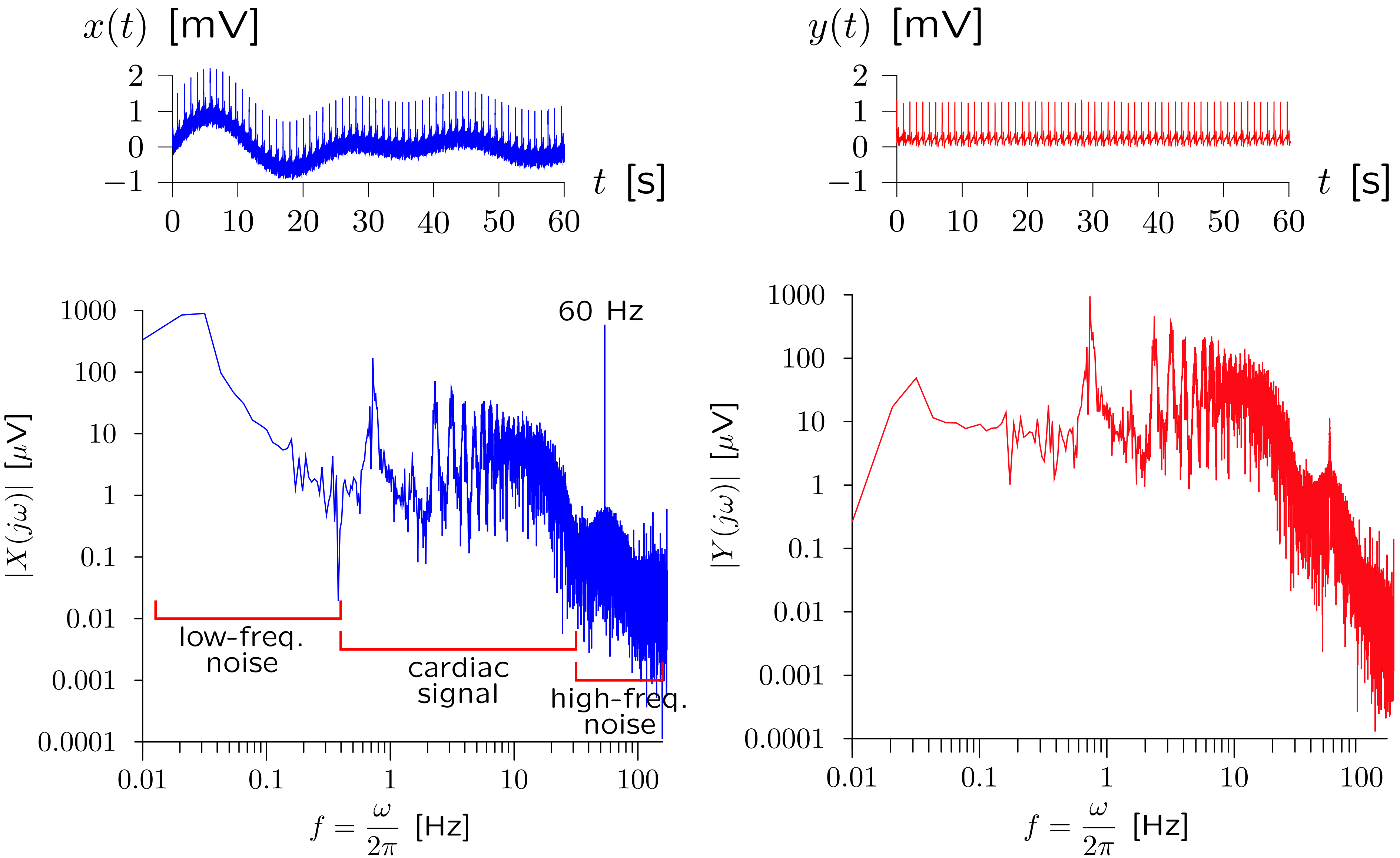
# Filtering Example: Electrocardiogram

By placing the poles of the notch filter very close to the zeros, the width of the notch can be made quite small.



# Filtering Example: Electrocardiogram

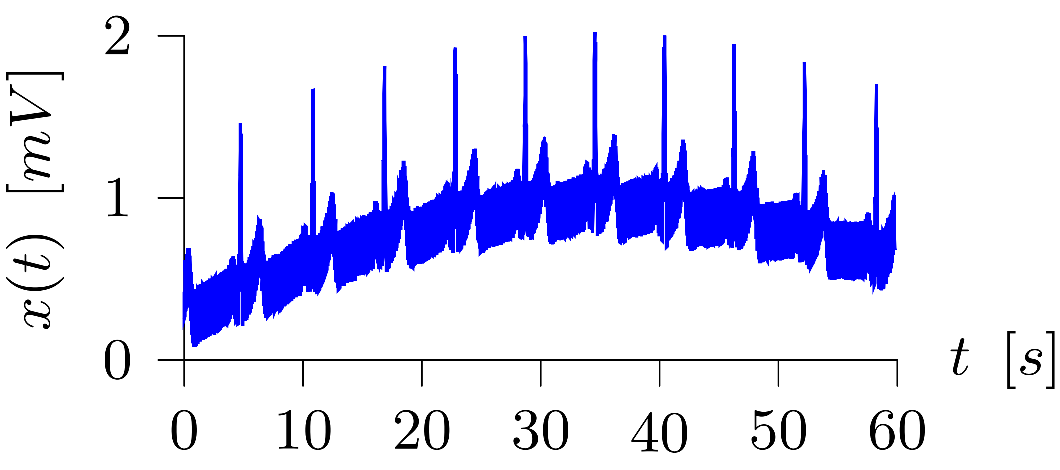
Comparision of filtered and unfiltered electrocardiograms.



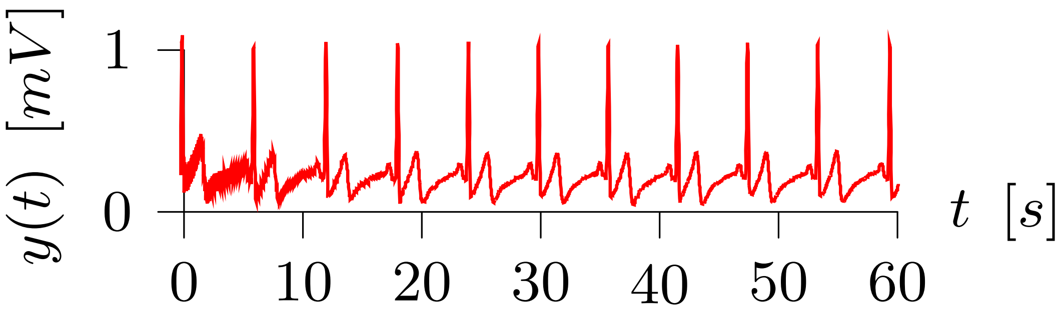
# Filtering Example: Electrocardiogram

Reducing the frequency components that are not generated by the heart simplifies the output, making it easier to diagnose cardiac problems.

Unfiltered ECG




Filtered ECG



# Continuous-Time Fourier Transform: Summary

Fourier transforms represent signals by their frequency content.

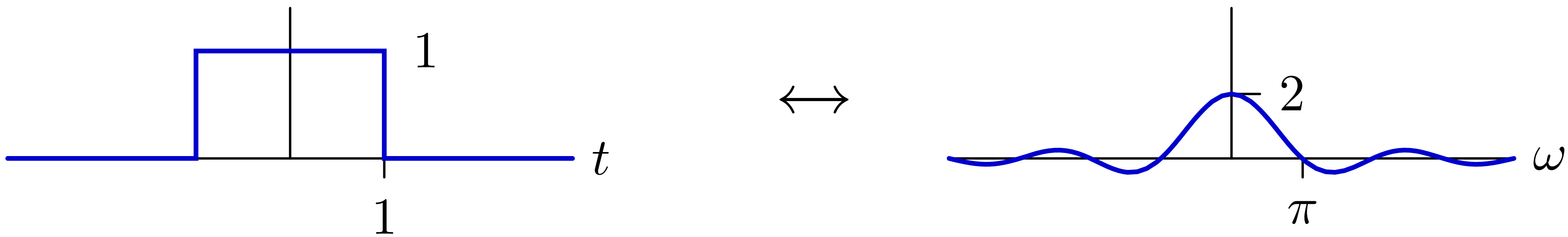
$\rightarrow$ useful for many signals, e.g., electrocardiogram.

$\rightarrow$ motivates representing a system as a filter.

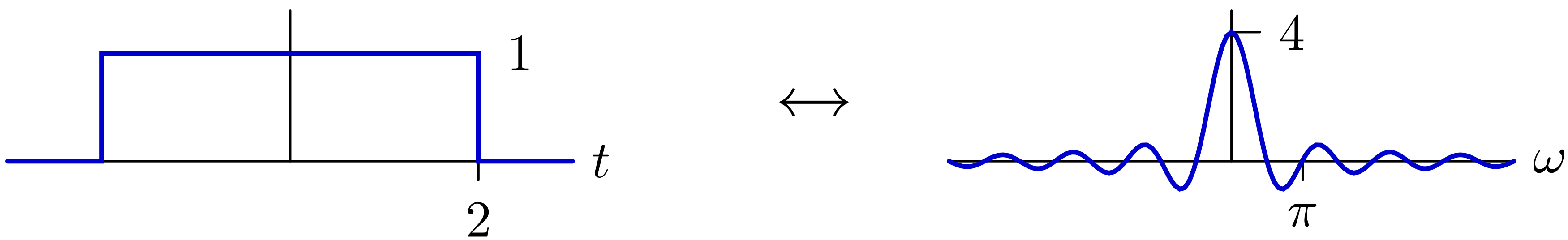
  $\rightarrow$ useful for many systems.

# Visualizing the Fourier Transform

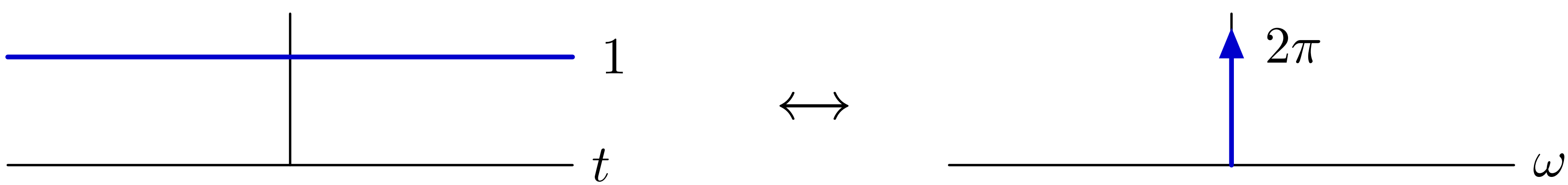
Fourier transforms provide alternate **views** of signals.




Pulses contain all frequencies except harmonics of $2\pi$/width.



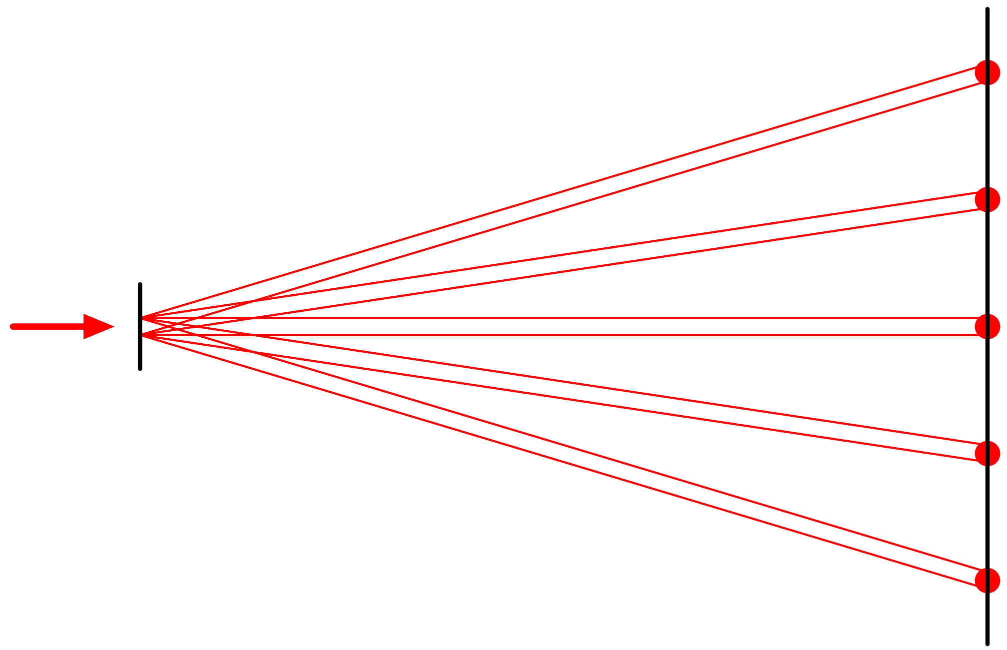

Wider pulses contain more low frequencies than narrow pulses.




Constants (in time) contain only frequencies at $\omega = 0$.

# Fourier Transforms in Physics: Diffraction

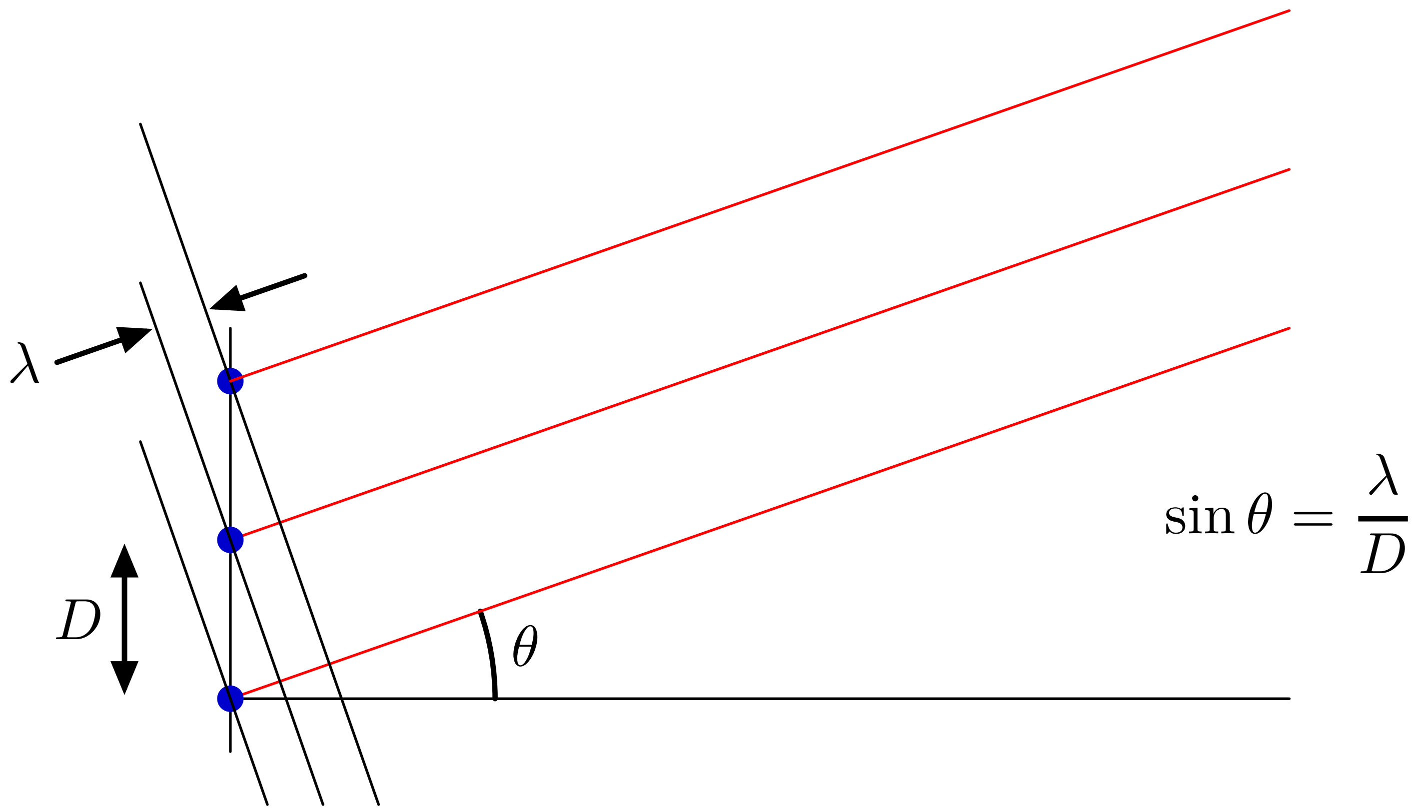
A diffraction grating breaks a laser beam input into multiple beams.



Demonstration.

# Fourier Transforms in Physics: Diffraction

The grating has a periodic structure (period $= D$).




The "far field" image is formed by interference of scattered light.

Viewed from angle $\theta$, the scatterers are separated by $D\sin\theta$.

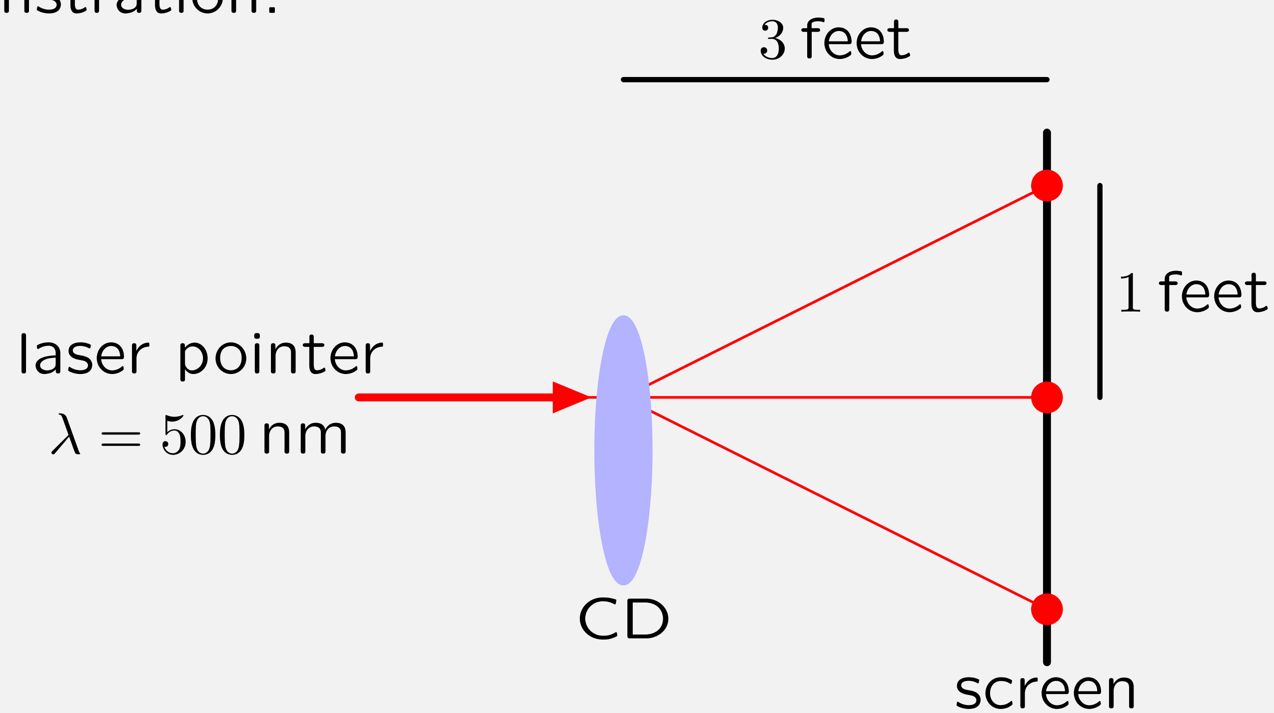
If this distance is an integer number of wavelengths $\lambda \rightarrow$ constructive interference.

# Fourier Transforms in Physics: Diffraction

CD demonstration.

# Check Yourself

CD demonstration.




What is the spacing of the tracks on the CD?

1. $160$ nm
2. $1600$ nm
3. $16\mu$m
4. $160\mu$m

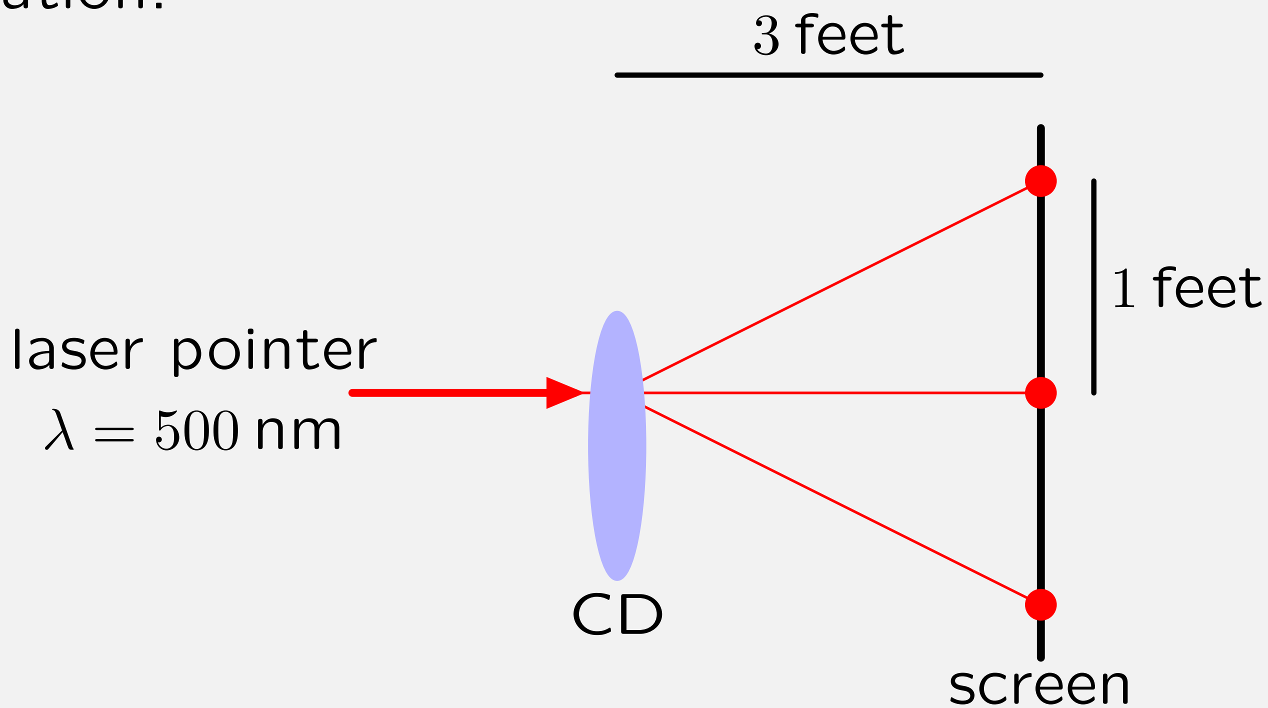
# Check Yourself

What is the spacing of the tracks on the CD?

| grating | $\tan\theta$ | $\theta$ | $\sin\theta$ | $D = \dfrac{500\,\text{nm}}{\sin\theta}$ | manufacturing spec. |
|---|---|---|---|---|---|
| CD | $\frac{1}{3}$ | $0.32$ | $0.31$ | $1613\,\text{nm}$ | $1600\,\text{nm}$ |

# Check Yourself

Demonstration.




What is the spacing of the tracks on the CD? 2.

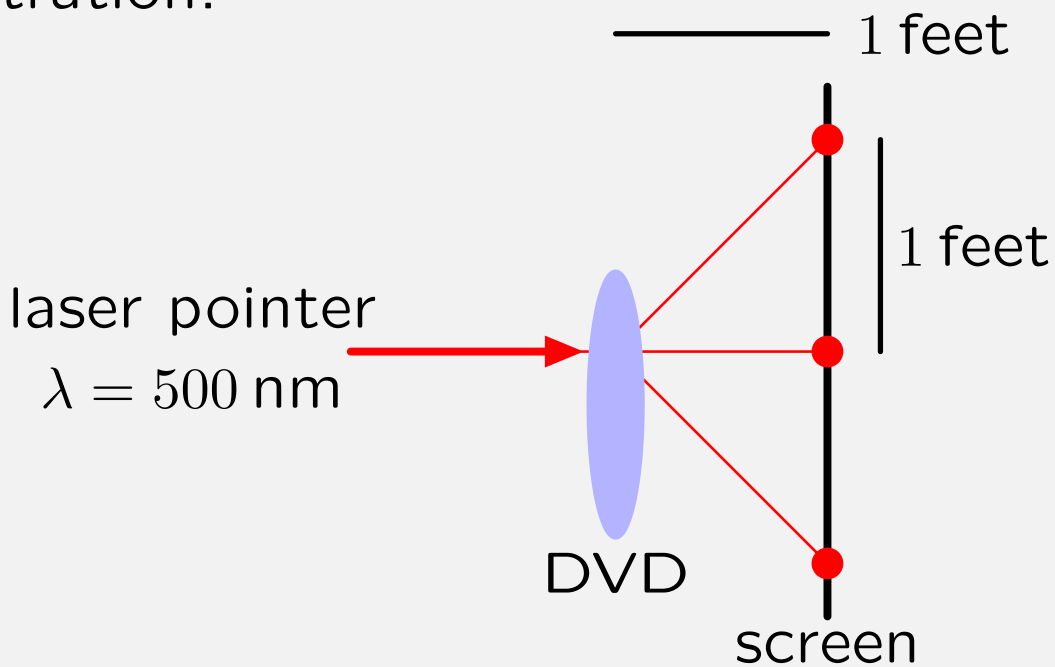
1. 160 nm
2. 1600 nm
3. 16$\mu$m
4. 160$\mu$m

# Fourier Transforms in Physics: Diffraction

DVD demonstration.

# Check Yourself

DVD demonstration.




What is track spacing on DVD divided by that for CD?

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

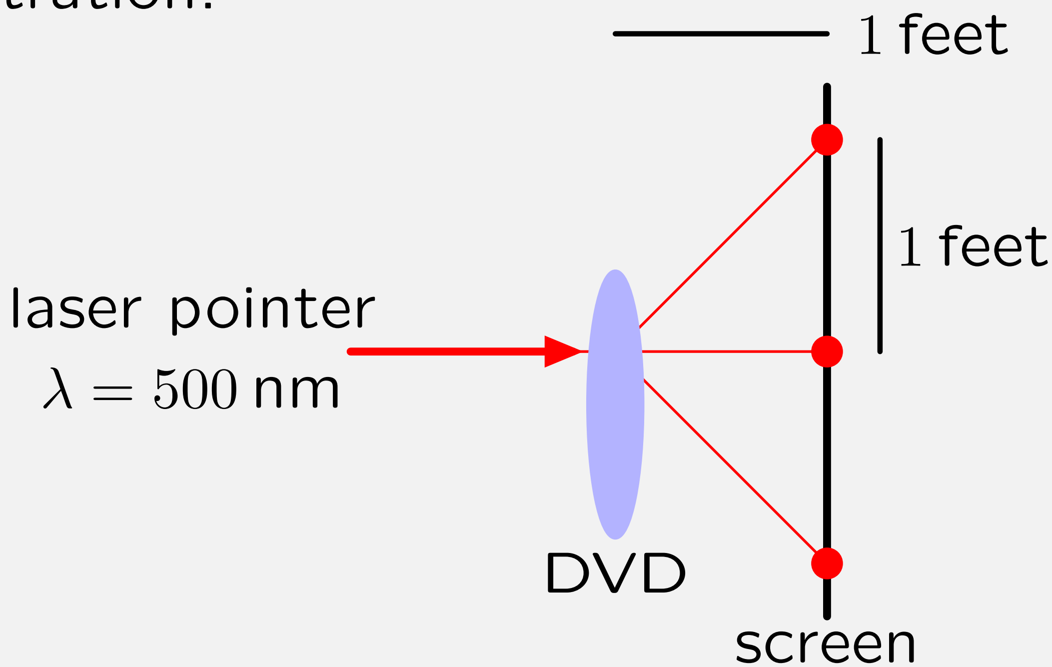
# Check Yourself

What is spacing of tracks on DVD divided by that for CD?

| grating | $\tan\theta$ | $\theta$ | $\sin\theta$ | $D = \dfrac{500\,\text{nm}}{\sin\theta}$ | manufacturing spec. |
|---|---|---|---|---|---|
| CD | $\frac{1}{3}$ | $0.32$ | $0.31$ | $1613\,\text{nm}$ | $1600\,\text{nm}$ |
| DVD | $1$ | $0.78$ | $0.71$ | $704\,\text{nm}$ | $740\,\text{nm}$ |

# Check Yourself

DVD demonstration.

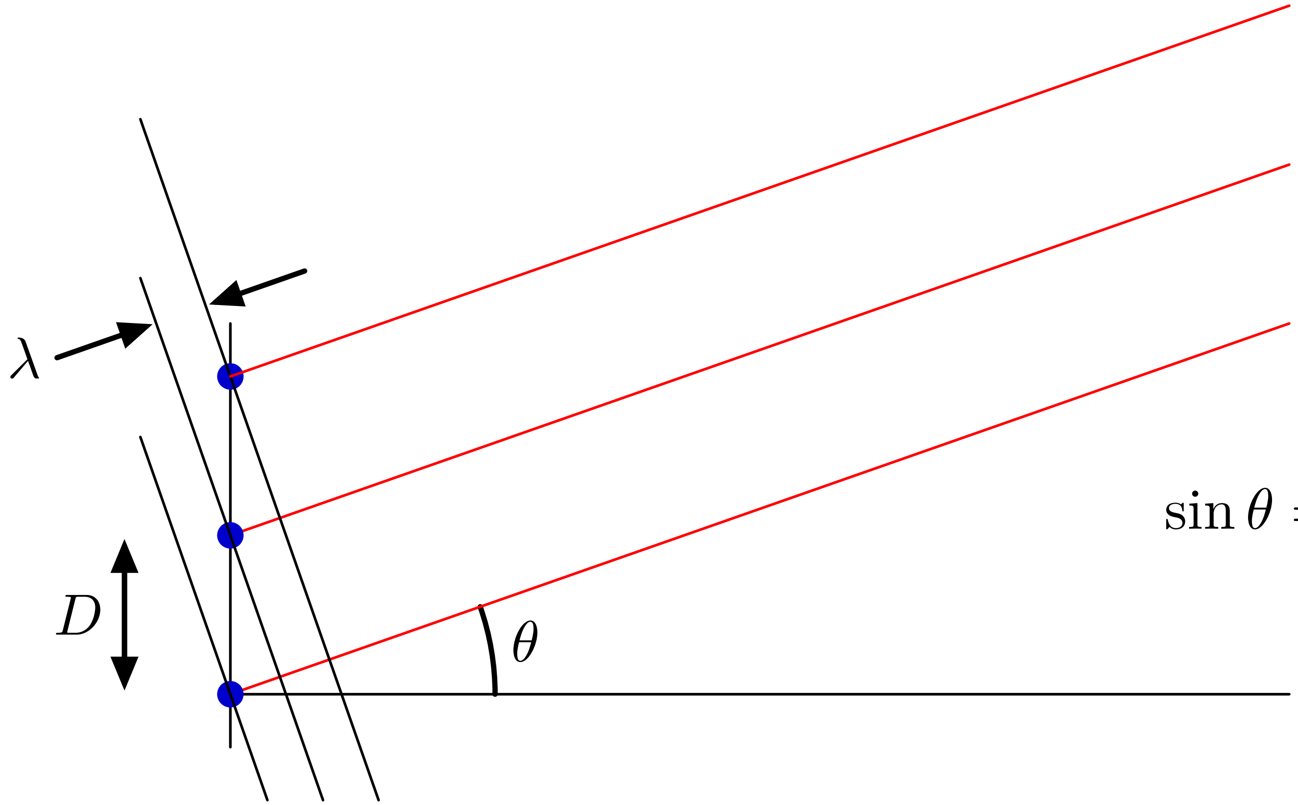



What is track spacing on DVD divided by that for CD? 3

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

# Fourier Transforms in Physics: Diffraction

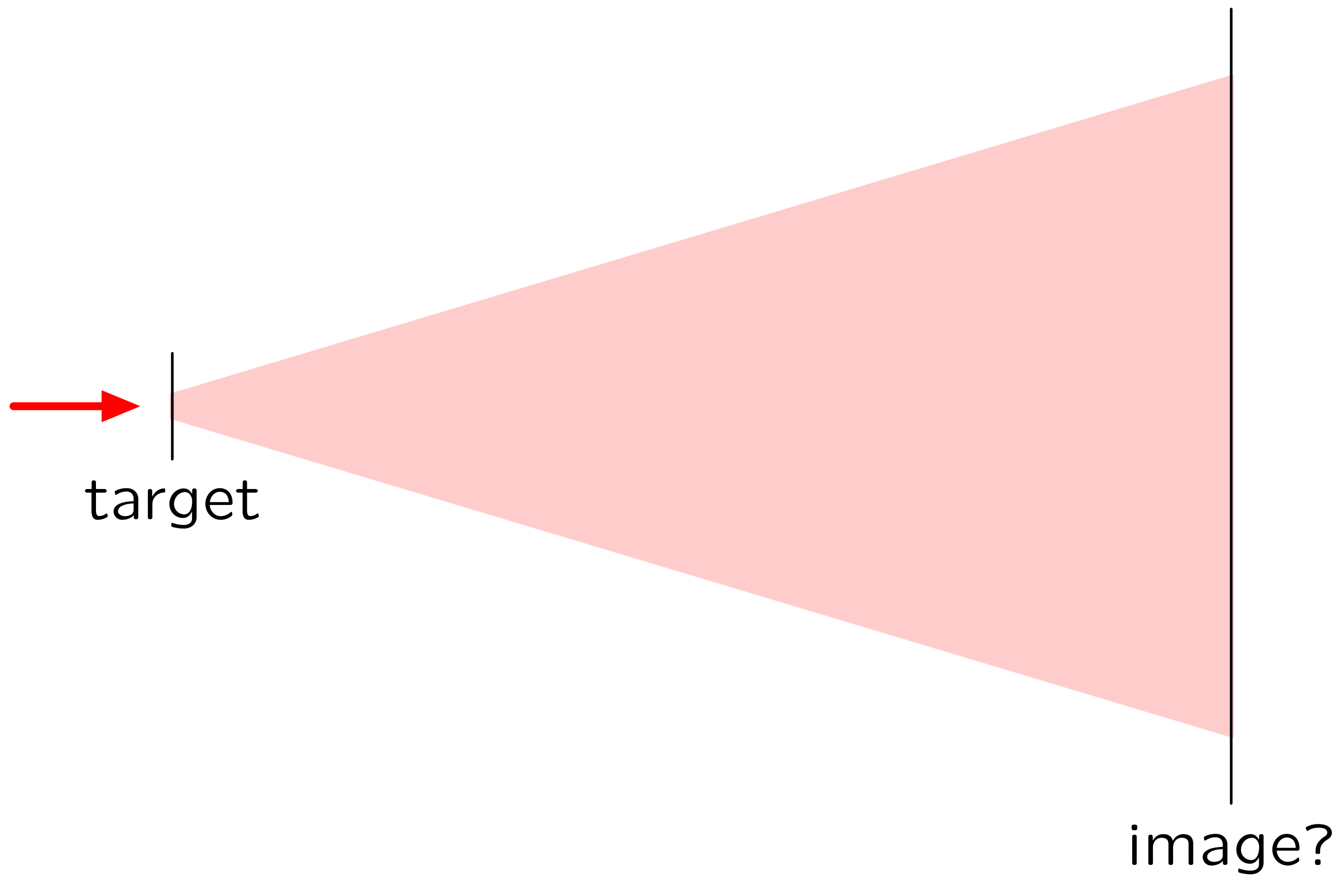
Macroscopic information in the far field provides microscopic (invisible) information about the grating.




$$\sin\theta = \frac{\lambda}{D}$$

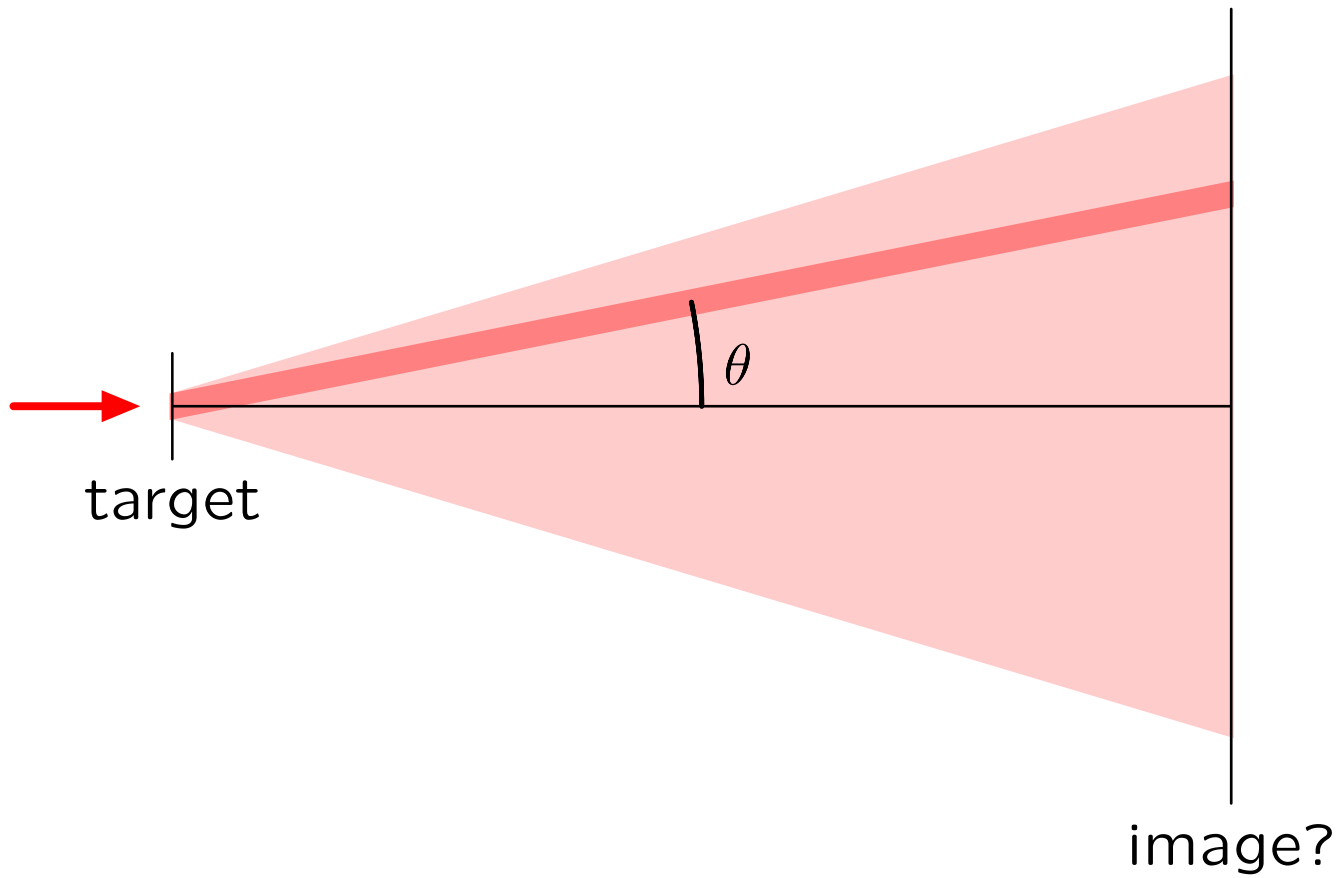
# Fourier Transforms in Physics: Crystallography

What if the target is more complicated than a grating?



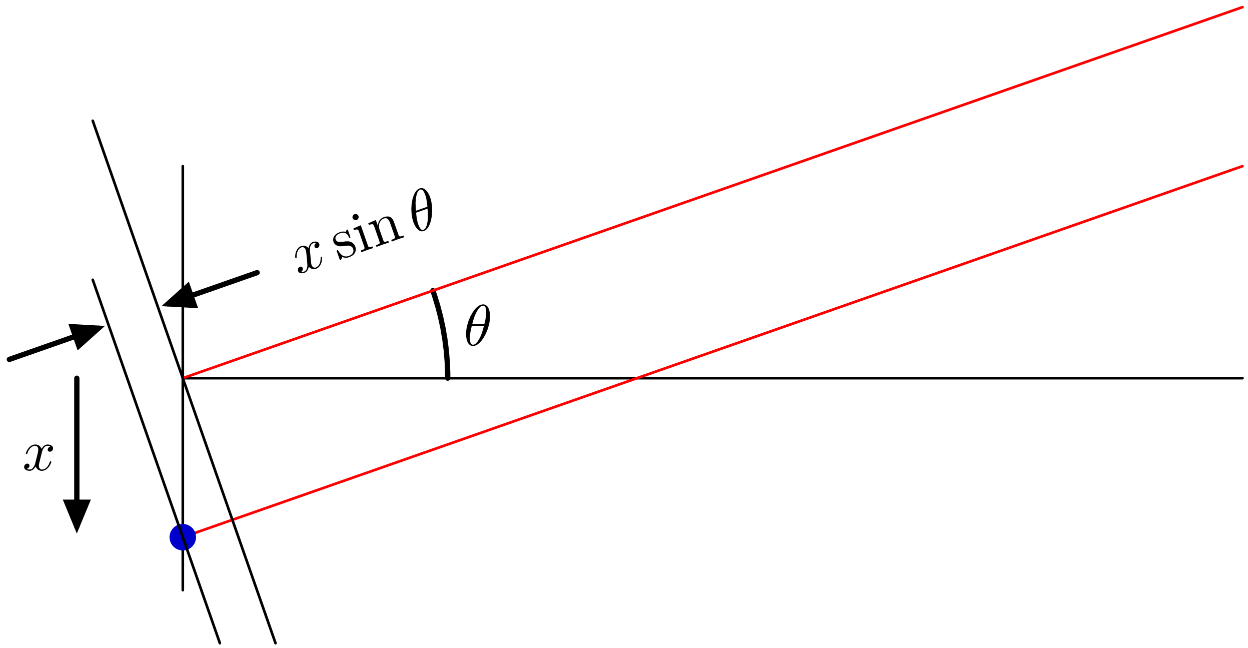
# Fourier Transforms in Physics: Crystallography

Part of image at angle $\theta$ has contributions for all parts of the target.



# Fourier Transforms in Physics: Crystallography

The phase of light scattered from different parts of the target undergo different amounts of phase delay.




Phase at a point $x$ is delayed (i.e., negative) relative to that at $0$:

$$\phi = -2\pi \frac{x \sin\theta}{\lambda}$$

# Fourier Transforms in Physics: Crystallography

Total light $F(\theta)$ at angle $\theta$ is the integral of amount scattered from each part of the target ($f(x)$) appropriately shifted in phase.

$$F(\theta) = \int f(x) e^{-j2\pi \frac{x \sin\theta}{\lambda}} dx$$

Assume small angles so $\sin\theta \approx \theta$.

Let $\omega = 2\pi \frac{\theta}{\lambda}$.

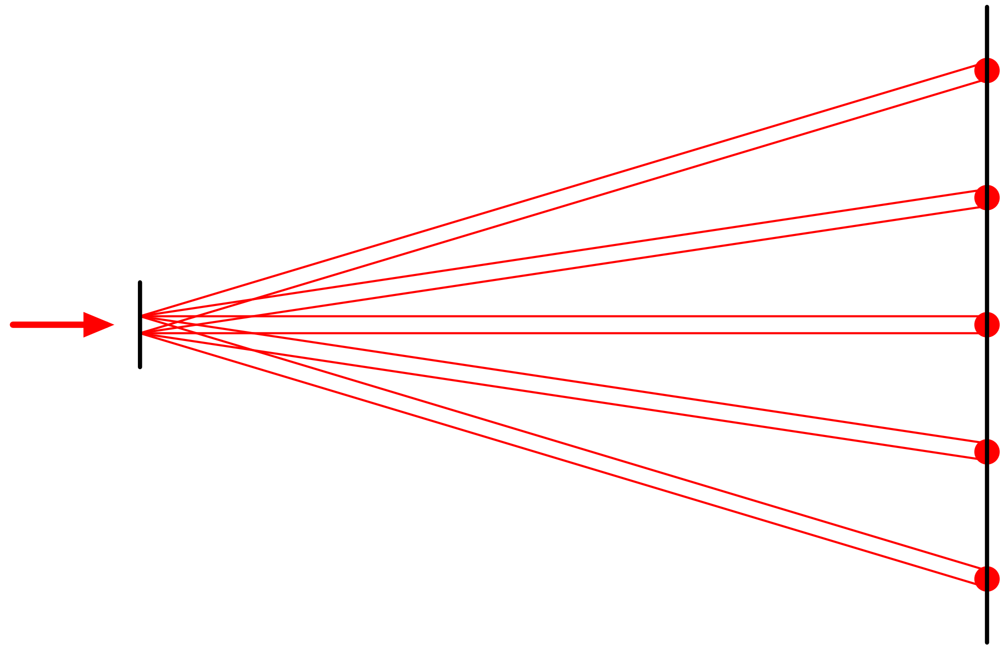
Then the pattern of light at the detector is

$$F(\omega) = \int f(x) e^{-j\omega x} dx$$

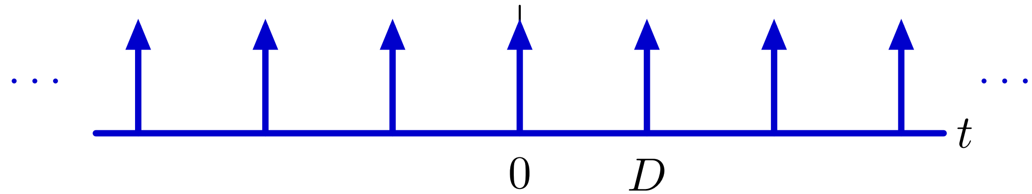
which is the Fourier transform of $f(x)$!

# Fourier Transforms in Physics: Diffraction

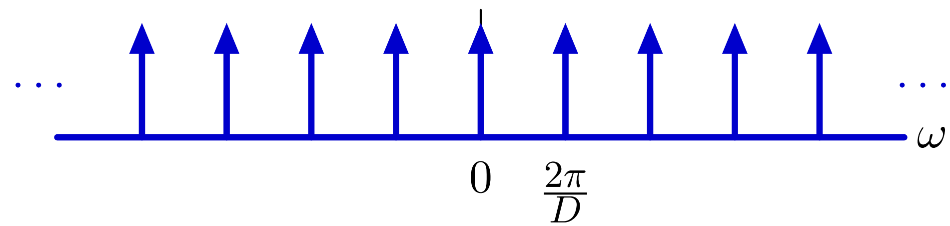
There is a Fourier transform relation between this structure and the far-field intensity pattern.



grating $\approx$ impulse train with pitch $D$



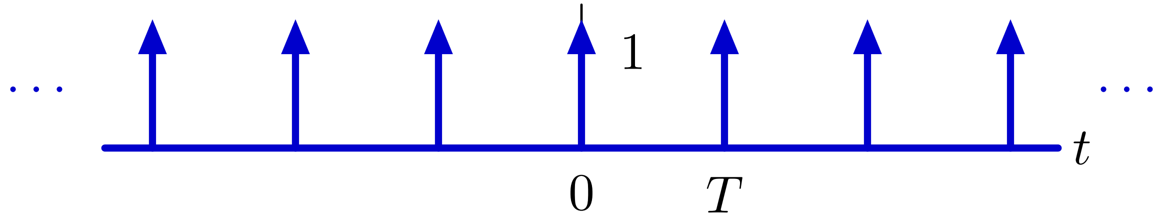

far-field intensity $\approx$ impulse train with reciprocal pitch $\propto \frac{\lambda}{D}$



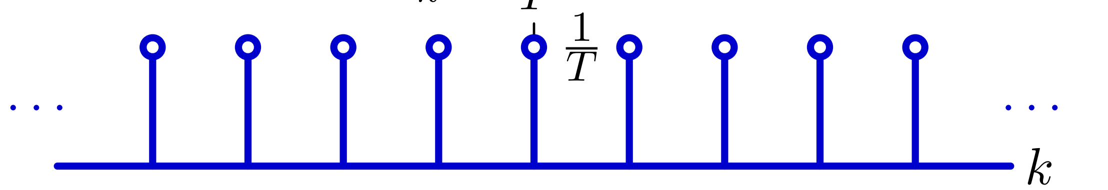
# Impulse Train

The Fourier transform of an impulse train is an impulse train.

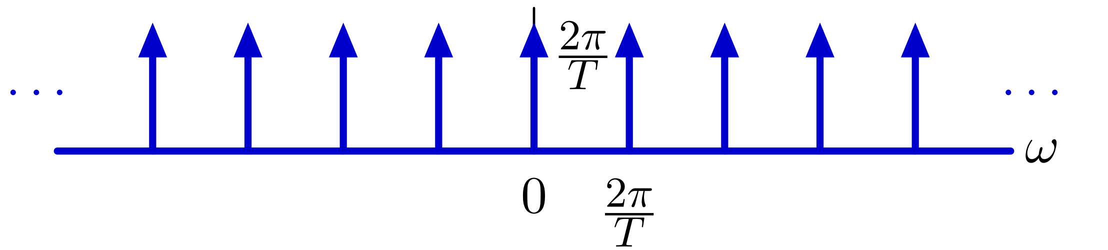
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$




$$a_k = \frac{1}{T} \quad \forall \ k$$




$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$

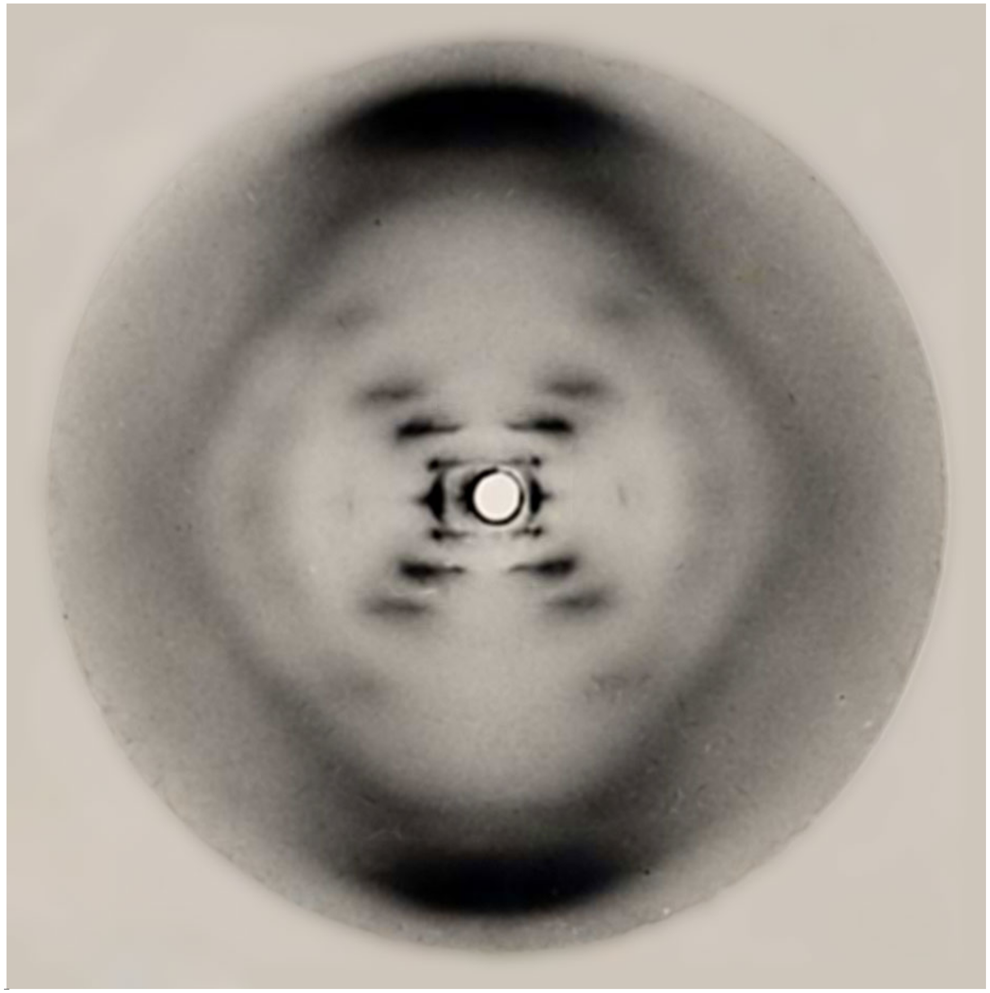


# Two Dimensions

Demonstration: 2D grating.

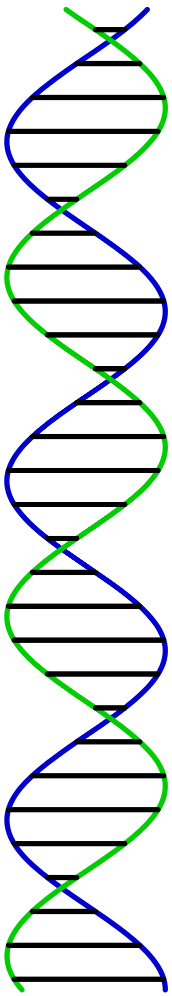
# An Historic Fourier Transform

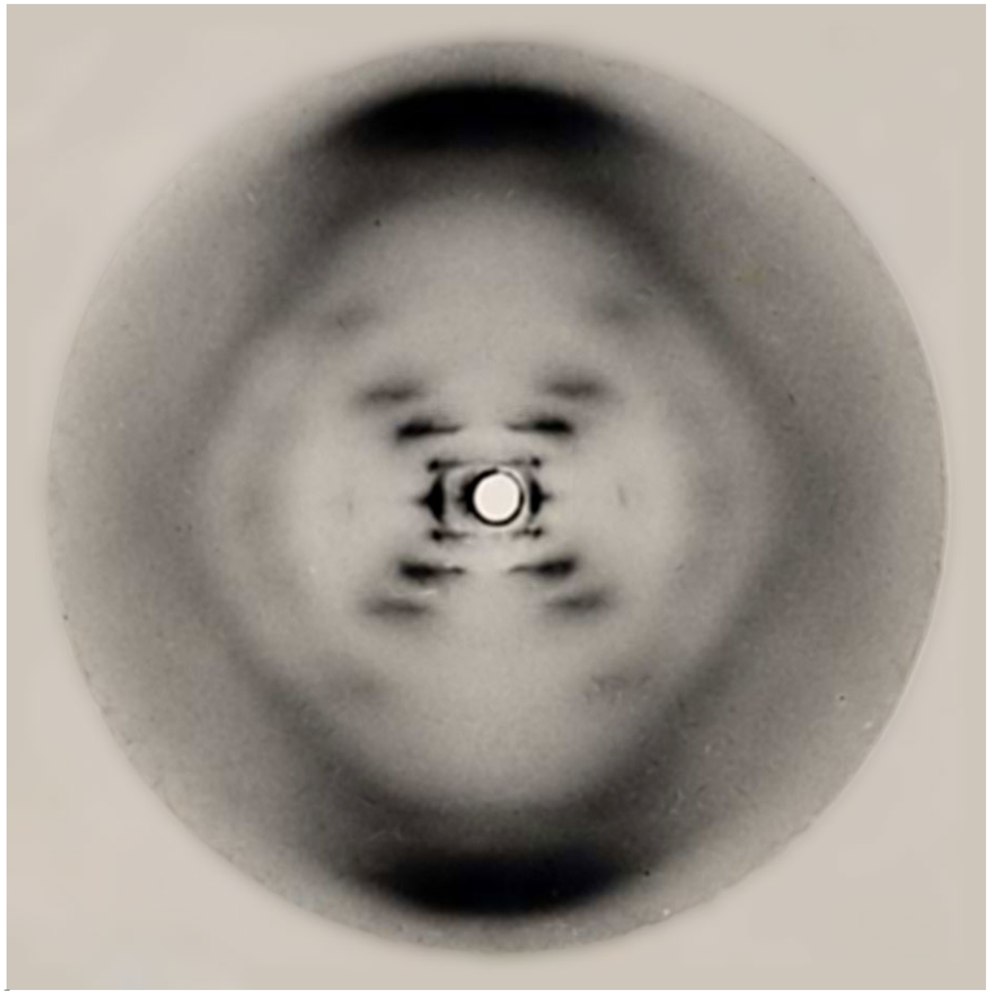
Taken by Rosalind Franklin, this image sparked Watson and Crick's insight into the double helix.



# An Historic Fourier Transform

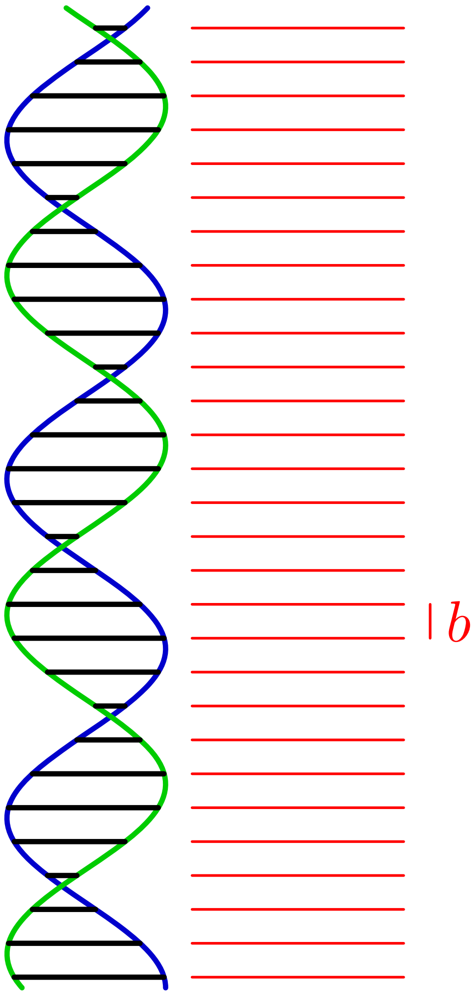
This is an x-ray crystallographic image of DNA, and it shows the Fourier transform of the structure of DNA.

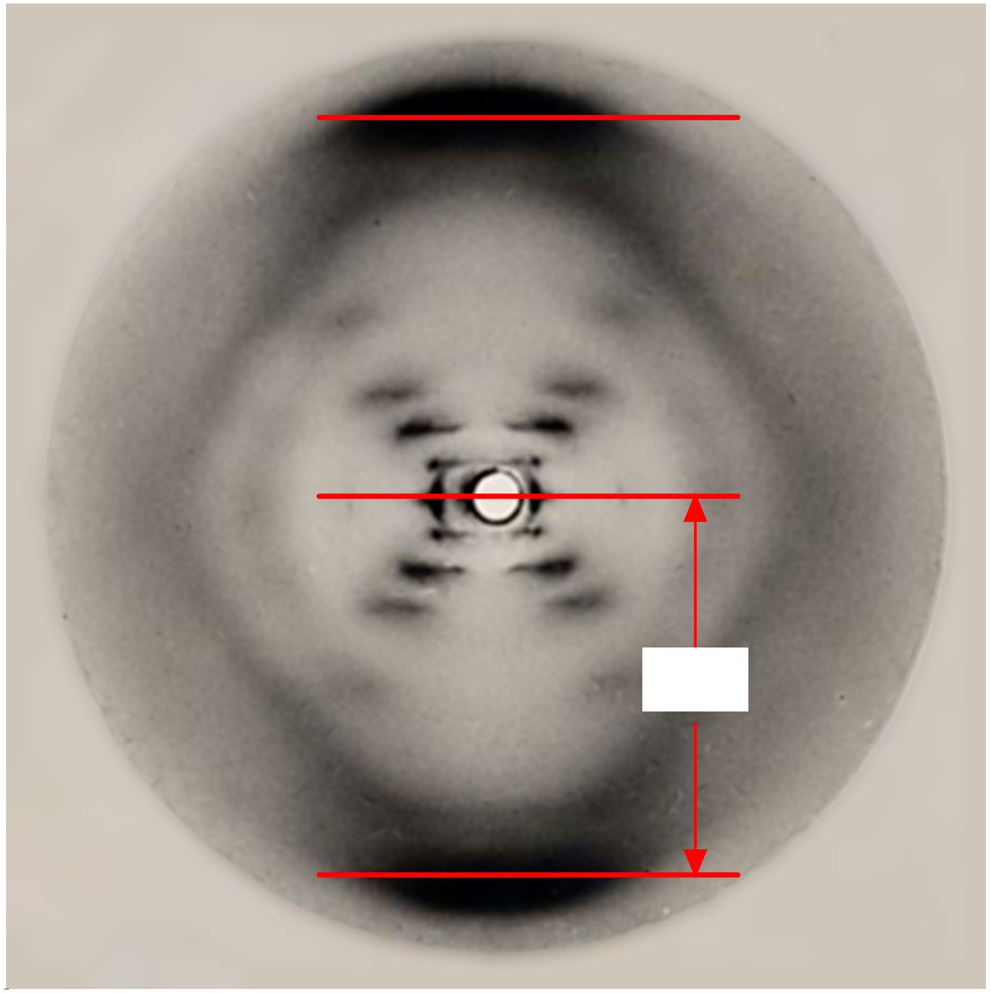




# An Historic Fourier Transform

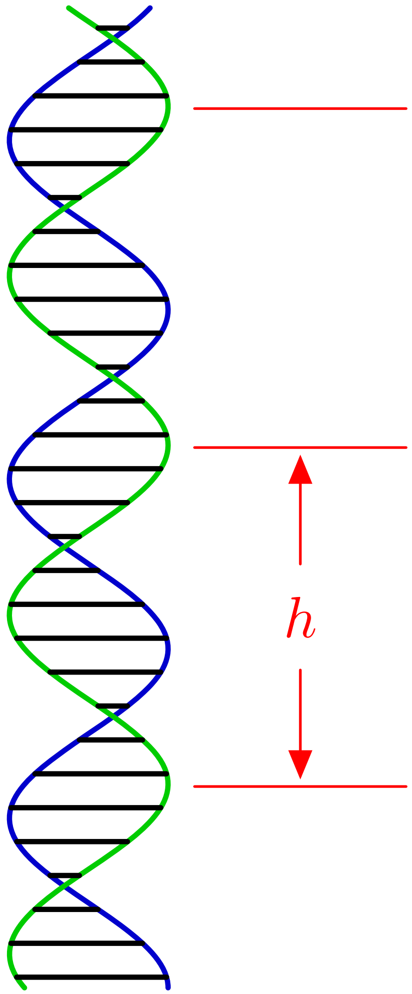
High-frequency bands indicate repreating structure of base pairs.

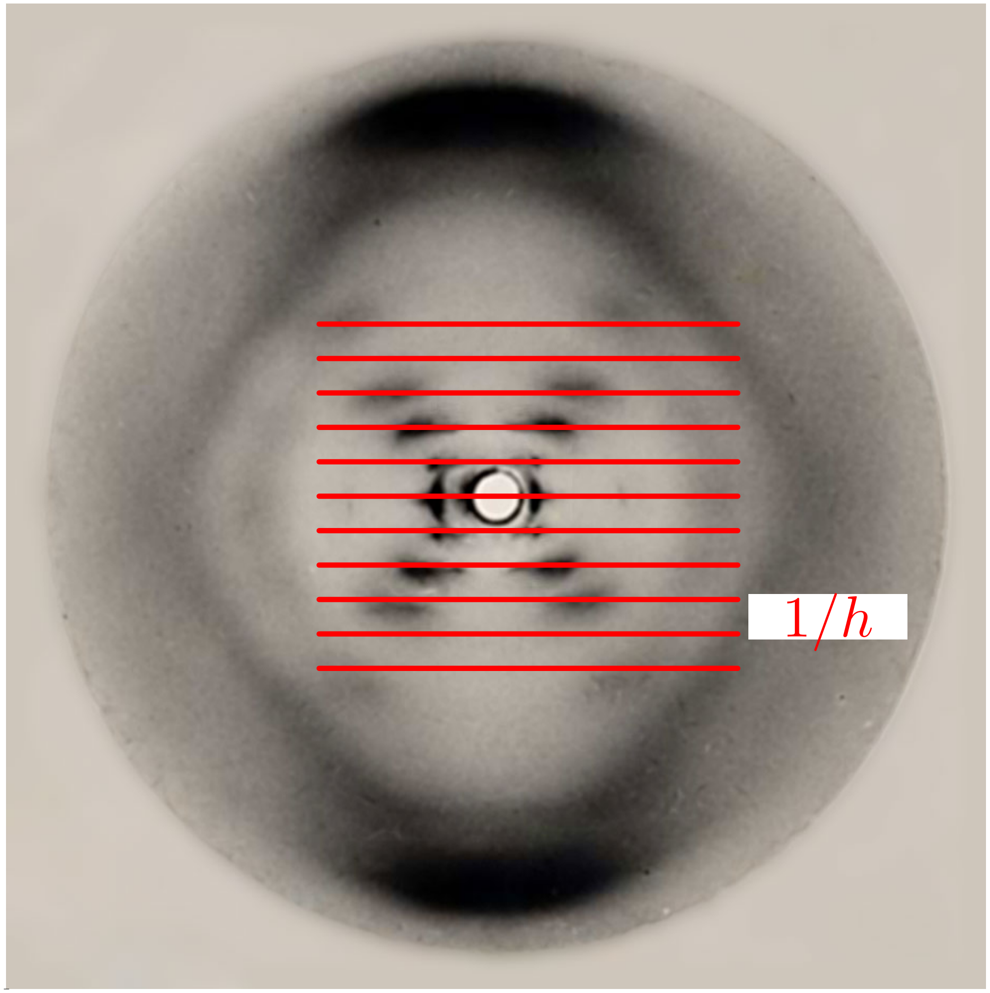


# An Historic Fourier Transform

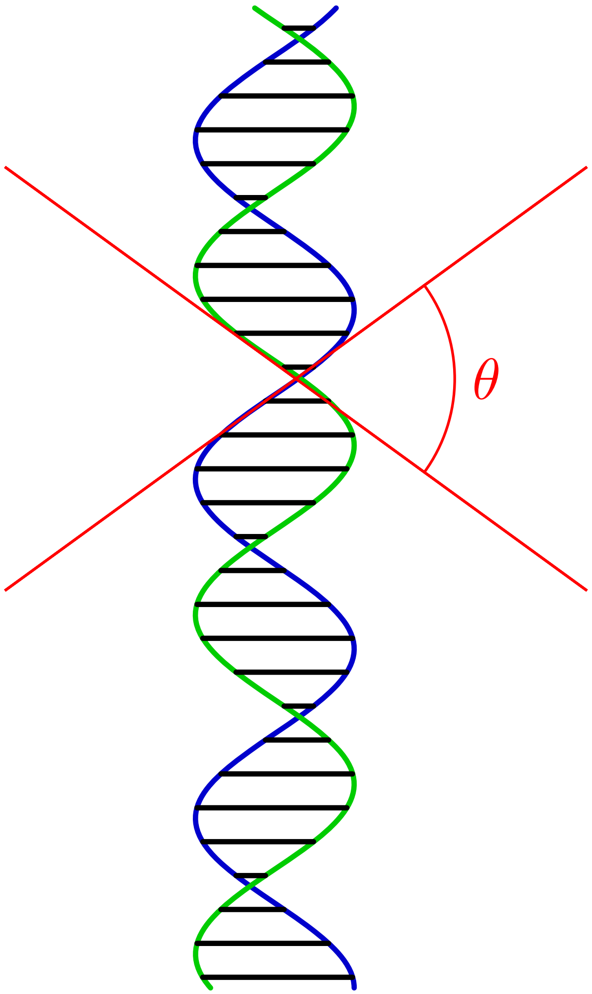
Low-frequency bands indicate a lower frequency repeating structure.

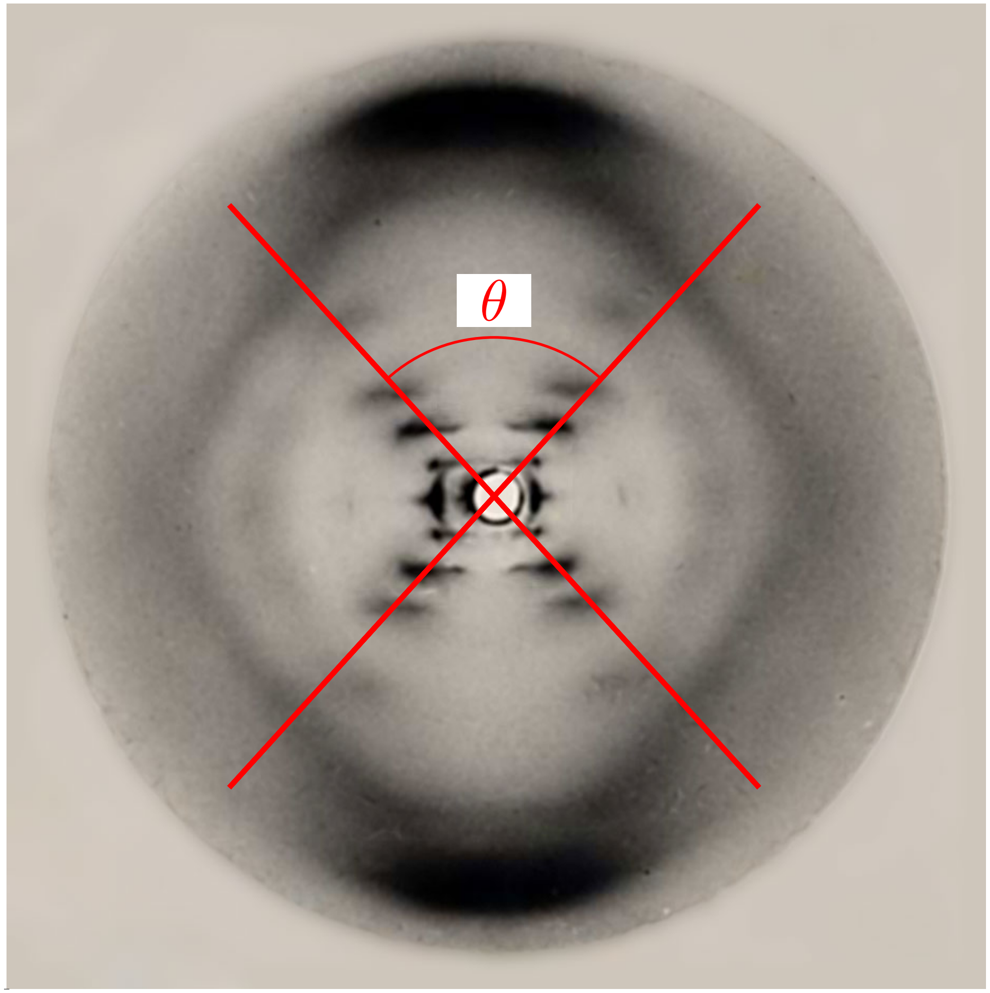


# An Historic Fourier Transform

Tilt of low-frequency bands indicates tilt of low-frequency repeating structure: the double helix!



# Simulation

Easy to calculate relation between structure and Fourier transform.

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Left: double helix drawing. Right: x-ray diffraction image.

# Fourier Transform Summary

Represent signals by their frequency content.

Key to “filtering,” and to signal-processing in general.

Important in many physical phenomenon: x-ray crystallography.