Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Recitation 10 Solutions
March 23, 2006

1. a) To find the transform, we integrate the density function over its full domain, against an exponential. This is often expressed as finding the expected value of the function e^{-rx}.

$$\begin{aligned} E[e^{-rx}] &= \int_a^b \frac{e^{-rx}}{b-a} \\ &= \frac{e^{-ra}-e^{-rb}}{r(b-a)}. \end{aligned}$$

b) To find the mean and the variance we use the moment generating properties of the transform, namely:

$$E[X^n] = (-1)^n \frac{d}{dr} E[e^{-rx}]\Big|_{r=0}$$

Thus we have:

$$\begin{aligned} E[X] &= -\frac{d}{dr}E[e^{-rx}]\Big|_{r=0} \\ &= -\left\{ \left(\frac{1}{b-a}\right)\frac{e^{-rb}-e^{-ra}}{r^2} + \left(\frac{1}{b-a}\right)\frac{be^{-rb}-ae^{-ra}}{r} \right\}\Bigg|_{r=0} \\ (L'H\hat{o}pital) \quad &= -\frac{b^2-a^2}{b-a} - \frac{a^2-b^2}{b-a} \\ &= \frac{b+a}{2}. \end{aligned}$$

To find the Variance we need to find $E[X^2]$ and thus we need to take the second derivative of the transform and evaluate at $r = 0$,

$$\begin{aligned} E[X^2] &= \frac{d^2}{dr^2}E[e^{-rx}]\Big|_{r=0} \\ &= \left\{ \left(\frac{2}{b-a}\right)\frac{e^{-ra}-e^{-rb}}{r^3} + \left(\frac{2}{b-a}\right)\frac{ae^{-ra}-be^{-rb}}{r^2} + \left(\frac{1}{b-a}\right)\frac{a^2e^{-ra}-b^2e^{-rb}}{r} \right\}\Bigg|_{r=0} \\ (L'H\hat{o}pital) \quad &= \frac{1}{3}\frac{b^3-a^3}{b-a} + \frac{a^3-b^3}{b-a} + \frac{b^3-a^3}{b-a} \\ &= \frac{1}{3}(b^2+ab+a^2) \end{aligned}$$

and therefore we have:

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2$$

2. The transform for nonegative integer valued random variables is defined as:

$$p_x^T(z) = \sum_{i=1}^{\infty} z^{x_i} P(X = x_i) = E[z^X]$$

and therefore we have:

$$E[z^X] = \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{4}z^3.$$

b) We observe from above that if we take n derivatives of the transform and evaluate at $z = 1$ then we will have a linear combination of the first n moments.

$$\begin{aligned}
\frac{d}{dz}E[z^X]\Big|_{z=1} &= E[X] \\
\frac{d^2}{dz^2}E[z^X]\Big|_{z=1} &= E[X^2] - E[X] \\
\frac{d^3}{dz^3}E[z^X]\Big|_{z=1} &= E[X^3] - 3E[X^2] + 2E[X]
\end{aligned}$$

and therefore we find:

$$\begin{aligned}
E[X] &= \frac{d}{dz}E[z^X]\Big|_{z=1} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{7}{4}
\end{aligned}$$

and similarly,

$$\begin{aligned}
E[X^2] &= \frac{d^2}{dz^2}E[z^X]\Big|_{z=1} + E[X] \\
&= \frac{1}{2} + \frac{3}{2} + \frac{7}{4} = \frac{15}{4}
\end{aligned}$$

and finally,

$$\begin{aligned}
E[X^3] &= \frac{d^3}{dz^3}E[z^X]\Big|_{z=1} + 3E[X^2] - 2E[X] \\
&= \frac{6}{4} + \frac{45}{4} + \frac{14}{4} = \frac{37}{4}
\end{aligned}$$

c) Direct computation thankfully produces the same results.

3. a) Note that by the definition of the transform,

$$M_X(s) = \sum_x e^{sx} p_X(x)$$

and therefore when evaluated at $s = 0$, the transform should equal 1. We see that only the second option satisfies this requirement.
b) It is observed that the transform is that of a Poisson random variable with parameter $\lambda = 2$. Hence the pdf is given as follows:

$$\begin{aligned}
p_X(k) &= e^{-\lambda}\frac{\lambda^k}{k!} \\
p_X(0) &= e^{-2}
\end{aligned}$$