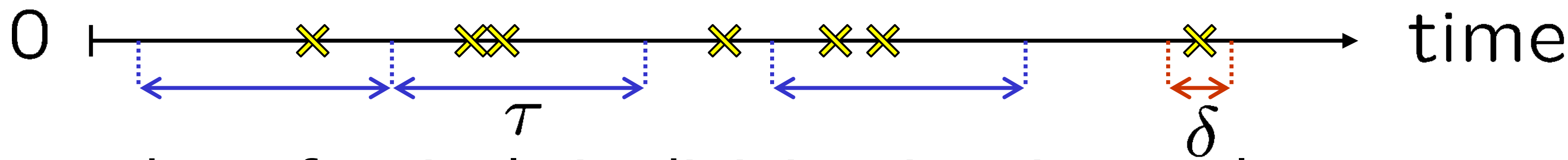
LECTURE 18

- Readings: Finish Section 5.2

Lecture outline

- Review of the Poisson process
- Properties
  - Adding Poisson Processes
  - Splitting Poisson Processes
- Examples

The Poisson Process: Review



- Number of arrivals in disjoint time intervals are independent, λ = "arrival rate"

$$\mathbf{P}(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k>0 \end{cases} \qquad \text{(for very small } \delta\text{)}$$

$$\mathbf{P}(k,\tau) = \frac{(\lambda t)^k}{k!}e^{-\lambda t} \qquad \mathbf{E}(N) = \lambda\tau \qquad \text{(Poisson)}$$

- Interarrival times (k =1):

$$f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0 \qquad \text{(Exponential)}$$

- Time to the k^{th} arrival:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0 \qquad \text{(Erlang)}$$

Example: **Poisson Catches**

- Catching fish according to Poisson $\lambda = 0.6/\text{hour}$.
- Fish for two hours, but if there's no catch, continue until the first one.

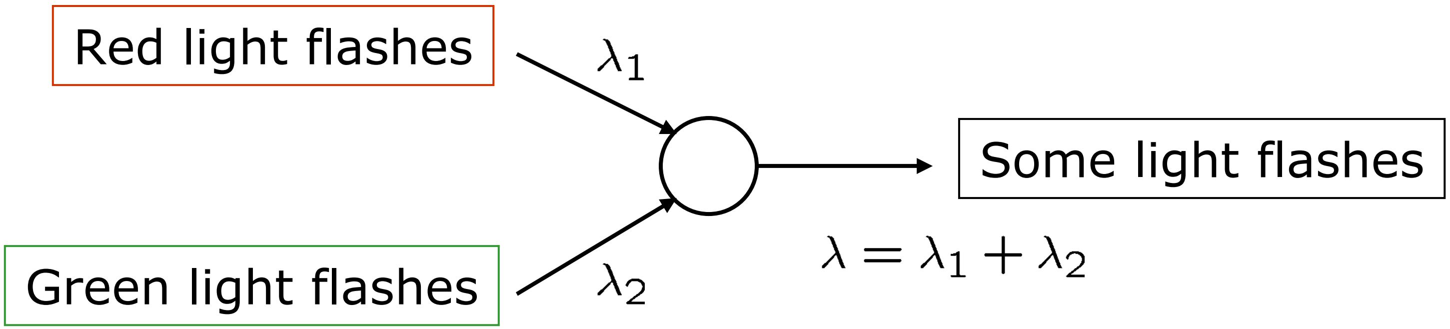
  - $\mathbf{P}(\text{fish more than 2 hrs}) =$
  - $\mathbf{P}(\text{fish more than 2 but less than 5 hrs}) =$
  - $\mathbf{P}(\text{catch at least 2 fish}) =$

Example: **Poisson Catches**

- Catching fish according to Poisson $\lambda = 0.6/\text{hour}$.
- Fish for two hours, but if there's no catch, continue until the first one.
  - $\mathbf{E}[\text{number of fish}] =$
  - $\mathbf{E}[\text{future fishing time} \mid \text{fished for 4 hrs}] =$
  - $\mathbf{E}[\text{total fishing time}] =$

Adding (Merging) Poisson Processes

- Sum of independent Poisson **random variables** is Poisson.
- Sum of independent Poisson **processes** is Poisson.

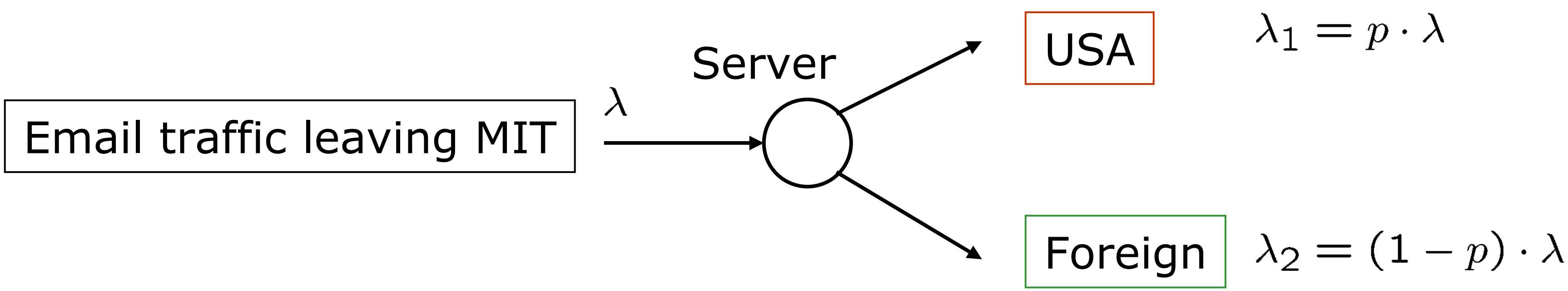


- What is the probability that the next arrival comes from the first process?

$$\frac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Splitting of Poisson Processes

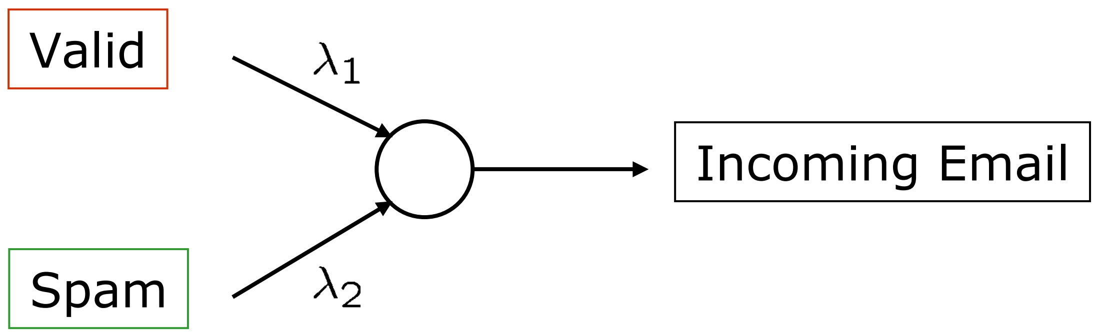
- Each message is routed along the first stream with probability p, and along the second stream with probability $1-p$.
  - Routing of different messages are independent.



  - Each output stream is Poisson.

Example: **Email Filter** (1)

- You have incoming email from two sources: valid email, and spam. We assume both to be Poisson.
- Your receive, on average, 2 valid emails per hour, and 1 spam email every 5 hours.



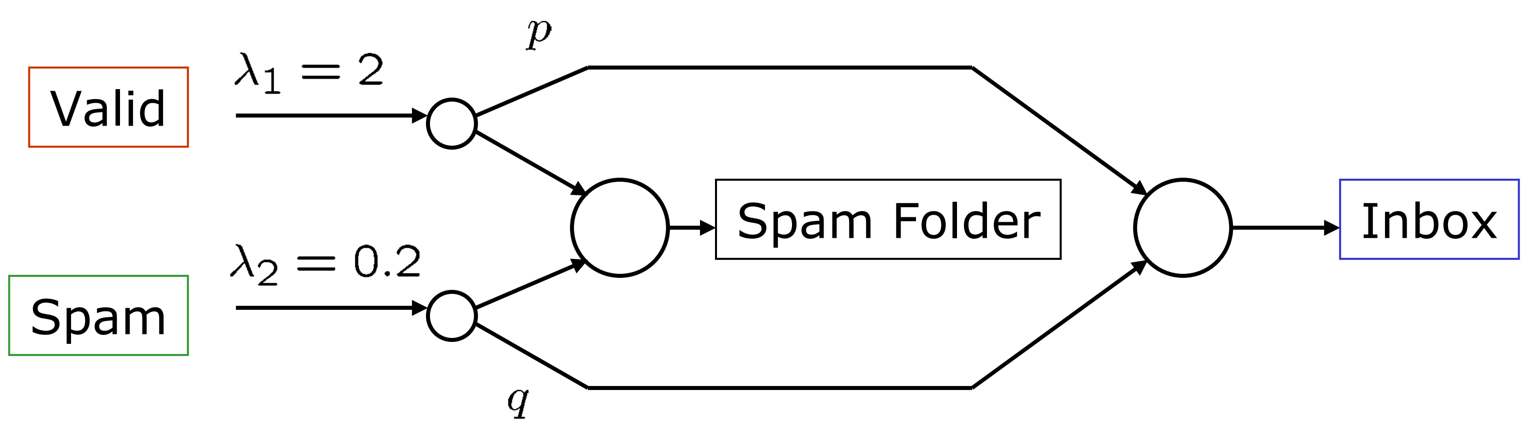
- Total incoming email rate =

$$\lambda = \lambda_1 + \lambda_2 = 2.2 \text{ emails per hour.}$$

- Probability that a received email is spam = $\frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{0.2}{2.2} \approx 0.09$

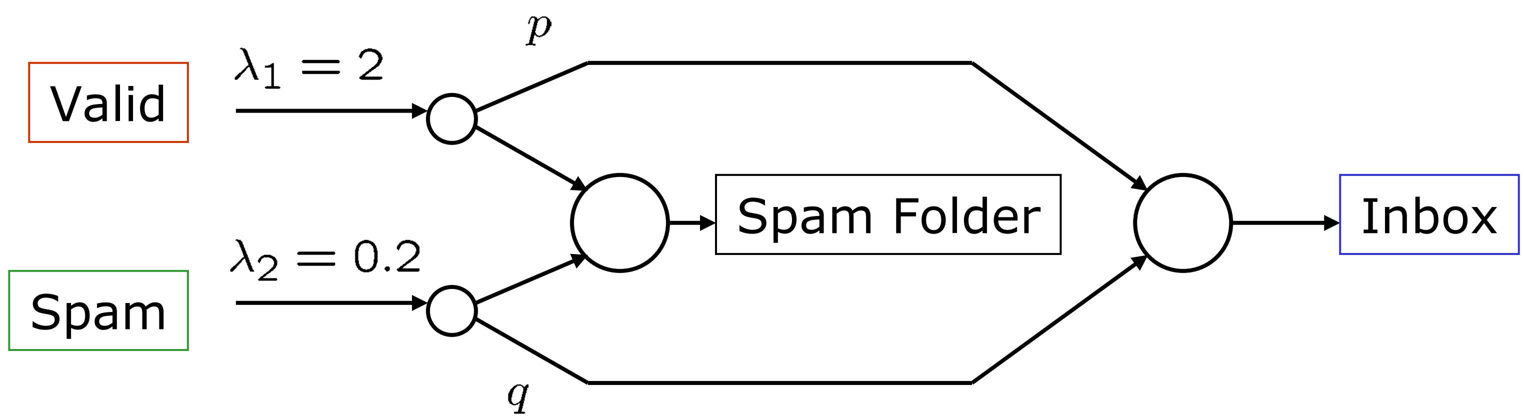
Example: **Email Filter** (2)

- You install a spam filter, that filters out spam email correctly 80% of the time, but also identifies a valid email as spam 5% of the time.



- $p = 0.95 \qquad q = 0.2$
- Inbox email rate = $p\lambda_1 + q\lambda_2 = 0.95 \cdot 2 + 0.2 \cdot 0.2 = 1.94$
- Spam folder email rate = $2.2 - 1.94 = 0.26$

Example: **Email Filter** (3)



- Probability that an email in the inbox is spam = $\frac{q\lambda_2}{p\lambda_1 + q\lambda_2} = \frac{0.2 \cdot 0.2}{1.94} \approx 0.02$

- Probability that an email in the spam folder is valid = $\frac{(1-p)\lambda_1}{(1-p)\lambda_1 + (1-q)\lambda_2} = \frac{0.05 \cdot 2}{0.26} \approx 0.38$

- Every how often should you check your spam folder, to find one valid email, on average?

$$\mathbf{E}(N) = \lambda_1(1-p)\tau = 1 \Rightarrow \tau = \frac{1}{0.05 \cdot 2} = 10 \text{ hrs.}$$