Midterm Quiz

- You have 110 minutes to complete the quiz.

- This is a closed-book quiz, except that three $8.5'' \times 11''$ sheets of notes are allowed.

- Calculators are allowed (provided that erasable memory is cleared), but will probably not be useful.

- There are two problems on the quiz. The first is a seven-part problem, each part worth 10 points. The second consists of three unrelated true-false questions, each worth 10 points.

- The problems are not necessarily in order of difficulty.

- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.

- If we can't read it, we can't grade it.

- If you don't understand a problem, please ask.

Problem M.1 (70 points)

Recall that an M-simplex signal set is a set of M signals $\mathcal{A} = \{\mathbf{a}_j \in \mathbb{R}^{M-1}, 1 \leq j \leq M\}$ in an $(M-1)$-dimensional real space $\mathbb{R}^{M-1}$, such that, for some $E_{\mathcal{A}} > 0$,

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \begin{cases} E_{\mathcal{A}}, & \text{if } i = j; \\ -\frac{E_{\mathcal{A}}}{M-1}, & \text{if } i \neq j. \end{cases}$$

Initially we will assume that M is a power of 2, $M = 2^m$, for some integer m.

(a) Compute the nominal spectral efficiency $\rho(\mathcal{A})$ and the nominal coding gain $\gamma_c(\mathcal{A})$ of an M-simplex signal set $\mathcal{A}$ on an AWGN channel as a function of $M = 2^m$.

(b) What is the limit of the effective coding gain $\gamma_{\text{eff}}(\mathcal{A})$ of an M-simplex signal set $\mathcal{A}$ as $M \to \infty$, at a target error rate of $\Pr(E) \approx 10^{-5}$?

(c) Give a method of implementing an $(M = 2^m)$-simplex signal set $\mathcal{A}$ in which each signal $\mathbf{a}_j$ is a sequence of points from a 2-PAM signal set $\{\pm\alpha\}$.

Now consider a concatenated coding scheme in which

- the outer code is an (n, k, d) linear code $\mathcal{C}$ over a finite field $\mathbb{F}_q$ with $q = 2^m$, which has N_d codewords of minimum nonzero weight;

- outer q-ary code symbols are mapped into a q-simplex signal set $\mathcal{A}$ via a one-to-one map $s : \mathbb{F}_q \to \mathcal{A}$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-tuple in $(\mathbb{F}_q)^n$, then $s(\mathbf{x}) = (s(x_1), s(x_2), \ldots, s(x_n))$ will be called the Euclidean image of $\mathbf{x}$. Let $\mathcal{A}' = s(\mathcal{C}) = \{s(\mathbf{x}), \mathbf{x} \in \mathcal{C}\}$ denote the signal set consisting of the Euclidean images of all codewords $\mathbf{x} \in \mathcal{C}$.

(d) Compute the nominal spectral efficiency $\rho(\mathcal{A}')$ of the concatenated signal set $\mathcal{A}'$ on an AWGN channel. Is this signal set appropriate for the power-limited or the bandwidth-limited regime?

(e) Compute $d_{\min}^2(\mathcal{A}')$, $K_{\min}(\mathcal{A}')$, and $\gamma_c(\mathcal{A}')$. Give a good estimate of an appropriately normalized error probability for $\mathcal{A}'$.

Now consider the case in which $\mathcal{C}$ is an $(n = q + 1, k = 2, d = q)$ linear code over $\mathbb{F}_q$.

(f) Show that a code $\mathcal{C}$ with these parameters exists whenever q is a prime power, $q = p^m$. Show that all nonzero codewords in $\mathcal{C}$ have the same Hamming weight.

(g) Show that the Euclidean image $\mathcal{A}' = s(\mathcal{C})$ of $\mathcal{C}$ is a q^2-simplex signal set.

Problem M.2 (30 points)

For each of the propositions below, state whether the proposition is true or false, and give a proof of not more than a few sentences. No credit will be given for correct answers without an adequate explanation.

(a) Let $p(t)$ be a complex $\mathcal{L}_2$ signal with Fourier transform $P(f)$. If the set of time shifts $\{p(t - kT), k \in \mathbb{Z}\}$ is orthonormal for some $T > 0$, then $P(0) \neq 0$.

(b) Let $s(\mathcal{C})$ be the Euclidean-space image of a binary linear block code $\mathcal{C}$ under a 2-PAM map $s : \{0, 1\} \to \{\pm\alpha\}$. Then the mean $\mathbf{m}$ of the signal set $s(\mathcal{C})$ is $\mathbf{0}$, unless there is some coordinate in which all codewords of $\mathcal{C}$ have the value 0.

(c) A polynomial $f(z) \in \mathbb{F}_q[z]$ satisfies $f(\beta) = 0$ for all $\beta \in \mathbb{F}_q$ if and only if $f(z)$ is a multiple of $z^q - z$.