6.451 Principles of Digital Communication II — Wednesday, February 9, 2005
MIT, Spring 2005 — Handout #5
Due: Wednesday, February 16, 2005

Problem Set 2

Problem 2.1 (Cartesian-product constellations)

(a) Show that if $\mathcal{A}' = \mathcal{A}^K$, then the parameters $N, \log_2 M, E(\mathcal{A}')$ and $K_{\min}(\mathcal{A}')$ of $\mathcal{A}'$ are K times as large as the corresponding parameters of $\mathcal{A}$, whereas the normalized parameters ρ, E_s, E_b and $d^2_{\min}(\mathcal{A})$ are the same as those of $\mathcal{A}$. Verify that these relations hold for $(M \times M)$-QAM constellations.

(b) Show that if the signal constellation is a Cartesian product $\mathcal{A}^K$, then MD detection can be performed by performing independent MD detection on each of the K components of the received KN-tuple $\mathbf{y} = (y_1, y_2, \ldots, y_K)$. Using this result, sketch the decision regions of a (4×4)-QAM signal set.

(c) Show that if $\Pr(E)$ is the probability of error for MD detection of $\mathcal{A}$, then the probability of error for MD detection of $\mathcal{A}'$ is

$$\Pr(E)' = 1 - (1 - \Pr(E))^K,$$

Show that $\Pr(E)' \approx K \Pr(E)$ if $\Pr(E)$ is small.

Problem 2.2 ($\Pr(E)$ invariance to translation, orthogonal transformations, or scaling)

Let $\Pr(E \mid \mathbf{a}_j)$ be the probability of error when a signal $\mathbf{a}_j$ is selected equiprobably from an N-dimensional signal set $\mathcal{A}$ and transmitted over a discrete-time AWGN channel, and the channel output $\mathbf{Y} = \mathbf{a}_j + \mathbf{N}$ is mapped to a signal $\hat{\mathbf{a}}_j \in \mathcal{A}$ by a minimum-distance decision rule. An error event E occurs if $\hat{\mathbf{a}}_j \neq \mathbf{a}_j$. $\Pr(E)$ denotes the average error probability.

(a) Show that the probabilities of error $\Pr(E \mid \mathbf{a}_j)$ are unchanged if $\mathcal{A}$ is translated by any vector $\mathbf{v}$; *i.e.,* the constellation $\mathcal{A}' = \mathcal{A} + \mathbf{v}$ has the same $\Pr(E)$ as $\mathcal{A}$.

(b) Show that $\Pr(E)$ is invariant under orthogonal transformations; *i.e.,* $\mathcal{A}' = U\mathcal{A}$ has the same $\Pr(E)$ as $\mathcal{A}$ when U is any orthogonal $N \times N$ matrix (*i.e.,* $U^{-1} = U^T$).

(c) Show that $\Pr(E)$ is unchanged if both $\mathcal{A}$ and $\mathbf{N}$ are scaled by $\alpha > 0$.

Problem 2.3 (optimality of zero-mean constellations)

Consider an arbitrary signal set $\mathcal{A} = \{\mathbf{a}_j, 1 \leq j \leq M\}$. Assume that all signals are equiprobable. Let $\mathbf{m}(\mathcal{A}) = \frac{1}{M}\sum_j \mathbf{a}_j$ be the average signal, and let $\mathcal{A}'$ be $\mathcal{A}$ translated by $\mathbf{m}(\mathcal{A})$ so that the mean of $\mathcal{A}'$ is zero: $\mathcal{A}' = \mathcal{A} - \mathbf{m}(\mathcal{A}) = \{\mathbf{a}_j - \mathbf{m}(\mathcal{A}), 1 \leq j \leq M\}$. Let $E(\mathcal{A})$ and $E(\mathcal{A}')$ denote the average energies of $\mathcal{A}$ and $\mathcal{A}'$, respectively.

(a) Show that the error probability of an MD detector is the same for $\mathcal{A}'$ as it is for $\mathcal{A}$.

(b) Show that $E(\mathcal{A}') = E(\mathcal{A}) - ||\mathbf{m}(\mathcal{A})||^2$. Conclude that removing the mean $\mathbf{m}(\mathcal{A})$ is always a good idea.

(c) Show that a binary antipodal signal set $\mathcal{A} = \{\pm\mathbf{a}\}$ is always optimal for $M = 2$.

Problem 2.4 (Non-equiprobable signals).

Let $\mathbf{a}_j$ and $\mathbf{a}_{j'}$ be two signals that are not equiprobable. Find the optimum (MPE) pairwise decision rule and pairwise error probability $\Pr\{\mathbf{a}_j \to \mathbf{a}_{j'}\}$.

Problem 2.5 (UBE for M-PAM constellations).

For an M-PAM constellation $\mathcal{A}$, show that $K_{\min}(\mathcal{A}) = 2(M-1)/M$. Conclude that the union bound estimate of $\Pr(E)$ is

$$\Pr(E) \approx 2\left(\frac{M-1}{M}\right) Q\left(\frac{d}{2\sigma}\right).$$

Observe that in this case the union bound estimate is exact. Explain why.