Numerical Methods for PDEs

Integral Equation Methods, Lecture 6
Discretization and Quadrature

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Outline

Everything is Galerkin

Reminder of 1-D and 3-D 2nd Kind

Collocation is Galerkin

- Single Point Quadrature
- Qualocation

Nystrom is Galerkin

- N point quadrature

Multidimensional Quadrature

Equation Examples

1-D

"Volume" Integral Equations

First Kind

$$\Psi(x) = \int_{-1}^{1} G(x, x')\sigma(x')dx' \quad x \in [-1, 1]$$

Second Kind

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} G(x, x')\sigma(x')dx' \quad x \in [-1, 1]$$

Equation Examples

3-D

"Surface" 3-D Potential Integral Equations

First Kind

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Second Kind

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}) + \int_\Gamma \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Discretization

2nd Kind

Collocation

Introduce a Basis Representation

$$\sigma_n(x) = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x)$$

Make Residual Zero at Collocation Points x_i

$$\Psi(x_i) = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x_i) - \int_{-1}^{1} G(x_i, x') \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x')dx'$$

Discretization

2nd Kind

Galerkin

Make Residual orthogonal to basis

$$\int_{P_i} \varphi_i(x)\Psi(x)dx =$$

$$\int_{P_i} \varphi_i(x) \sum_{j=1}^{n} \sigma_{nj}\varphi_j(x)dx$$

$$+ \int_{P_i} \varphi_i(x) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x,x')\varphi_j(x')dx'dx$$

Note: P_i is the support of $\varphi_i(x)$.

Discretization

2nd Kind

Galerkin Cont.

Assume Orthonormal Basis

Orthogonality

$$\int_{P_i U P_j} \varphi_i(x)\varphi_j(x)dx = 0$$

Normalization

$$\int_{P_i} \varphi_i(x)\varphi_i(x)dx = 1$$

Discretization

2nd Kind

Comparison

Collocation = Galerkin with one point quadrature

One point quadrature implies

$$\int_{P_i} \varphi_i(x)\Psi(x)dx \approx w_i \varphi_i(x_i)\Psi(x_i)$$

$x_i =$ quadrature point

$w_i =$ quadrature weight

Discretization

2nd Kind

Comparison Cont.

One point quadrature implies

$$\int_{P_i} \varphi_i(x) \sum_{j=1}^{n} \sigma_{nj} \varphi_j(x) dx \approx w_i \varphi_i(x_i) \sum_{j=1}^{n} \sigma_{nj} \varphi_j(x_i)$$

$$\int_{P_i} \varphi_i(x) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x, x') \varphi_j(x') dx' dx \approx$$

$$w_i \varphi_i(x_i) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x_i, x') \varphi_j(x') dx'$$

Discretization

2nd Kind

Comparison Cont II.

Putting together

$$w_i \varphi_i(x_i) \Psi(x_i) =$$

$$w_i \varphi_i(x_i) \sum_{j=1}^{n} \sigma_{nj} \varphi_j(x_i)$$

$$+ w_i \varphi_i(x_i) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x_i, x') \varphi_j(x') dx'$$

Discretization

2nd Kind

Comparison final.

Dividing by $w_i\varphi_i(x_i)$ and reorganizing

$$\Psi(x_i) = \sum_{j=1}^{n} \sigma_{nj}\varphi_j(x_i) + \int \sum_{j=1}^{n} \sigma_{nj} G(x_i, x')\varphi_j(x')dx'$$

which is precisely the collocation equations.

Discretization

Qualocation

Alternative

$$\int_{P_i} \varphi_i(x) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x, x')\varphi_j(x')dx'dx \approx$$

$$w_i \varphi_i(x_i) \sum_{j=1}^{n} \sigma_{nj} \int_{P_j} G(x_i, x')\varphi_j(x')dx'$$

or

$$\sum_{j=1}^{n} \sigma_{nj} w_j \varphi_j(x_j) \int_{P_i} G(x', x_j)\varphi_i(x')dx'$$

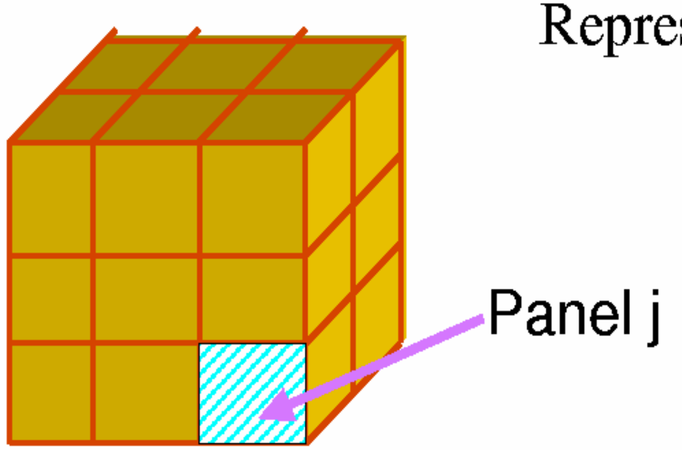
Discretization

Qualocation

3-D Piecewise Constant Basis

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}) + \int_\Gamma \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma'$$

Discretize Surface into Panels



Represent $\sigma(x) \approx \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

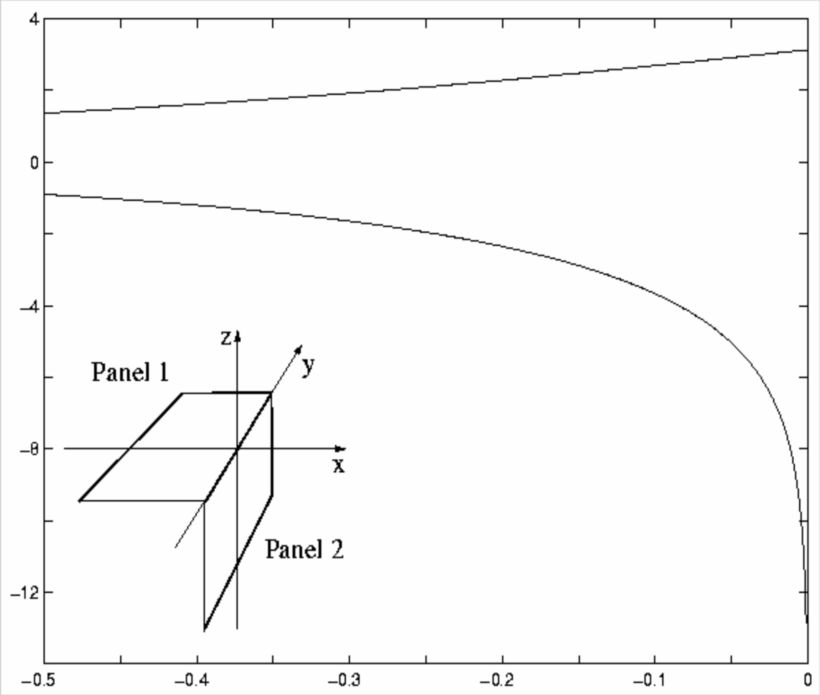
$\varphi_j(x) = 1$ if x is on panel j

$\varphi_j(x) = 0$ otherwise

Discretization

Qualocation

Integration Difficulty



top curve:

$$\int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} d\Gamma'$$

Bottom curve:

$$\int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} d\Gamma'$$

Discretization

Qualocation

Nystrom Method

Set **quadrature points = collocation points**

$$\Psi(x_1) = \sigma_{n1} + \sum_{j=1}^{n} w_j G(x_1, x_j)\sigma_{nj}$$

$$\vdots$$

$$\Psi(x_n) = \sigma_{n1} + \sum_{j=1}^{n} w_j G(x_n, x_j)\sigma_{nj}$$

System of n equations in n unknowns

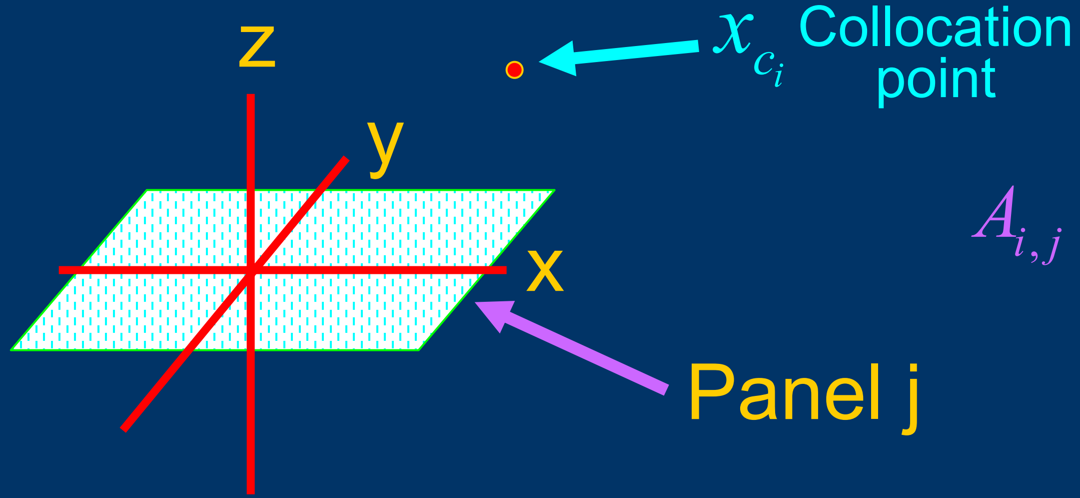
Collocation equation per quad/colloc point

Nystrom = Galerkin with n point quadrature

2-D Integration (from 3-D problems)

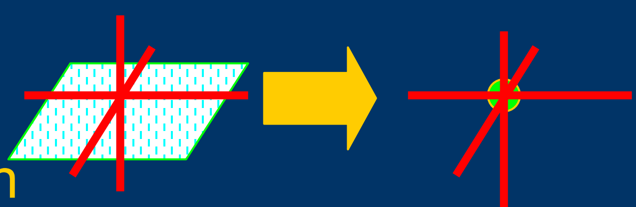
Reminder

Calculating Matrix Elements



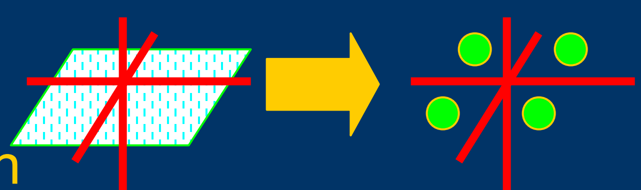
$$A_{i,j} = \int_{panel\ j} G\left(x_{c_i}, x'\right) dS'$$

One point quadrature Approximation



$$A_{i,j} \approx Area \bullet G\left(x_{c_i}, x_{centroid_j}\right)$$

Four point quadrature Approximation



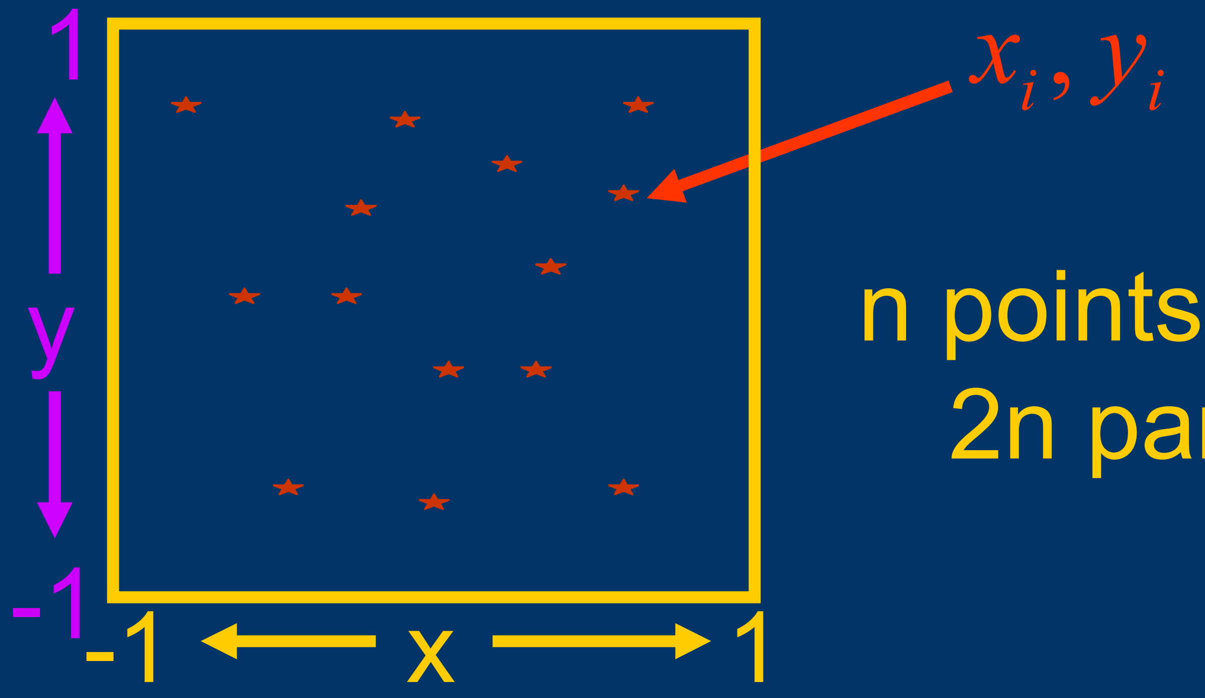
$$A_{i,j} \approx \sum_{j=1}^{4} \frac{Area}{4} * G\left(x_{c_i}, x_{point_j}\right)$$

Symmetrically Normalized 2-D Problem

Quadrature Scheme

General n-point formula

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx \simeq \sum_{i=1}^{n} w_i f(x_i, y_i)$$



n points, n weights
2n parameters

Symmetrically Normalized 2-D Problem

Quadrature Scheme

2-D Gaussian Quadrature

Exactness for l-th order polys

$$\int_{-1}^{1}\int_{-1}^{1} p_l(x,y)dx = \sum_{i=1}^{n} w_i p_l(x_i, y_i)$$

l-th order 2-D poly definition

$$p_l(x,y) = \sum_{(i+j)\le l} \alpha_{i,j} x^i y^j$$

$$\text{Number of terms} = \frac{(l+1)(l+2)}{2}$$

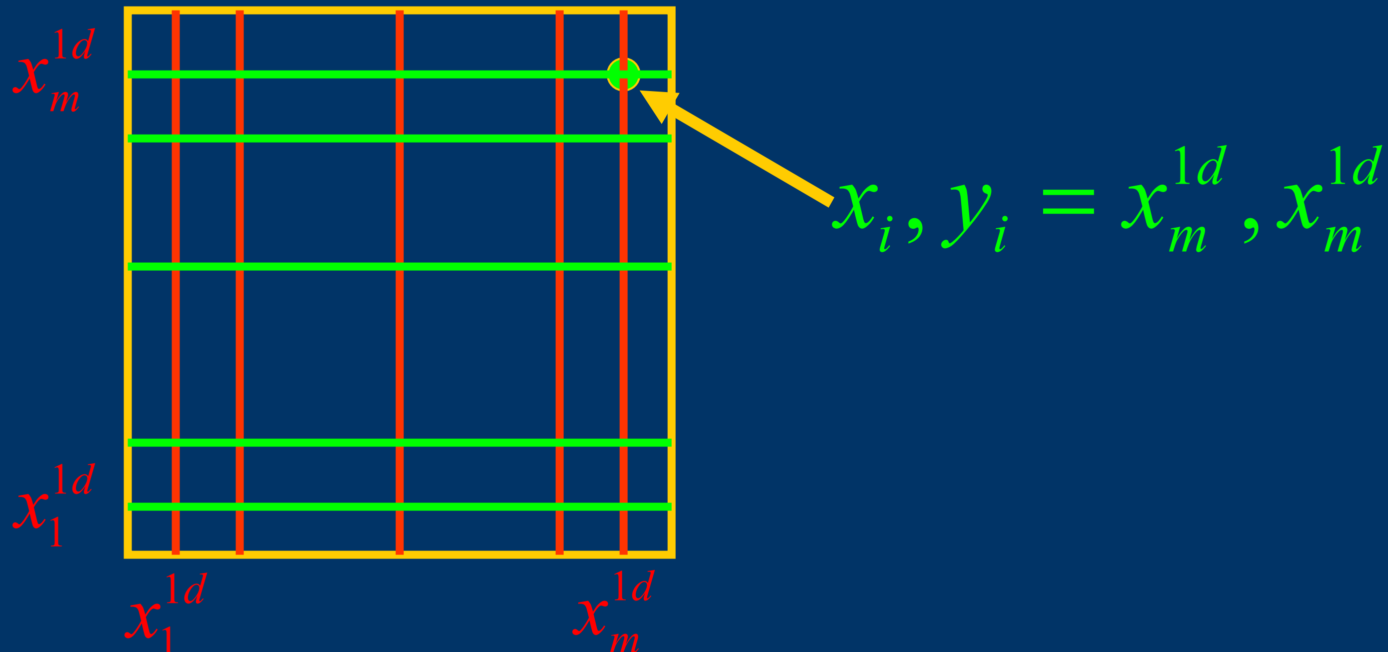
Symmetrically Normalized 2-D Problem

Quadrature Scheme

Product Method

1) Get a 1-d formula $\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{m} w_i^{1d} f\left(x_i^{1d}\right)$

2) Form a "product" grid



Symmetrically Normalized 2-D Problem

Quadrature Scheme

Product Method Continued

3) Determine the weights

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dx \simeq \sum_{i=1}^{m}\sum_{i=1}^{m} w_{i,j} f\left(x_i^{1d}, x_j^{1d}\right)$$

$$w_{i,j} = w_i^{1d} w_j^{1d}$$

Note that n = mxm

Symmetrically Normalized 2-D Problem

Quadrature Scheme

Product Method Theorem

Theorem: The product method is exact for all 2-d polys up to order 2m

Proof:

Using the l-th order poly def

$$\int_{-1}^{1}\int_{-1}^{1} p_m(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} \sum_{(i+j)\leq m} \alpha_{i,j} x^i y^j dxdy$$

Using the properties of integration

$$\int_{-1}^{1}\int_{-1}^{1}\sum_{(i+j)\le m}\alpha_{i,j}x^i y^j dxdy = \sum_{(i+j)\le m}\alpha_{i,j}\int_{-1}^{1}\left(\int_{-1}^{1}x^i dx\right)y^j dy$$

$$= \sum_{(i+j)\le m}\alpha_{i,j}\left(\int_{-1}^{1}x^i dx\right)\left(\int_{-1}^{1}y^j dy\right)$$

Since the 1-d quadrature is exact for polys of order less than 2m

$$\sum_{(i+j)\le m} \alpha_{i,j} \left(\int_{-1}^{1} x^i dx \right) \left(\int_{-1}^{1} y^j dy \right) =$$

$$\sum_{(i+j)\le m} \alpha_{i,j} \left(\sum_{k=1}^{m} w_k^{1d} \left(x_k^{1d} \right)^i \right) \left(\sum_{l=1}^{m} w_l^{1d} \left(x_l^{1d} \right)^j \right)$$

Symmetrically Normalized 2-D Problem

Quadrature Scheme

Product Theorem Cont.

Rearranging the sums

$$\sum_{(i+j)\le m} \alpha_{i,j} \left(\sum_{k=1}^{m} w_k^{1d} \left(x_k^{1d} \right)^i \right) \left(\sum_{l=1}^{m} w_l^{1d} \left(x_l^{1d} \right)^j \right) =$$

$$\sum_{l=1}^{m} \sum_{k=1}^{m} w_k^{1d} w_l^{1d} \sum_{(i+j)\le m} \alpha_{i,j} \left(x_k^{1d} \right)^i \left(x_l^{1d} \right)^j =$$

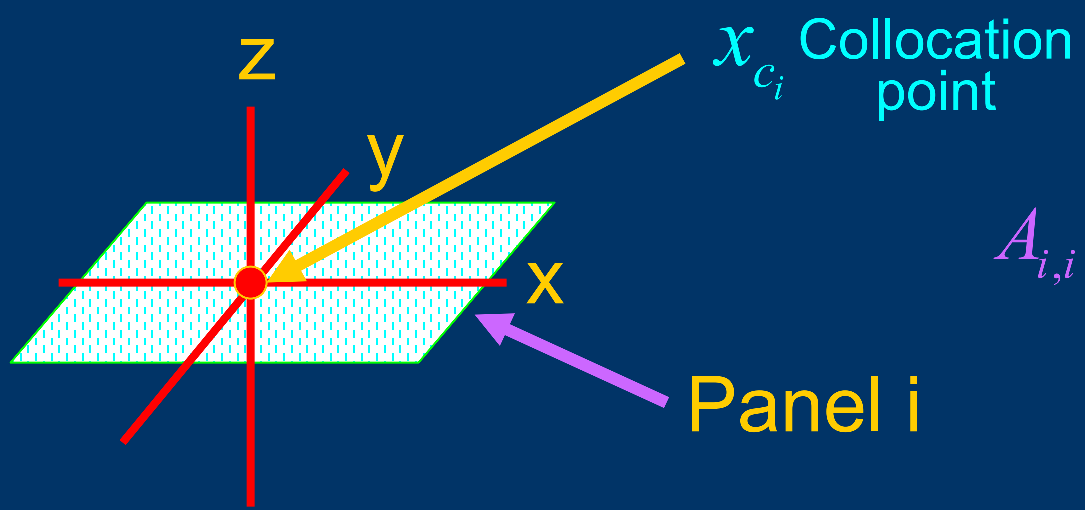
$$\sum_{l=1}^{m} \sum_{k=1}^{m} w_k^{1d} w_l^{1d} p_m \left(x_k^{1d}, x_l^{1d} \right)$$

Which proves the theorem

3-D Laplace's Equation

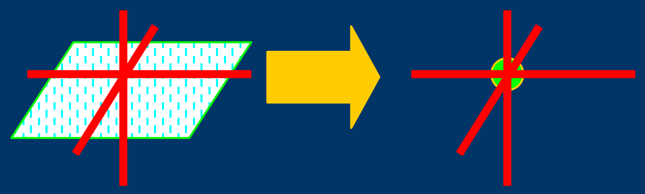
Basis Function Approach

Calculating "Self-Term"



$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

One point quadrature Approximation



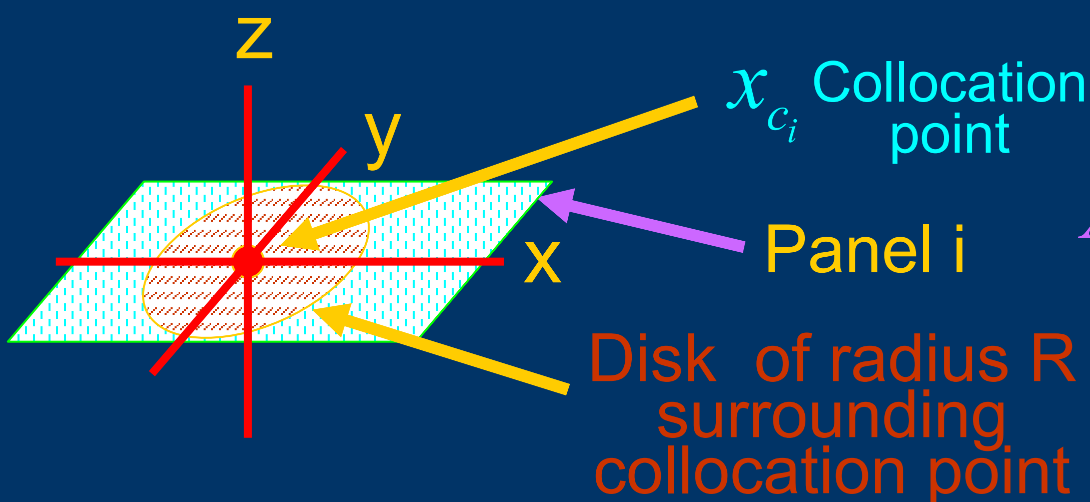
$$A_{i,i} \approx \frac{Panel\ \ Area}{\underbrace{\left\| x_{c_i} - x_{c_i} \right\|}_{0}}$$

$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' \quad \text{is an integrable singularity}$$

3-D Laplace's Equation

Basis Function Approach

Calculating "Self-Term"
Tricks of the trade



$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

Integrate in two pieces

$$A_{i,i} = \int_{disk} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' + \int_{rest\ of\ panel} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

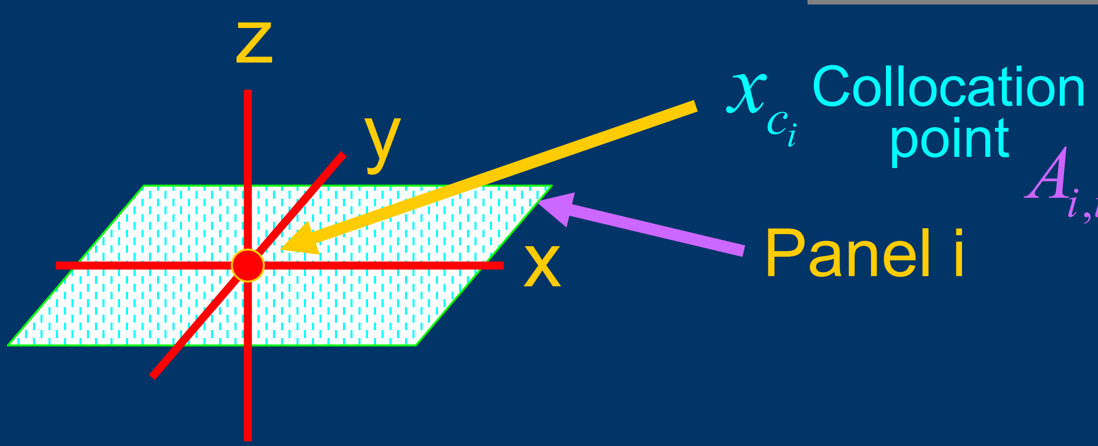
Disk Integral has singularity but has analytic formula

$$\int_{disk} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' = \int_0^R \int_0^{2\pi} \frac{1}{r} r\,dr\,d\theta = 2\pi R$$

3-D Laplace's Equation

Basis Function Approach

Calculating "Self-Term"
Other Tricks of the trade



$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

Integrand is singular

1) If panel is a flat polygon, analytical formulas exist

2) Curve panels can be handled with projection