Numerical Methods for PDEs

Integral Equation Methods, Lecture 1
Discretization of Boundary Integral Equations

Notes by Suvranu De and J. White

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Outline for this Module

Overview of Integral Equation Methods

Important for many exterior problems

(Fluids, Electromagnetics, Acoustics)

Quadrature and Cubature for computing integrals

One and Two dimensional basics

Dealing with Singularities

1^{st} and 2^{nd} Kind Integral Equations

Collocation, Galerkin and Nystrom theory

Alternative Integral Formulations

Ansatz approach and Green's theorem

Outline for this Module

Fast Solvers

Fast Multipole and FFT-based methods.

Outline

Integral Equation Methods

- Exterior versus interior problems
- Start with using point sources

Standard Solution Methods

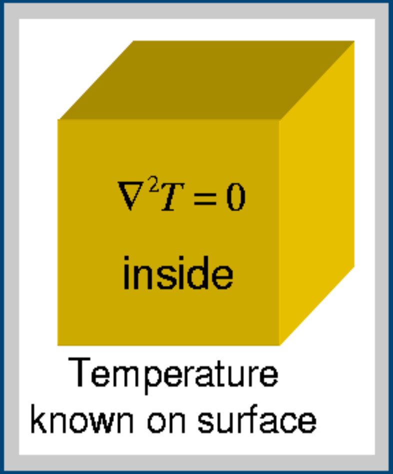
- Collocation Method
- Galerkin Method

Some issues in 3D

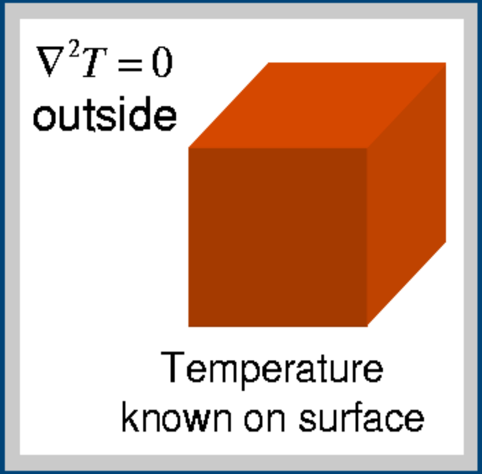
- Singular integrals

Interior Vs Exterior Problems

Interior



Exterior



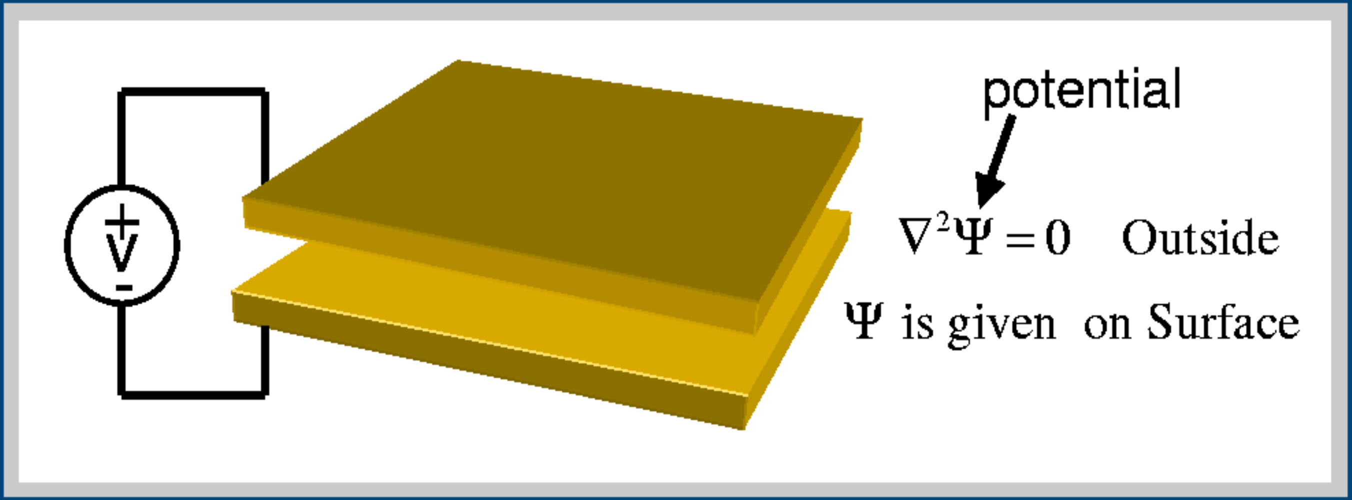
"Temperature in a tank" "Ice cube in a bath"

What is the heat flow?

Heat flow = Thermal conductivity $\int_{surface} \frac{\partial T}{\partial n}$

Examples

Computation of Capacitance

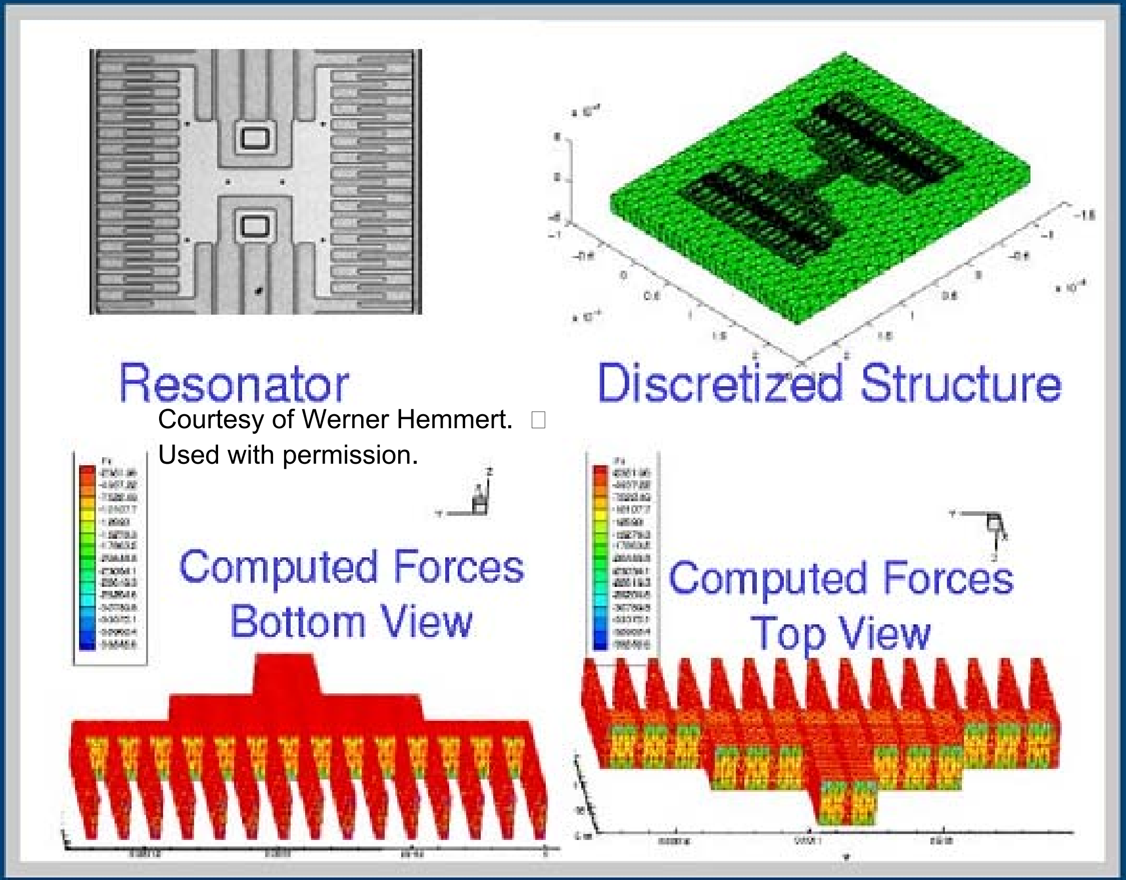


What is the capacitance?

Capacitance = Dielectric Permittivity $\int \frac{\partial \Psi}{\partial n}$

Examples

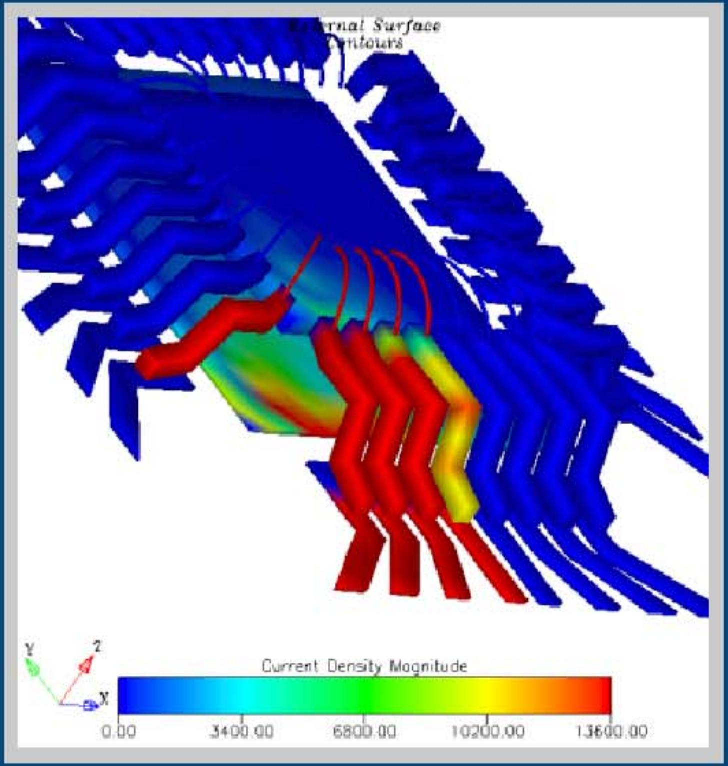
Drag Force in a Microresonator



Courtesy of Werner Hemmert. Used with permission.

Examples

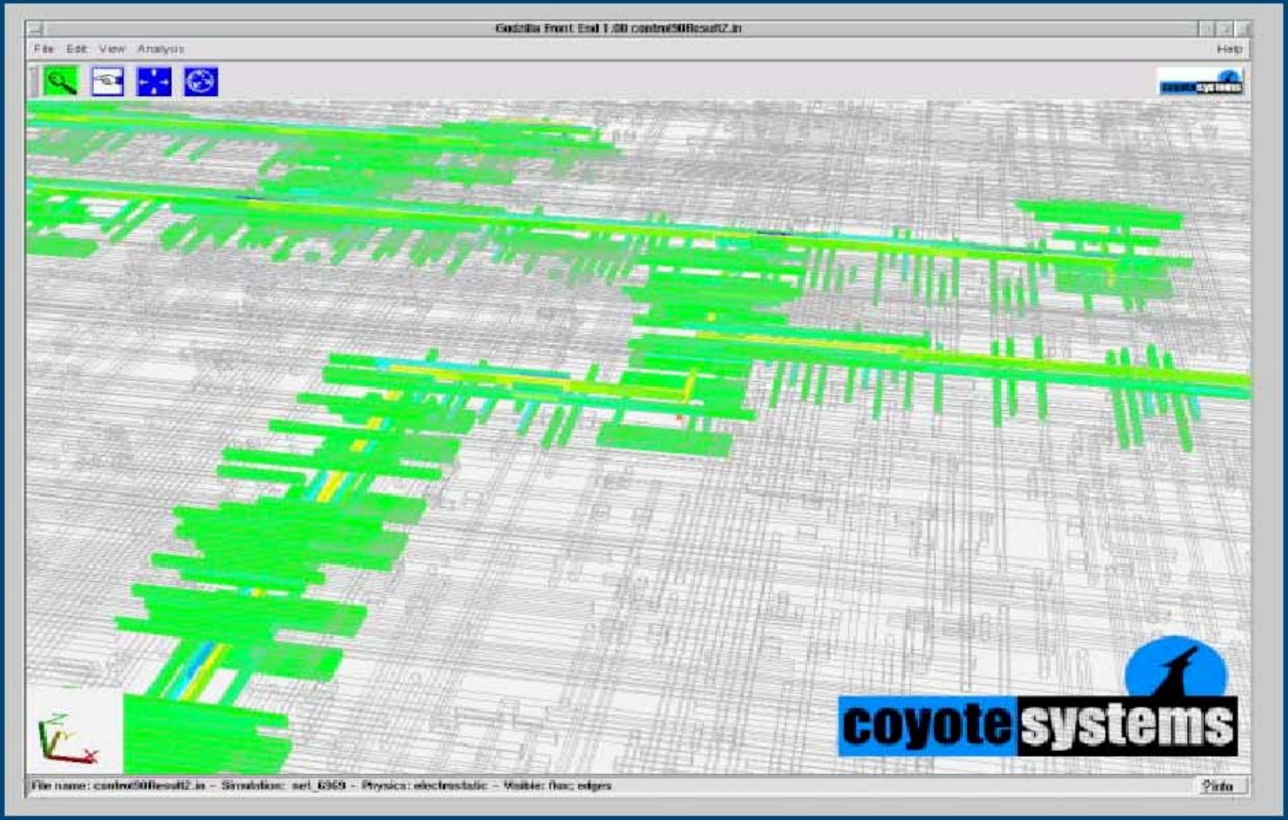
Electromagnetic Coupling in a Package



Picture Thanks to Coventor.

Examples

Capacitance of Microprocessor Signal Lines



What is common about these problems?

Exterior Problems

MEMS device - fluid (air) creates drag

Package - Exterior fields create coupling

Signal Line - Exterior fields.

Quantities of interest are on surface

MEMS device - Just want surface traction force

Package - Just want coupling between conductors

Signal Line - Just want surface charge.

Exterior problem is linear and space-invariant

MEMS device - Exterior Stoke's flow equation (linear)

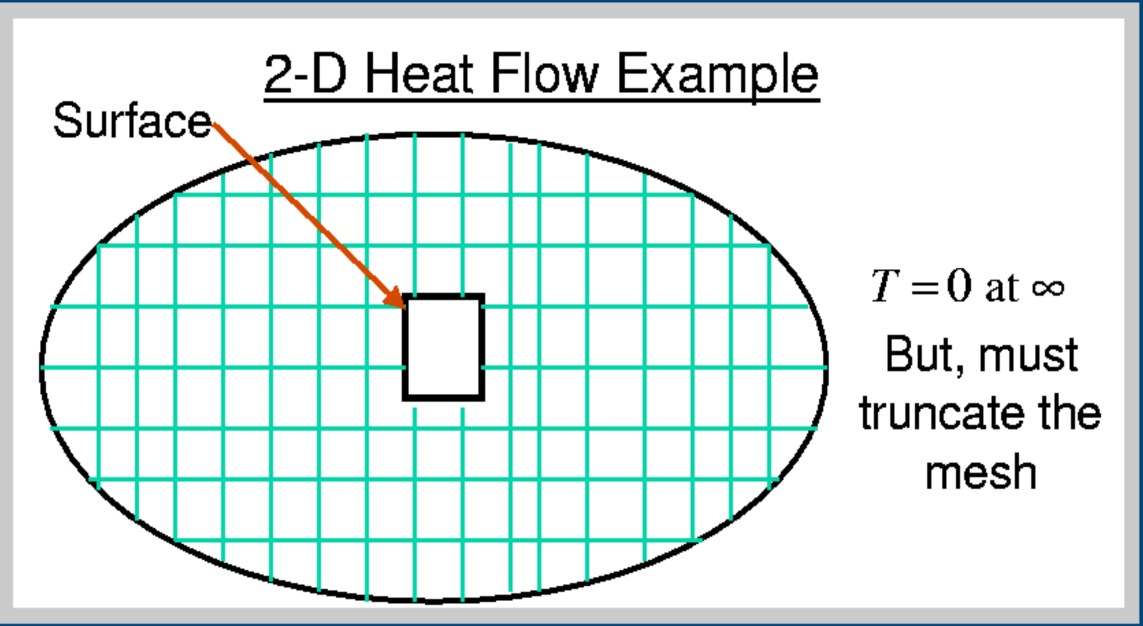
Package - Maxwell's equations in free space (linear)

Signal line - Laplace's equation in free spce (linear)

But problems are geometrically very complex

Exterior Problems

Why not use FDM / FEM?



Only need $\frac{\partial T}{\partial n}$ on the surface, but T is computed everywhere.

Must truncate the mesh, $\Rightarrow T(\infty) = 0$ becomes $T(R) = 0$.

Laplace's Equation

Green's Function

In 2D

If $u = log\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$

then $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \forall \quad (x,y)\neq(x_0,y_0)$

In 3D

If $u = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$

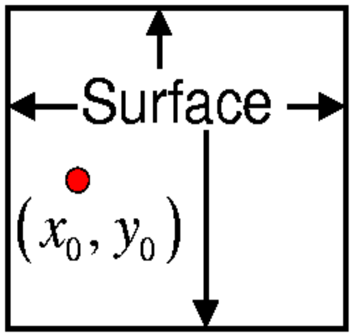
then $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad \forall \quad (x,y,z)\neq(x_0,y_0,z_0)$

Proof: Just differentiate and see!

Laplace's Equation in 2D

Simple idea

u is given on surface



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

$$\text{Let } u = \log\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

~~Problem Solved~~

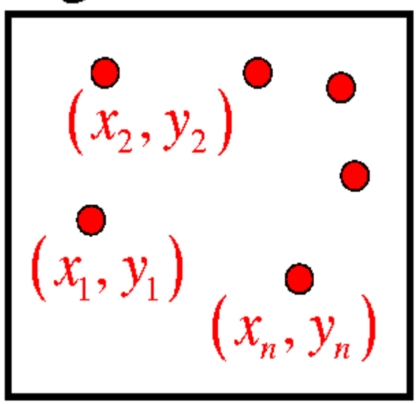
Does not match boundary conditions!

Laplace's Equation in 2D

Simple idea

"More points"

u is given on surface



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

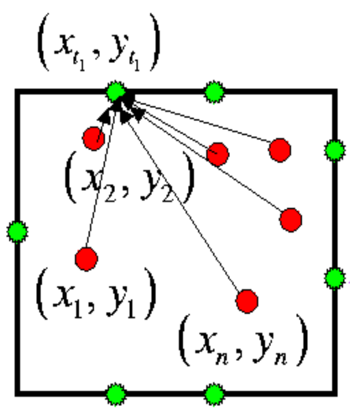
Let $u = \sum_{i=1}^{n} \alpha_i \log\left(\sqrt{(x - x_i)^2 + (y - y_i)^2}\right) = \sum_{i=1}^{n} \alpha_i G(x - x_i, y - y_i)$

Pick the α_i's to match the boundary conditions!

Laplace's Equation in 2D

Simple idea

"More points equations"



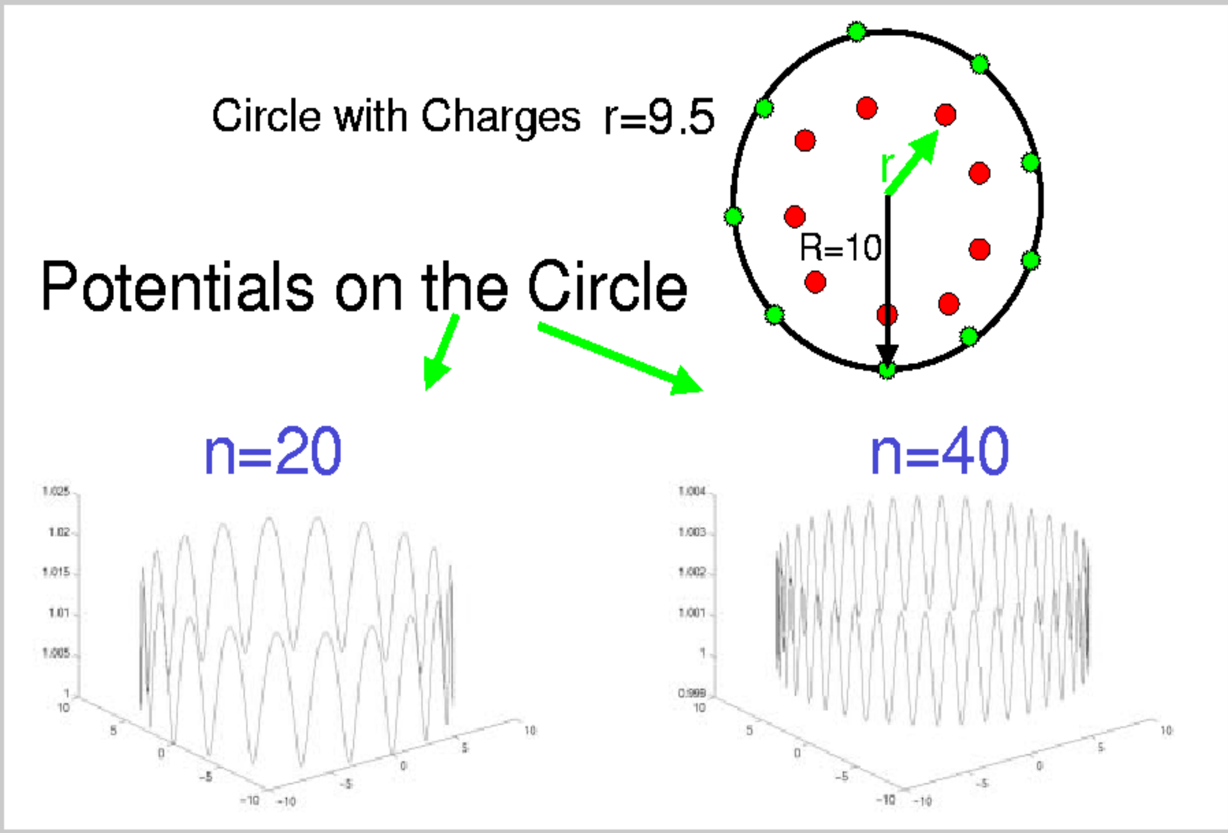
Source Strengths selected to give correct potential at **test** points.

$$\begin{bmatrix} G(x_{t_1}-x_1, y_{t_1}-y_1) & \cdots & \cdots & G(x_{t_1}-x_n, y_{t_1}-y_n) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ G(x_{t_n}-x_1, y_{t_n}-y_1) & \cdots & \cdots & G(x_{t_n}-x_n, y_{t_n}-y_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}, y_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_n}, y_{t_n}) \end{bmatrix}$$

Laplace's Equation in 2D

Simple idea

Computational Results

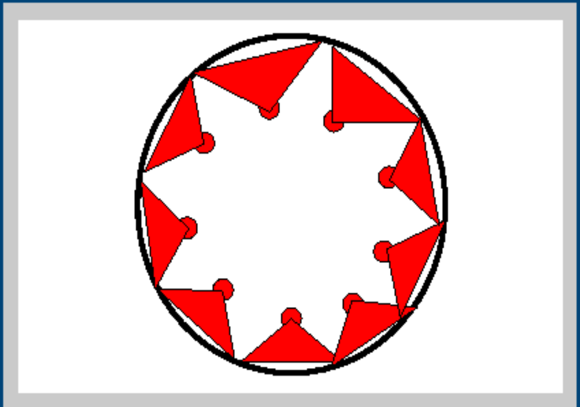


Laplace's Equation in 2D

Integral Formulation

Limiting Argument

Want to smear point charges to the surface



Results in an integral equation

$$\Psi(x) = \int_{surface} G(x, x')\sigma(x')dS'$$

How do we solve the integral equation?

Laplace's Equation in 2D

Basis Function Approach

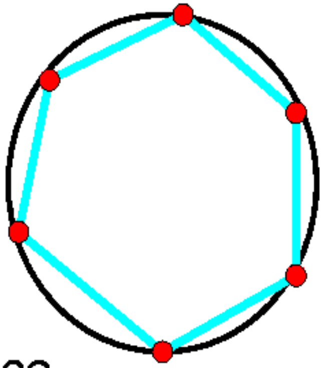
Basic Idea

Represent $\sigma(x) = \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

Example Basis

Represent circle with straight lines

Assume σ is constant along each line



The basis functions are "on" the surface

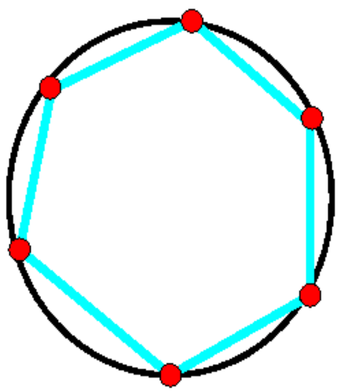
Can be used to approximate the density

May also approximate the geometry.

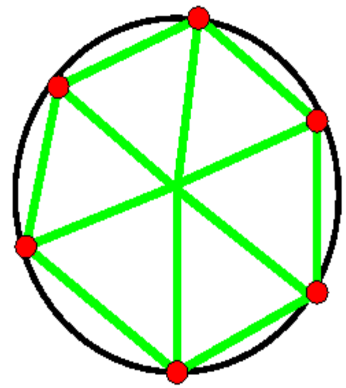
Laplace's Equation in 2D

Basis Function Approach

Geometric Approximation is Not New



Piecewise Straight surface basis Functions approximate the circle



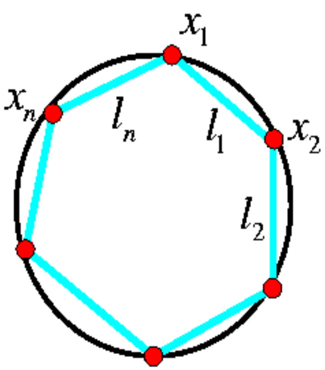
Triangles for 2-D FEM approximate the circle too!

$$\Psi(x) = \int_{\substack{\text{approx}\\\text{surface}}} G(x, x') \sum_{i=1}^{n} \alpha_i \varphi_i(x')\, dS'$$

Laplace's Equation in 2D

Basis Function Approach

Piecewise Constant Straight Sections Example



1) Pick a set of n Points on the surface
2) Define a new surface by connecting points with n lines.
3) Define $\varphi_i(x) = 1$ if x is on line l_i otherwise, $\varphi_i(x) = 0$

$$\Psi(x) = \int_{\substack{\text{approx} \\ \text{surface}}} G(x,x') \sum_{i=1}^{n} \alpha_i \varphi_i(x') dS' = \sum_{i=1}^{n} \alpha_i \int_{line\ l_i} G(x,x') dS'$$

How do we determine the α_i's?

Laplace's Equation in 2D

Basis Function Approach

Residual Definition and Minimization

$$R(x) = \Psi(x) - \int_{\substack{\text{approx}\\ \text{surface}}} G(x,x') \sum_{i=1}^{n} \alpha_i \varphi_i(x') dS'$$

We pick the α_i's to make $R(x)$ small

General Approach: Pick a set of test functions $\phi_1, \ldots, \phi_n$ and force $R(x)$ to be orthogonal to the set

$$\int \phi_i(x) R(x) dS = 0 \quad \text{for all i}$$

Laplace's Equation in 2D

Basis Function Approach

Residual Minimization Using Test Functions

$$\boxed{\int \phi_i(x) R(x) dS = 0} \Rightarrow$$

$$\int \phi_i(x)\Psi(x)dS - \int \int_{\substack{\text{approx} \\ \text{surface}}} \phi_i(x) G(x,x') \sum_{j=1}^{n} \alpha_j \varphi_j(x') dS' dS = 0$$

We will generate different methods by choosing the $\phi_1, \ldots, \phi_n$

Collocation : $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

Galerkin Method : $\phi_i(x) = \varphi_i(x)$ (basis = test)

Weighted Residual Method : $\phi_i(x) = 1\ if\ \varphi_i(x) \neq 0$ (averages)

Collocation: $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

$\int \delta(x - x_{t_i}) R(x) dS = R(x_{t_i}) = 0$ $\Rightarrow$

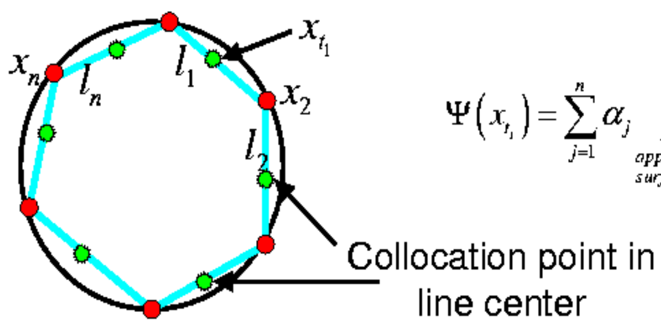
$$\sum_{j=1}^{n} \alpha_j \overbrace{\int_{\substack{\text{approx}\\ \text{surface}}} G(x_{t_i}, x') \varphi_j(x') dS'}^{A_{i,j}} = \Psi(x_{t_i})$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_n}) \end{bmatrix}$$

Laplace's Equation in 2D

Basis Function Approach

Centroid Collocation for Piecewise Constant Bases



$$\Psi(x_{t_i}) = \sum_{j=1}^{n} \alpha_j \int_{\substack{approx \\ surface}} G(x_{t_i}, x')\varphi_j(x')\,dS'$$

$$\begin{bmatrix} A_{1,1} & \cdots\cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots\cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_1}) \end{bmatrix}$$

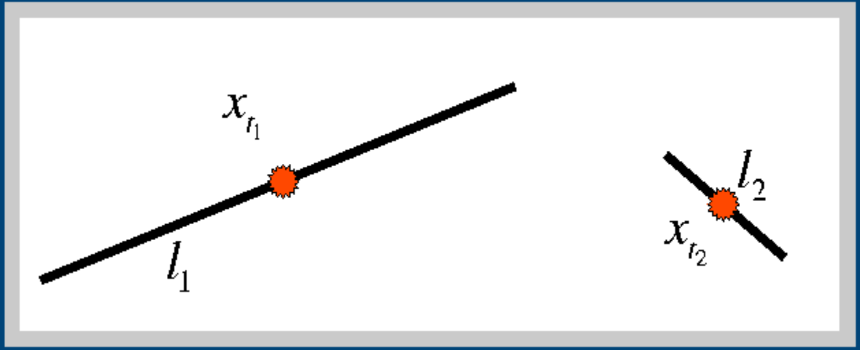
$$\Psi(x_{t_i}) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{line\,j} G(x_{t_i}, x')\,dS'}_{A_{i,j}}$$

Laplace's Equation in 2D

Basis Function Approach

Centroid Collocation Generates Nonsymmetric A

$$\Psi(x_{t_i}) = \sum_{j=1}^{n} \alpha_j \overbrace{\int_{line j} G(x_{t_i}, x')\, dS'}^{A_{i,j}}$$



$$A_{1,2} = \int_{line2} G(x_{t_1}, x')\, dS' \neq \int_{line1} G(x_{t_2}, x')\, dS' = A_{2,1}$$

Laplace's Equation in 2D

Basis Function Approach

Galerkin

Galerkin: $\phi_i(x) = \varphi_i(x)$ (test=basis)

$$\int \varphi_i(x)\, R(x)\, dS = \int \varphi_i(x)\, \Psi(x)\, dS - \int_{\substack{approx\\surface}} \int \varphi_i(x) G(x,x') \sum_{j=1}^{n} \alpha_j \varphi_j(x')\, dS'\, dS = 0$$

$$\underbrace{\int_{\substack{approx\\surface}} \varphi_i(x) \Psi(x)\, dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{\substack{approx\\surface}} \int_{\substack{approx\\surface}} G(x,x') \varphi_i(x) \varphi_j(x')\, dS'\, dS}_{A_{i,j}}$$

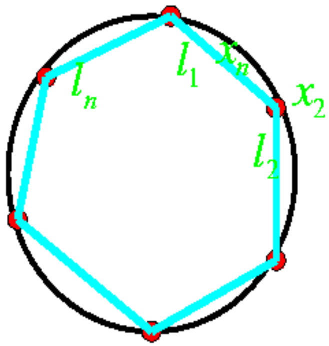
$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

If $G(x,x') = G(x',x)$ then $A_{i,j} = A_{j,i} \Rightarrow$ **A is symmetric**

Laplace's Equation in 2D

Basis Function Approach

Galerkin for Piecewise Constant Bases



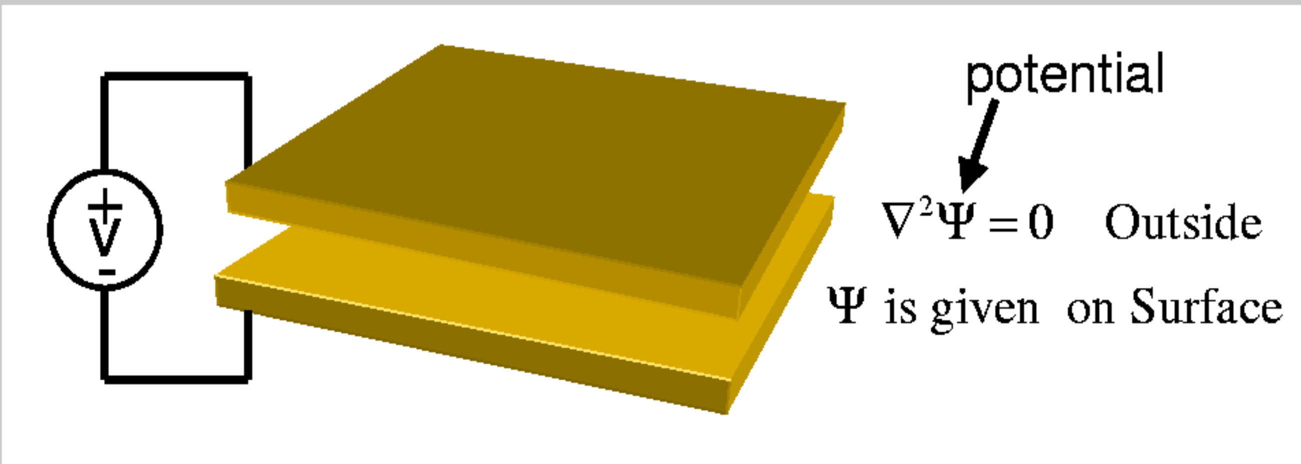
$$\underbrace{\int_{line_i} \Psi(x)\, dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{line_i} \int_{line_j} G(x, x')\, dS' dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Laplace's Equation in 3D

Electrostatics Example

Dirichlet Problem



First kind integral equation for charge:

$$\underbrace{\Psi(x)}_{Potential} = \int_{surface} \underbrace{\frac{1}{\|x - x'\|}}_{Green's\ function} \underbrace{\sigma(x')}_{Charge\ density} dS'$$

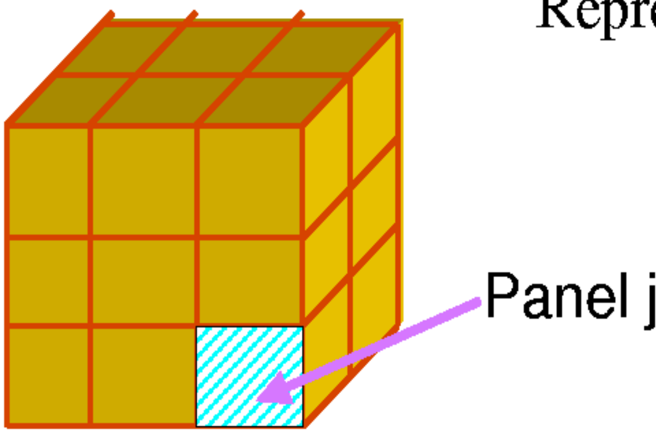
Laplace's Equation in 3D

Basis Function Approach

Piecewise Constant Basis

Integral Equation : $\Psi(x) = \int_{surface} \frac{1}{\|x - x'\|} \sigma(x') dS'$

Discretize Surface into Panels



Represent $\sigma(x) \approx \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

$$\varphi_j(x) = 1 \quad \text{if } x \text{ is on panel j}$$
$$\varphi_j(x) = 0 \quad \text{otherwise}$$

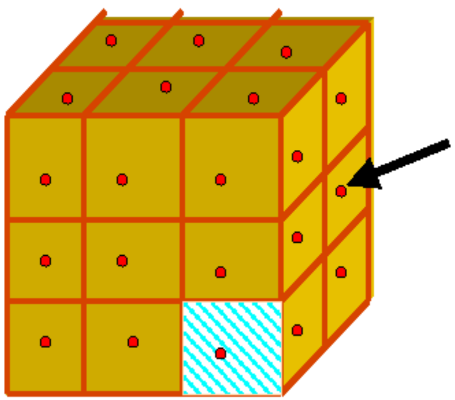
Laplace's Equation in 3D

Basis Function Approach

Centroid Collocation

Put collocation points at panel centroids

$$\Psi\left(x_{c_i}\right) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{panel\ j} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'}_{A_{i,j}}$$

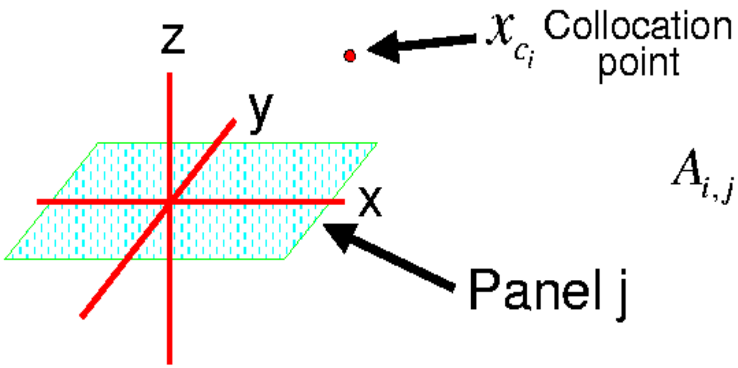


$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi\left(x_{c_1}\right) \\ \vdots \\ \vdots \\ \Psi\left(x_{c_n}\right) \end{bmatrix}$$

Laplace's Equation in 3D

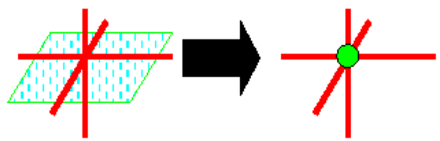
Basis Function Approach

Calculating Matrix Elements



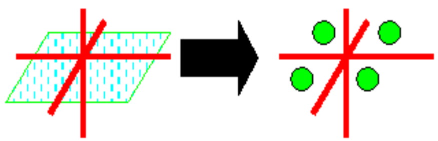
$$A_{i,j} = \int_{panel\ j} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

One point quadrature Approximation



$$A_{i,j} \approx \frac{Panel\ Area}{\left\| x_{c_i} - x_{centroid_j} \right\|}$$

Four point quadrature Approximation

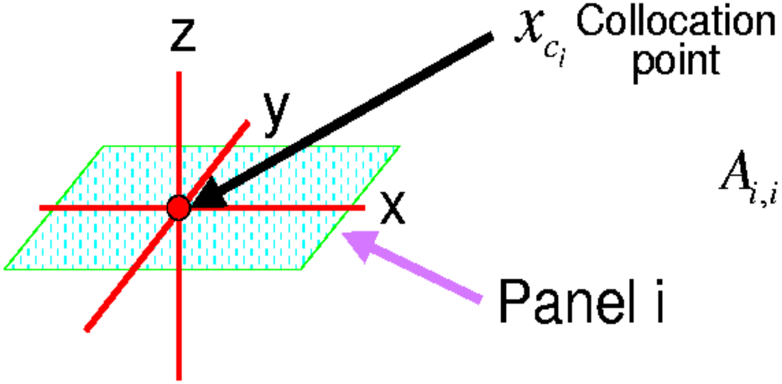


$$A_{i,j} \approx \sum_{j=1}^{4} \frac{0.25 * Area}{\left\| x_{c_i} - x_{point_j} \right\|}$$

Laplace's Equation in 3D

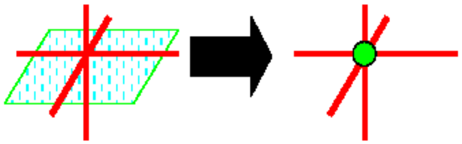
Basis Function Approach

Calculating "Self Term"



$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

One point quadrature Approximation



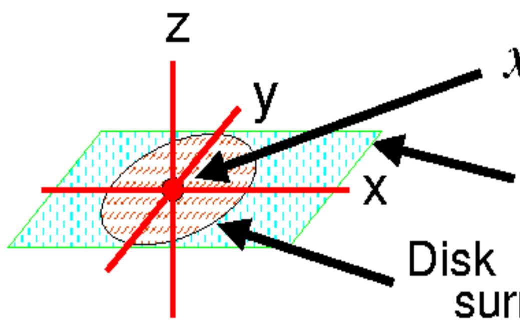
$$A_{i,i} \approx \frac{Panel\ \ Area}{\underbrace{\left\| x_{c_i} - x_{c_i} \right\|}_{0}}$$

$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' \text{ is an integrable singularity}$$

Laplace's Equation in 3D

Basis Function Approach

Calculating "Self Term"Tricks of the Trade



$$A_{i,i} = \int_{panel\ i} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

Integrate in two pieces

$$A_{i,i} = \int_{disk} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' + \int_{rest\ of\ panel} \frac{1}{\left\| x_{c_i} - x' \right\|} dS'$$

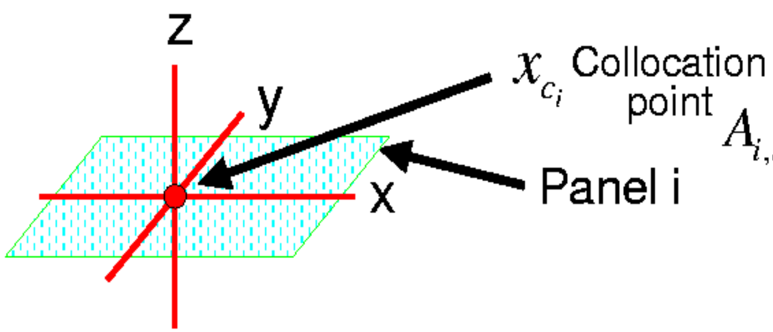
Disk Integral has singularity but has analytic formula

$$\int_{disk} \frac{1}{\left\| x_{c_i} - x' \right\|} dS' = \int_0^R \int_0^{2\pi} \frac{1}{r} r\,dr\,d\theta = \boxed{2\pi R}$$

Laplace's Equation in 3D

Basis Function Approach

Calculating "Self Term" Other Tricks of the Trade



$$A_{i,i} = \int_{panel\ i} \underbrace{\frac{1}{\left\| x_{c_i} - x' \right\|}}_{\text{Integrand is singular}} dS'$$

1. If panel is a flat polygon, analytical formulas exist.
2. Curved panels can be handled with projection.

Summary

Integral Equation Methods

- Exterior versus interior problems
- Start with using point sources

Standard Solution Methods

- Collocation Method
- Galerkin Method

Integrals for 3D Problems

- Singular Integrals

We will examine computing integrals next time, and then examine integral equation convergence theory.