Numerical Methods for PDEs

Integral Equation Methods, Lecture 1
Discretization of Boundary Integral Equations

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1 Outline for this Module

Overview of Integral Equation Methods
Important for many exterior problems
(Fluids, Electromagnetics, Acoustics)
Quadrature and Cubature for computing integrals
One and Two dimensional basics
Dealing with Singularities
1st **and** 2nd **Kind Integral Equations**
Collocation, Galerkin and Nystrom theory
Alternative Integral Formulations
Ansatz approach and Green's theorem
Fast Solvers
Fast Multipole and FFT-based methods.

2 Outline

Integral Equation Methods
Exterior versus interior problems
Start with using point sources
Standard Solution Methods
Collocation Method
Galerkin Method
Some issues in 3D
Singular integrals

3 Interior Vs Exterior Problems

Interior   Exterior

$\nabla^2 T = 0$
outside

$\nabla^2 T = 0$
inside

Temperature
known on surface

Temperature
known on surface

"Temperature in a tank"   "Ice cube in a bath"

<u>What is the heat flow?</u>
Heat flow = Thermal conductivity $\int_{surface} \frac{\partial T}{\partial n}$

Note 1   ***Why use integral equation methods?***

For both of the heat conduction examples in the above figure, the temperature, T, is a function of the spatial coordinate, x, and satisfies $\nabla^2 T(x) = 0$. In both

problems $T(x)$ is given on the surface, defined by Γ, and therefore both problems are Dirichlet problems. For the "temperature in a tank" problem, the problem domain, Ω is the interior of the cube, and for the "ice cube in a bath" problem, the problem domain is the infinitely extending region exterior to the cube. For such an exterior problem, one needs an additional boundary condition to specify what happens sufficiently far away from the cube. Typically, it is assumed there are no heat sources exterior to the cube and therefore

$$\lim_{\|x\|\to\infty} T(x) \to 0.$$

For the cube problem, we might only be interested in the net heat flow from the surface. That flow is given by an integral over the cube surface of the normal derivative of temperature, scaled by a thermal conductivity. It might seem inefficient to use the finite-element or finite-difference methods discussed in previous sections to solve this problem, as such methods will need to compute the temperature everywhere in Ω. Indeed, it is possible to write an integral equation which relates the temperature on the surface directly to its surface normal, as we shall see shortly.

In the four examples below, we try to demonstrate that it is quite common in applications to have exterior problems where the known quantities and the quantities of interest are all on the surface.

4 Examples

4.1 Computation of Capacitance

potential

$\nabla^2\Psi = 0$ Outsi

Ψ is given on S
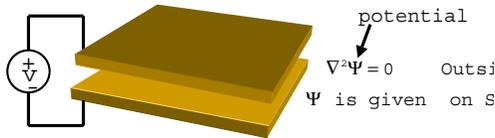
What is the capacitance?

Capacitance = Dielectric Permittivity $\int \frac{\partial\Psi}{\partial n}$

Note 2   ***Example 1: Capacitance problem***

In the example in the slide, the yellow plates form a parallel-plate capacitor with an applied voltage V. In this 3-D electrostatics problem, the electrostatic potential Ψ satisfies $\nabla^2\Psi(x) = 0$ in the region exterior to the plates, and the potential is known on the surface of the plates. In addition, far from the plates, $\Psi \to 0$. What is of interest is the capacitance, C, which satisfies

$$q = CV$$

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where q, the net charge on one of the plates, is given by the surface normal of the potential integrated over one plate and scaled by a dielectric permittivity.

4.2 Drag Force in a Microresonator

Resonator   Discretized Stru

Computed Forces Bottom View   Computed Forces Top View
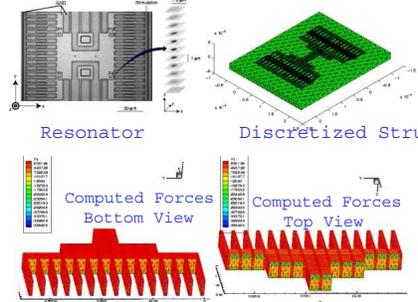
Note 3	Example 2: Drag force in a MEMS device

The example in the slide is a microresonator, it is a structure that can be made to vibrate using electrostatic forces. The changing character of those vibrations can be used to sense rotation. The particulars of how the microresonator operates is not directly relevant to our discussion of integral equations, except for one point. In order to determine how much energy is needed to keep the microresonator vibrating, it is necessary to determine the fluid drag force on comb structures shown in the bottom part of theslide. The fluid is the air surrounding the structure, and at the micron-scale of these devices, air satisfies the incompressible Stokes equation,

$$\nabla^2 u(x) = \nabla p(x) \tag{1}$$
$$\nabla \cdot u(x) = 0$$

where u is the fluid velocity and p is the pressure.By specifying the comb velocity, and then computing the surface pressure and the normal derivative of velocity, one can determine the net drag force on the comb. Once again, this is a problem in which the known quantities and the quantities of interest are on the surface.

3

4.3 Electromagnetic Coupling in a Package
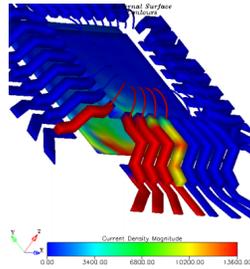
Picture Thanks to Coventor.

Note 4   **Example 3: Electromagnetic coupling in a package**

In the 40 lead electronic package pictured in the slide, it is important to determine the extent to which signals on different package leads interact. To determine the magnetic interaction between signal currents flowing on different wires, one must solve

$$\nabla^2 H(x) = J(x) \tag{2}$$
$$\nabla \cdot J(x) = 0$$

where H is the magnetic field and J is the signal current density. By specifying the current, and then computing the magnetic field at the surfaces of the leads, one can determine the magnetic interaction. Again, this is a problem in which the known quantities and the quantities of interest are on the surface.

4.4 Capacitance of Microprocessor Signal Lines
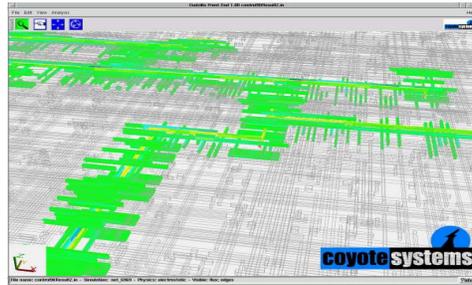
Note 5   **Example 4: Capacitance of microprocessor signal lines**

This last example in the above slide is a picture of the wiring on a microprocessor integrated circuit. A typical microprocessor has millions of wires, so we are only looking at a small piece of a processor. The critical problem in this example

is determining how long signals take to get from the output of a logical gate to the input of the next gate. To compute that delay, one must determine the capacitance on each of the wires given in the slide picture. To do so requires computing charges given electrostatic potentials as noted above.

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5 What is common about these problems?

Exterior Problems
MEMS device - fluid (air) creates drag
Package - Exterior fields create coupling
Signal Line - Exterior fields.
Quantities of interest are on surface
MEMS device - Just want surface traction force
Package - Just want coupling between conductors
Signal Line - Just want surface charge.
Exterior problem is linear and space-invariant
MEMS device - Exterior Stoke's flow equation (linear)
Package - Maxwell's equations in free space (linear)
Signal line - Laplace's equation in free spce (linear)

But problems are geometrically very complex

6 Exterior Problems

6.1 Why not use FDM / FEM?

2-D Heat Flow Example

Surface

$T = 0$ at ∞
But, must truncate t mesh
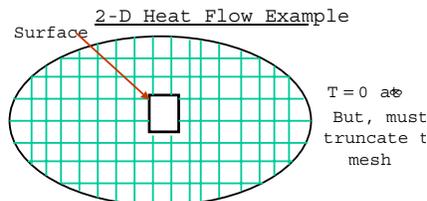
Only need $\frac{\partial T}{\partial n}$ on the surface, but T is computed everywhere.
Must truncate the mesh, $\Rightarrow T(\infty) = 0$ becomes $T(R) = 0$.

Note 6   ***Heat conduction in 2D***

In this slide above, we consider a two dimensional exterior heat conduction problem in which the temperature is known on the edges, or surface, of a square. Here, the quantity of interest might be the total heat flow out of the square. The temperature T satisfies

$$\nabla^2 T(x) = 0 \ x \in \Omega \tag{3}$$

5

$$T(x) \ given \ x \in \Gamma$$
$$lim_{\|x\| \to \infty} T(x) = 0$$

where Ω is the infinite domain outside the square and Γ is the region formed by the edges of the square.

Using finite-element or finite-difference methods to solve this problem requires introducing an additional approximation beyond discretization error. It is not possible to discretize all of Ω, as it is infinite, and therefore the domain must be truncated with an artificial finite boundary. In the slide,the artificial boundary is a large ellipse on which we assume the temperature is zero. Clearly, as the radius of the ellipse increases, the truncated problem more accurately represents the domain problem, but the number of unknowns in the discretization increases.

7 Laplace's Equation

7.1 Green's Function

In 2D

If $u = log\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$

then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \forall \quad (x,y) \neq (x_0, y_0)$

In 3D

If $u = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$

then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \forall \quad (x,y,z) \neq (x_0, y_0, z_0)$

Proof: Just differentiate and see!

Note 7   ***Green's function for Laplace's equation***
_____

In the next few slides, we will use an informal semi-numerical approach to deriving the integral form of Laplace's equation. We do this inpart because such a derivation lends insight to the subsequent numerical procedures.

To start, recall from basic physics that the potential due to a point charge is related only to the distance between the point charge and the evaluation point. In 2-D the potential is given by the log of the distance, and in 3-D the potential is inversely proportion to the distance. The precise formulas are given on the slide. A little more formally, direct differentiation reveals that

$$u(x,y) = \log\sqrt{(x-x_0)^2 + (y-y_0)^2} \tag{4}$$

satisfies the 2-D Laplace's equation everywhere except $x = x_0$, $y = y_0$ and

$$u(x,y,z) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \tag{5}$$

satisfies the 3-D Laplace's equation everywhere except $x = x_0$, $y = y_0$, $z = z_0$. These functions are sometimes referred to as Green's functions for Laplace's equation.

▷ **Exercise 1** Show by direct differentiation that the functions in (4) and (5) satisfy $\nabla^2 u = 0$, in the appropriate dimension almost everywhere. ∎

8 Laplace's Equation in 2D

8.1 Simple idea

u is given on surface
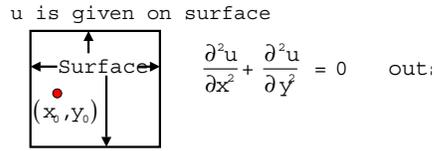
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{out}$$

Let $u = \log\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside} \quad \text{Problem Solved}$$

Does not match boundary conditions!

Note 8   ***Simple idea for solving Laplace's equation in 2D***

Here is a simple idea for computing the solution of Laplace's equation outside the square. Simply let

$$u(x,y) = \alpha \log \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

where x_0, y_0 is a point inside the square. Clearly such a u will satisfy $\nabla^2 u = 0$ outside the square, but u may not match the boundary conditions. By adjusting α, it is possible to make sure to match the boundary condition at at least one point.

▷ ***Exercise 2*** Suppose the potential on the surface of the square is a constant. Can you match that constant potential everywhere on the perimeter of the square by judiciously selecting α. ∎

8.1.1 "More points"

u is given on surface

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{out}$$
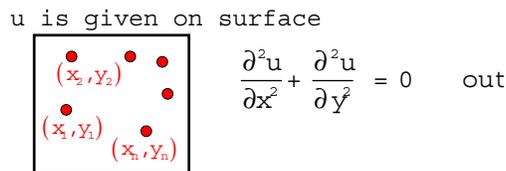
Let $u = \sum_{i=1}^{n} \alpha_i \log \left(\sqrt{(x-x_i)^2 + (y-y_i)^2} \right) = \sum_{i=1}^{n} \alpha_i G(x-x_i, y-y_i)$

Pick the α_i's to match the boundary conditions!

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To construct a potential that satisfies Laplace's equation and matches the boundary conditions at more points, let u be represented by the potential due to a sum of n weighted point charges in the square's interior. As shown in the slide, we can think of the potential due to a sum of charges as a sum of Green's functions. Of course, we have to determine the weights on the n point charges, and the weight on the ith charge is denoted hereby α_i.

8.1.2 "More points equations"
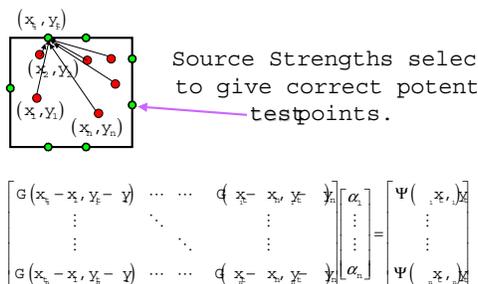
To determine a system of n equations for the n α_i's, consider selecting a set of n test points, as shown in the slide above. Then, by superposition, for each test point x_{t_i}, y_{t_i},

$$u(x_{t_i}, y_{t_i}) = \sum_{i=1}^{n} \alpha_i \log \sqrt{(x_{t_i} - x_0)^2 + (y_{t_i} - y_0)^2} = \sum_{i=1}^{n} \alpha_i G(x_{t_i} - x_0, y_{t_i} - y_0).$$
(6)

Writing an equation like (6) for each test point yields the matrix equation on the slide.

The matrix A in the slide has some properties worth noting:

- A is dense, that is $A_{i,j}$ never equals zero. This is because every charge contributes to every potential.

- If the test points and the charge points are ordered so that the ith test point is nearest the ith charge, then $A_{i,i}$ will be larger than $A_{i,j}$ for all j.

Item 2 above seems to suggest that A is diagonally dominant, but this is *not* the case. Diagonal dominance requires that the absolute *sum* of the off-diagonal entries is smaller than the magnitude of the diagonal. The matrix above easily violates that condition.

▷ **Exercise 3** Determine a set of test points and charge locations for the 2-D square problem that generates an A matrix where the magnitude of the diagonals are bigger than the absolute value of the off-diagonals, but the magnitude of the diagonal is smaller than the absolute sum of the off-diagonals. ∎

8.1.3 Computational Results

Circle with Charges
Potentials on the Circle
n=20   n=40
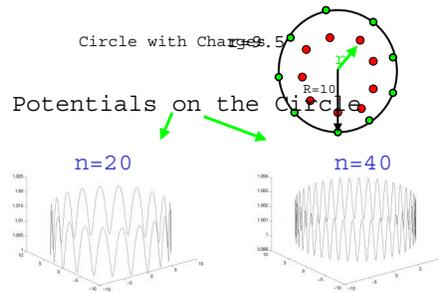
Note 11   *results*

It is possible to construct a numerical scheme for solving exterior Laplace problems by adding progressively more point charges so as to match more boundary conditions. In the slide above, we show an example of using such a method to compute the potential exterior to a circle of radius 10, where the potential on the circle is given to be unity. In the example, charges are placed uniformly on a circle of radius 9.5, and test points are placed uniformly on the radius 10 circle. If 20 point charges are placed in a circle of radius 9.5, then the potential produced will be exactly one only at the 20 test points on the radius 10 circle. The potential produced by the twenty point charges on the radius 10 circle is plotted in the lower left corner of the slide above. As might be expected, the potential produced on the radius 10 circle is exactly one at the 20 test points, but then oscillates between 1 and 1.2 on the radius 10 circle. If 40 charges and test points are used, the situation improves. The potential on the circle still oscillates, as shown in the lower right hand corner, but now the amplitude is only between 1 and 1.004.

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8.2 Integral Formulation

8.2.1 Limiting Argument

<u>Want to smear point charges to the surface</u>

Results in an integral equation

$$\Psi(x) = \int_{surface} G(x, x')\sigma(x')dS'$$

How do we solve the integral equation?

Note 12   ***Single layer potential***

The oscillating potential produced by the point charge method is due to the rapid change in potential as the separation between evaluation point and point charge shrinks. If the point charges could be smeared out, so that the produced potential did not rise to infinity with decreasing separation, then the resulting computed potential would not have the oscillation noted on the previous slide. In addition, it makes the most sense to smear the point charges onto the surface, as then the charge density and the known potential have the same associated geometry. The result is the integral equation given on the bottom of the above slide, where now the unknown is a charge *density* on the surface and the potential due to that charge density is given by the well-known superposition integral. In the case of two or three dimensional Laplace problems, $G(x, x')$ can be written as $\hat{G}(x - x')$, as the potential is only a function of distance to the charge density and not a function of absolute position. For such a Green's function,

$$\Psi(x) = \int_{surface} \hat{G}(x - x')\sigma(x')dS',$$

which one may recognize from system theory as a convolution. This connection is quite precise. A space-invariant system has an impulse response, which is usually referred to as a Green's function. The output, in this case the potential, is a convolution of the impulse response with the input, in this case the charge density. Such an integral form of the potential is referred to as a *single layer potential*.

Note 13   ***Types of integral equations***

The single layer potential is an example of a class of integral equations known as "Fredholm integral equation of the First Kind". A Fredholm integral equation

of the Second Kind results when the unknown charge density exists not only under the integral sign but also outside it. An example of such an equation is

$$\Psi(x) = \sigma(x) + \int_{surface} K(x - x')\sigma(x')dS',$$

Fredholm integral equations, in which the domain of integration is fixed, usually arise out of boundary value problems. Initial value problems typically give rise to the so-called Volterra integral equations, where the domain of integration depends on the output of interest. For example, consider the initial value problem

$$\frac{dx(t)}{dt} = tx(t); \quad t \in [0, T], \; T > 0.$$

$$x(t = 0) = x_0$$

The "solution" of this equation is the following Volterra integral equation:

$$x(t) = x_0 + \int_0^t \xi x(\xi)d\xi$$

8.3   Basis Function Approach

8.3.1   Basic Idea

Represent $\sigma(x) = \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

Example Basis
Represent circle with straight lines
Assume σ is constant along each line
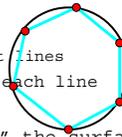
The basis functions are "on" the surfac

Can be used to approximate the density
May also approximate the geometry.

Note 14	*Numerical solution of the single layer potential*

As we have studied extensively in the finite-element section of the course, one approach to numerically computing solutions to partial differential equations is to represent the solution approximately as a weighted sum of basis functions. Then, the original problem is replaced with the problem of determining the basis function weights. In finite-element methods, the basis functions exist in a volume, for integral equations they typically exist on a surface. For 2-D problems that means the basis functions are restricted to curves and in 3-D the basic functions are on physical surfaces.

As an example, consider the circle in the slide above. One could try to represent the charge density on the circle by breaking the circle into n sub-arcs, and then

assume the charge density is a constant on each sub-arc. Such an approach is not commonly used. Instead the geometry is usually approximated along with the charge density. In this example case, shown in the center right of the slide, the sub-arcs of the circle are replaced with straight sections, thus forming a polygon. The charge density is assumed constant on each edge of the polygon.The result is a piecewise constant representation of the charge density on a polygon.

8.3.2   Geometric Approximation is Not New

Piecewise Straight surface Basis Triangles for 2-D I
Functions approximate the approximate the circ

$$\Psi(x) = \int\limits_{\substack{approx \\ surface}} G(x,\acute{x}) \sum_{i=1}^{n} \alpha_i \varphi_i(\acute{x}) \, d\acute{S}$$

Note 15

The idea that both the geometry and the unknown charge density has been approximated is not actually a new issue. As shown in the figure in the above slide, if FEM methods are used to solve an interior problem, and triangular elements are used, then the circle is approximated to exactly the same degree as when straight sections replace the sub-arcs for the surface integral equation. As shown at the bottom of the above slide, we can substitute the basis function representation into the integral equation, but then we should also note that the integral is now over the approximated geometry. It is common, but not mathematically justified, to ignore the errors generated by the geometry approximation. We will also ignore the error in the geometric approximation, and justify ourselves by assuming that there are no circles, but only polygons, as then there is no geometric approximation.

8.3.3   Piecewise Constant Straight Sections Example

1) Pick a set of n Point
surface
2) Define a new surface
connecting points with n
3) Define $\varphi_i(x) =$ x1 if   i≜
otherwise $\varphi_i(x) =$ 0

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$$\Psi(x) = \int_{\substack{\text{approx} \\ \text{surface}}} G(x, x') \sum_{i=1}^{n} \alpha_i \varphi_i(x') dS' = \sum_{i=1}^{n} \alpha_i \int_{line \ l_i} G(x, x') dS'$$

How do we determine the α_i's?

Note 16

In the above slide, we complete the description of using constant charge densities on straight sections as the basis. If we substitute this example basis function into the integral equation, as is done at the bottom of the slide, the result is to replace the original integration of the product of the Green's function and the density with a weighted sum of integrals over straight lines of just the Green's function. The next step is then to develop an approach for determining the weights, denoted here by α_i's.

8.3.4 Residual Definition and Minimization

$$R(x) = \Psi(x) - \int_{\substack{\text{approx} \\ \text{surface}}} G(x, x') \sum_{i=1}^{n} \alpha_i \varphi_i(x') dS'$$

We pick the α_i's to make $R(x)$ small

General Approach: Pick a set of test functions $\phi_1, \ldots, \phi_n$ and force $R(x)$ to be orthogonal to the set

$$\int \phi_i(x) R(x) dS = 0 \quad \text{for all i}$$

Note 17

One way of assessing the accuracy of the basis function based approximation of the charge density is to examine how well the approximation satisfies the integral equation. To be more precise, we define the residual associated with the integral equation and an approximate solution at the top of the above slide. Note that $R(x)$ is just the difference between the given potential on the surface and the potential produced by the approximated charge density. Note also that the equation is now over the approximate geometry and therefore x and x' are both on the approximated surface.

If the representation satisfies the integral equation exactly, then the residual $R(x)$ will be zero for all x and the approximate solution is equal to the exact solution (provided the integral equation has a unique exact solution, more on this later). However this will usually not be possible, and instead we will try to pick the basis function weights, the α_i's, to somehow minimize $R(x)$. One

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approach to minimizing $R(x)$ is to make it orthogonal to a collection of test functions, which may or may not be related to the basis functions. As noted on the bottom of the slide, enforcing orthogonality in this case means ensuring that the integral of the product of $R(x)$ and $\phi(x)$ over the surface is zero.

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8.3.5 Residual Minimization Using Test Functions

$$\boxed{\int \phi_i(x)R(x)dS = 0} \Rightarrow$$

$$\int \phi_i(x)\Psi(x)dS - \int\int_{\substack{\text{approx} \\ \text{surface}}} \phi_i(x)G(x,x')\sum_{j=1}^{n}\alpha_j\varphi_j(x')dS'dS = 0$$

We will generate different methods by choosing the $\phi_1,\dots,\phi_n$

Collocation : $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

Galerkin Method : $\phi_i(x) = \varphi_i(x)$ (basis = test)

Weighted Residual Method : $\phi_i(x) = 1\ if\ \varphi_i(x) \neq 0$ (averages)

Note 18

As noted in the equation on the top of the above slide, by substituting the definition of the residual into the orthogonality equation given in the previous slide, it is possible to generate n equations, one for each test function. The generated equation has two integrals. The first is a surface integral of the product of the given potential with the test function. The second integral is a double integral over the surface. The integrand of the double integral is a product of the test function, the Green's function, and the charge density representation.

As noted in the slide, and as we will describe in more detail below, three different numerical techniques can be derived by altering the test functions.

8.3.6 Collocation

$$\text{Collocation: } \phi_i(x) = \delta(x - x_{t_i}) \text{ (point matching)}$$

$$\boxed{\int \delta(x-x_{t_i})R(x)dS = R(x_{t_i})=0} \Rightarrow$$

$$\sum_{j=1}^{n}\alpha_j \overbrace{\int_{\substack{\text{approx} \\ \text{surface}}} G(x_{t_i},x')\varphi_j(x')dS'}^{A_{i,j}} = \Psi(x_{t_i})$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_n}) \end{bmatrix}$$

Note 19 *Collocation*

The collocation method, described in the above slide, uses shifted impulse functions as test functions, $\phi_i(x) = \delta(x - x_i)$. Impulse functions, as called "delta"

functions, have a *sifting* property when integrated with a smooth function $f(x)$,

$$\int f(x)\delta(x - x_i)dx = f(x_i).$$

Impulse functions are also referred to as generalized functions, and they are specified only by their behavior when integrated with a smooth function. In the case of the impulse function, one can think of the function as being zero except for a very narrow interval around x_i, and then being so large in that narrow interval that $\int \delta(x - x_i)dx = 1$.

As the summation equation in the middle of the above slide indicates, testing with impulse functions is equivalent to insisting that $R(x_i) = 0$, or in words, that the potential produced by the approximated charge density should match the given potential at n test points. That the potentials match at the test points gives rise to the method's name, the point where the potential is exactly matched is "co-located" with a set of test points.

The $n \times n$ matrix equation at the bottom of the above slide has as its right-hand side the potentials at the test points. The unknowns are the basis function weights. The jth matrix element for the ith row is the potential produced at test point x_i by a charge density equal to basis function φ_j.

8.3.7 Centroid Collocation for Piecewise Constant Bases

SLIDE 22

$$\Psi(x_t) = \sum_{j=1}^{n} \alpha_j \int_{\substack{approx \\ surface}} G(x_t, x)\varphi(\hat{x}) \, d\hat{S}$$

Collocation point in line center

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_1}) \end{bmatrix}$$

$$\Psi(x_{t_i}) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{line j} G(x_{t_i}, x')dS'}_{A_{i,j}}$$

Note 20

In the above slide, a specific collocation algorithm is described. First, the basis being used is the constant charge density on n straight sections or lines, as described above. Note that therefore the geometry is being approximated. Second, the collocation points being selected are the centroids of the basis functions, in this case just the center of each straight line. Note that the collocation point is on the approximated geometry, not the original geometry. So, one can think of the problem as having been restated to be on a polygon instead of the original circle. One could also have selected the collocation points on the original circle, but then the replacement interpretation does not hold.

In collocation, or point- matching, the charge densities on each of the straight lines are selected so that the resulting potential at the line centers matches the given potential. As the equations on this slide make clear, the matrix element $A_{i,j}$ is the potential at the center of line i due to a unit charge density along line j.

It should be noted that the matrix A is dense, the charge on line j contributes to the potential everywhere. Also note that if line j is far away from line i, then

$$A_{i,j} \approx length(line_j) * G(x_{t_i}, x_{t_j}) \tag{7}$$

▷ **Exercise 4** Suppose we are using piecewise constant centroid collocation to solve a 2-D Laplace problem, so $G(x, y, x', y') = \log \sqrt{(x - x')^2 + (y - y')^2}$. Roughly how far apart do line sections i and j have to be for (7) to be accurate to within one percent? Assume line j has length of one. Does your answer depend on the orientation of line j? Does your answer depend on the orientation of line i? (You should answer yes to one of these and no to the other, do you see why?) ∎

8.3.8 Centroid Collocation Generates Nonsymmetric A

$$\Psi(x_{t_i}) = \sum_{j=1}^{n} \alpha_j \overbrace{\int_{line j} G(x_{t_i}, x')dS'}^{A_{i,j}}$$

$$A_{1,2} = \int_{line2} G(x_{t_1}, x')dS' \neq \int_{line1} G(x_{t_2}, x')dS' = A_{2,1}$$

Note 21

Consider the two line sections, l_1 and l_2 given in the above slide. For Laplace problems, $G(x, x') = G(x', x)$, and this suggests a symmetry in the underlying integral equation which is not represented in the collocation discretization. This asymmetry is shown on the slide above by noting that $A_{1,2} \neq A_{2,1}$. That is, the potential at the center of l_2 due to a unit charge density on l_1 is not equal to the potential at the center of l_1 due to a unit charge on l_2.

It is possible to scale the variables to improve the symmetry, consider a change of variables

$$\hat{\alpha}_i = \alpha_i * length(line_j).$$

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In this change of variables, the unknowns $\hat{\alpha}_i$ are now the net line charges rather than the line charge densities. In this new system, $\hat{A}\hat{\alpha} = \Psi$, where the elements of the matrix $\hat{A}$ are given by

$$\hat{A}_{i,j} = \frac{1}{length(line_j)} \int_{line_j} G(x_{t_i}, x')dS'.$$

Under the change of variables, if line j is far away from line i, then

$$\hat{A}_{i,j} \approx G(x_{t_i}, x_{t_j}) \approx \hat{A}_{j,i}. \tag{8}$$

In other words, the elements of $\hat{A}$ corresponding to distant terms are approximately symmetric.

$\triangleright$ **Exercise 5** Give an example which shows that the scaled entries of $\hat{A}$ can be far from symmetric. Assume we are using piecewise constant straight sections with centroid collocation and the 2-D Laplace's equation Green's function. $\blacksquare$

8.3.9 Galerkin

Galerkin: $\phi_i(x) = \varphi_i(x)$ (test=basis)

$$\int \varphi_i(x) R(x) dS = \int \varphi_i(x) \Psi(x) dS \int \int_{\substack{approx \\ surface}} \varphi(x) x(G, x) \sum_{j=1}^{n} \alpha_j \varphi_j(x') x' dS dS$$

$$\underbrace{\int_{\substack{approx \\ surface}} \varphi_i(x) \Psi(x) dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{\substack{approx \\ surface}} \int_{\substack{approx \\ surface}} G(x, x) \varphi_i(x) \varphi_j(x') dS dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

If $G(x, x') = G(x', x)$ then $A_{i,j} = A_{j,i} \Rightarrow$ **A is symmetric**

Note 22   **Galerkin method**

In the slide above, we give the equations for the Galerkin method, in which the test functions are equal to the basis functions. In particular, one generates n equations for the basis function weights by insisting that $R(x)$ is orthogonal to each of the basis functions. Enforcing orthogonality corresponds to setting

$$\int \varphi(x)R(x)dS = 0$$

and substituting the definition of $R(x)$ into the orthogonality condition yields the equation in the center of the above slide.

Note that the Galerkin method yields a system of n equations, one for each

orthogonality condition, and n unknowns, one for each basis function weight. Also, the system does not have the potential explicitly as the right hand side. Instead, the ith right- hand side entry is the average of the product of the potential and the ith basis function.

8.3.10 Galerkin for Piecewise Constant Bases
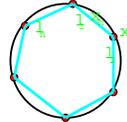
$$\underbrace{\int_{line} \Psi(x)\,dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{line}\int_{line} G(x,\acute{x})\,d\acute{S}\,dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Note 23

In the above slide, a specific Galerkin algorithm is described using the same example basis introduced above, constant charge density on n straight sections or lines. Once again, we will think of the problem as having been restated to be on a polygon instead of the original circle.

In the Galerkin method, the charge densities on each of the straight lines are selected so that the resulting line averaged potential matches the line averaged given potential. As the equations on this slide make clear, the matrix element $A_{i,j}$ is the average potential over line i, scaled by the length of $line_i$, due to a unit charge density along line j.

As with the collocation method, the matrix A is dense because the the charge on line j contributes to the averaged potentials everywhere. Also note that if line j is far away from line i, then

$$A_{i,j} \approx length(line_j) * length(line_i) * G(x_{t_i}, x_{t_j}) \tag{9}$$

▷ **Exercise 6** Suppose we are using piecewise constant centroid collocation to solve a 2-D Laplace problem, so $G(x, y, x', y') = \log \sqrt{(x - x')^2 + (y - y')^2}$. Roughly how far apart do line sections i and j have to be for (9) to be accurate to within one percent? Assume line j has length of one. Does your answer depend on the orientation of line j? Does your answer depend on the orientation of line i? (Your answer should be different than the answer you gave for the collocation method. Do you see why?) ■

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9 Laplace's Equation in 3D

9.1 Electrostatics Example

9.1.1 Dirichlet Problem
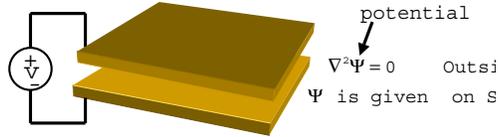
First kind integral equation for charge:

$$\underbrace{\Psi(x)}_{Potential} = \int_{surface} \underbrace{\frac{1}{||x - x'||}}_{Green's\ function} \underbrace{\sigma(x')}_{Charge\ density}\ dS'$$

Note 24

In the above slide, we give a typical application example, determining the charge density on the surface of conducting plates given an applied voltage. As noted next to the parallel plate figure in the above slide, in the domain exterior to the plates the potential, Ψ, satisfies Laplace's equation, and the potential is given on the surface of the two plates. In this particular example, the top plate potential is $\Psi = 0.5V$ and the bottom plate potential is $\Psi = -0.5V$, where V is the voltage noted in the figure. As mentioned in previous lectures, it is also important to note that

$$lim_{|x| \to \infty} \Psi(x) = 0.$$

That is, far away from the plates the potential decays to zero.

For this exterior Dirichlet problem, one can write an integral equation that relates the surface charge density on the plates σ to the potential on the plates. This integral equation, given at the bottom of the above slide, is often referred to by physicists as the superposition integral. In the integral equation, x is any point on the plate surfaces and the surface being integrated over is the union of the top and bottom plate surfaces. Note that the integration surface is not a connected domain, but this presents no difficulties.

9.2 Basis Function Approach

9.2.1 Piecewise Constant Basis

Integral Equation : $\Psi(x) = \int_{surface} \frac{1}{||x-x'||} \sigma(x') dS'$

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Discretize Surface into Panels

$$\text{Represent } \sigma(x) \approx \sum_{i=1}^{n} \alpha_i \ \underbrace{\varphi_i(x)}_{\text{Basis Function}}$$

Panel j

$\varphi_j(x) = 1$ if x is on

$\varphi_j(x) = 0$ otherwise
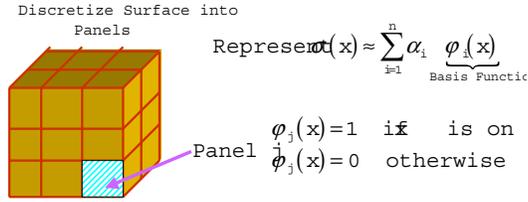
Note 25

Consider solving the integral on the top of the above slide where the surface is the surface of the cube shown in the lower left-hand corner. The first step, as we have mentioned in previous lectures, is to develop a basis in which to represent the surface charge density σ.

The cube pictured in the slide has had its surface divided into panels, and a basis is derived from the panels. In particular, one can associate a basis function φ_j with each panel j by assigning $\varphi_j(x)$ the value one when x is a point on panel j, and setting $\varphi_j(x) = 0$ otherwise. If σ is approximated by a weighted combination of these basis functions, then the approximation is a piecewise constant representation of the charge density on the surface of the cube.

A few aspects of this basis set should be noted.

- The basis functions are orthogonal, that is if $i \neq j$,

$$\int \varphi_j(x)\varphi_i(x)dx = 0.$$

- These basis functions are normalized with respect to l_∞, not l_2. That is, $\|\varphi\|_\infty = 1$ but

$$\|\varphi_j\|_2^2 = \int \varphi_i(x)\varphi_i(x)dx = panel \ \ area.$$

Finally, many papers in the literature on solving integral equations refer to "panel methods". The name is derived from the idea of breaking a surface into flat panels.

In the application area of analyzing ocean wave forces on ship hulls, panel methods are commonly used. However, it is not possible to represent a curved hull with quadrilateral flat panels. Researchers in the area often create a best fit panelled surface in which there are gaps between the edges of the panels. Such a discretization technique is often referred to as using "leaky panels", a very compelling image.

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9.2.2 Centroid Collocation

Put collocation points at panel centroids

$$\Psi\left(x_{c_i}\right) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{panel} \frac{1}{\left\|x_{c_i} - x_j'\right\|} dS'}_{A_{i,j}}$$

x_{c_i} Collocation point
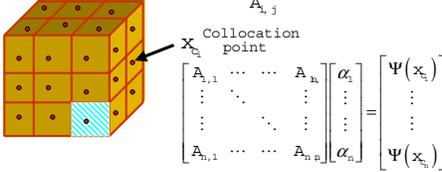
$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi\left(x_{c_1}\right) \\ \vdots \\ \vdots \\ \Psi\left(x_{c_n}\right) \end{bmatrix}$$

Note 26

After one has decide on a basis with which to approximately represent the surface charge density, the next step is to develop a system of equations from which to determine the basis weights, denoted in the slide above as the α_i's. The most commonly used approach to forming such a system is to use collocation, though Galerkin methods are also quite widely used. Recall that in collocation, the basis function weights are determined by insuring the the integral equation is exactly satisfied at a collection of "collocation" points. For panel methods, the most common choice for the position of the collocation points are the panel centroids, as shown in the cube diagram in the above slide.

The equation in the top right of the above slide relates the potential at collocation point x_{c_i} to the weights for the panel-based basis functions. To see how the equation was derived, consider evaluting the potential at the ith collocation point using the original integral equation

$$\Phi(x_{c_i}) = \int_{surface} \frac{1}{\|x_{c_i} - x'\|} \sigma(x') dS', \tag{10}$$

where Φ is the know potential on the problem surface and σ is the unknown charge density. Substituting the approximate representation for σ,

$$\sigma(x) \approx \sum_{j=1}^{n} \alpha_j \varphi_j(x)$$

into the integral equation results in

$$\Phi(x_{c_i}) = \int_{surface} G(x_{c_i}, x') \sum_{j=1}^{n} \alpha_j \varphi_j(x') dS', \tag{11}$$

where $G(x, x') \equiv \frac{1}{\|x-x'\|}$ is used to simplify the formula. Exploiting the fact that $\varphi_j(x) = 1$ if x is on panel j, and zero otherwise, results in the formula at the top right of the slide.

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The system of equations from which to determine the basis function weights is given in the lower right corner of the slide. The right hand side of the system is the vector of known potentials at the collocation points. The i,jth element of the matrix A is the potential produced at collocation point i due to a unit charge density on panel j. The vector of α's are the unknown panel charge densities.

$\triangleright$ **Exercise 7** Determine a scaling of the α's ($\hat{\alpha}_i = c_i \alpha_i$) such that the scaled matrix $\hat{A}$ has the property

$$A_{i,j} \approx \frac{1}{\|x_{c_i} - x_{c_j}\|}$$

when $\|x_{c_i} - x_{c_j}\|$ is much larger than a panel diameter. $\blacksquare$

9.2.3 Calculating Matrix Elements
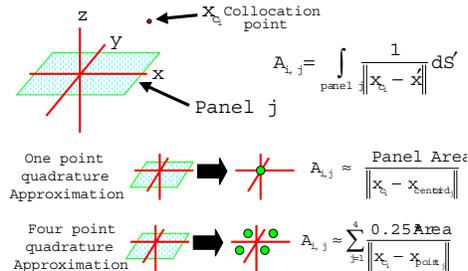
Note 27

In order to calculate the matrix entries for the system of equations described in the previous slide, recall that $A_{i,j}$ is the potential produced at collocation point i due to a unit charge density on panel j. The formula for $A_{i,j}$ is given on the top right side of the above slide.

The figure on the left of the above slide is a diagram of how one typically computes the panel integral given on the top right. First, consider a shift and rotatation of the coordinate system so that the panel lies in the $x - y$ plane at $z = 0$, with the panel's center at $x = 0$, $y = 0$. The figure in the top left shows the panel in the shifted and rotated coordinate system. Note that the collocation point must also be placed in the new coordinate system.

If panel j is reasonably well seperated from collocation point i, it is possible to approximate the integral given in the top right by a single point quadrature. More specifically, one could approximates the integral of $\frac{1}{\|x_{c_i} - x'\|}$ by a product of $\frac{1}{\|x_{c_i} - x_{centroid_j}\|}$ and the panel area. As show in the middle figure, a single

point quadrature is like treating the panel as if it were a point charge at the panel's centroid, where the point charge's strength is equal to the panel area.

If the collocation point is close to the panel, then a single point quadrature will be insufficiently accurate. Instead, a more accurate four point quadrature scheme would be to break the panel into four subpanels, and then treat each of the subpanels as point charges at their respective centers. This simple idea is shown in the figure at the bottom of the above slide. This four point scheme is equivalent to

$$\int_{panel_j} \frac{1}{\|x_{c_i} - x'\|} dS' \approx \sum_{j=1}^{4} \frac{0.25 * Area}{\|x_{c_i} - x_{point_j}\|}.$$

If the panel is a unit square in the x-y plane whose center is at the coordinate system origin, then the four x_{point_j}'s are $(x, y, z) = (0.25, 0.25, 0)$, $(x, y, z) = (-0.25, 0.25, 0)$, $(x, y, z) = (-0.25, -0.25, 0)$, and $(x, y, z) = (0.25, -0.25, 0)$.

9.2.4 Calculating "Self Term"
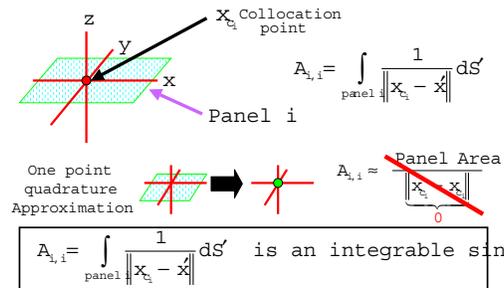
Note 28

The diagonal terms $A_{i,i}$ can not be computed using the quadrature approximation given on the previous slide. To see this, consider the figure at the top left of the above slide, where a panel has been shifted and rotated into the x-y plane, and the collocation point is the center of the panel. The integral that must be computed is given on the right side of the top of the above slide.

As shown in the middle of the slide, using a single point quadrature scheme will fail, because the distance between the point charge approximation to the panel and the collocation point will be zero. Therefore, the single point formula will require computing the reciprocal of zero, which is infinite. The problem is that the integrand in

$$\int_{panel_i} \frac{1}{\|x_{c_i} - x'\|} dS' \tag{12}$$

is singular. That is, the integrand approaches infinity at a point x' which is in the domain of integration. What is not so obvious is that (12) is an integrable

singularity. Therefore, even though the integrand approaches infinity at some point, the "area under the curve" is finite.

9.2.5 Calculating "Self Term"
Tricks of the Trade

In the above slide, we both show that

$$\int_{panel_i} \frac{1}{\|x_{c_i} - x'\|} dS'$$

is integrable, and also give an idea about how to compute the integral.

As shown in the slide, first rotate and shift the coordinate system so that the panel is in the x-y plane at $z = 0$, and so that the collocation point (or equivalently the panel centroid) is at the origin. In this new coordinate system, the integral can be written as

$$A_{i,i} = \int_{panel(rs)_i} \frac{1}{\|x'\|} dS'$$

where the notation $panel(rs)$ is used to indicate that the integral is over the rotated and shifted panel.

On the top left of the above slide, a circular disk of radius R and center at the collocation point is inscribed in the rotated panel. In the equations that follow the figure, it is noted that the panel integral can be recast as the sum of an integral over the disk plus an integral over the rest of the panel. The integrand in the integral over the rest of the panel is no longer singular, but the integrand in the integral over the disk is still singular.

The integral over the disk can be computed analytically by using a change of variables. After rotating and shifting the panel, the disk is in the x-y plane and its center, equal to the collocation point, is at zero. Therefore,

$$\int_{disk} \frac{1}{\|x_{c_i} - x'\|} dS' = \int_{disk} \frac{1}{\|x'\|} dS' = \int_0^R \int_0^{2\pi} \frac{1}{r} r\, dr\, d\theta$$

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where $r = \sqrt{x^2 + y^2}$, $rsin\theta = x$ and $rcos\theta = y$. As given in the bottom of the above slide, the integral over the disk is $2\pi R$.

9.2.6   Calculating "Self Term"
Other Tricks of the Trade
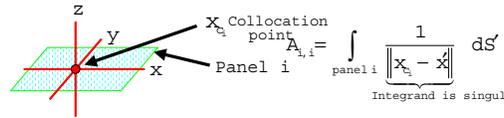
1. If panel is a flat polygon, analytical formulas exist.

2. Curved panels can be handled with projection.

10   Summary

Integral Equation Methods
Exterior versus interior problems
Start with using point sources
Standard Solution Methods
Collocation Method
Galerkin Method
Integrals for 3D Problems
Singular Integrals

We will examine computing integrals next time, and then examine integral equation convergence theory.

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