# Hyperbolic Equations : Scalar One-Dimensional Conservation Laws

## Lecture 11

# Scalar Conservation Laws

## Definitions

### Conservative Form

General form (1D):

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$u(x,t)$ : is the unknown conserved quantity (mass, momentum, heat, . . . )

$f(u)$ : is the flux

# Scalar Conservation Laws

## Definitions

### Primitive Form

Can also be written . . .

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial u}{\partial t} + \frac{df}{du}\frac{\partial u}{\partial x} = 0$$

$$\boxed{\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0}$$

where $a(u) = \dfrac{df}{du}$.

N1

# Scalar Conservation Laws

## Definitions

### Integral Form

Consider a *fixed* domain $\Omega \equiv [x_L, x_R] \in \mathbb{R}$

$$\int_\Omega \left( \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \right) dV = 0$$

$$\boxed{\frac{d}{dt} \int_\Omega u \, dV = -[f(u_R) - f(u_L)]}$$

# Scalar Conservation Laws

## Derivation Example

### Conservation of Mass...

Consider a volume $\Omega$ enclosed by surface $\partial\Omega$ containing fluid of density $\rho(\boldsymbol{x}, t)$ and known velocity $\boldsymbol{v}(\boldsymbol{x}, t)$

RATE OF CHANGE OF MASS INSIDE $\Omega$ $\equiv$ MASS FLUX OF FLUID THROUGH $\partial\Omega$

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \, dV = -\int_{\partial\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{n} \, dS$$

$$= -\int_{\Omega} \nabla \cdot (\rho \boldsymbol{v}) \; dV$$

# Scalar Conservation Laws

## Derivation Example

### ...Conservation of Mass

$$\int_{\Omega} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right] dV = 0$$

holds for all $\Omega$, so we can write

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0}$$

This is the **differential form** of the conservation law.

# Scalar Conservation Laws

## Examples

### Linear Advection Equation

Model convection of a concentration $\rho(x,t)$:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho a}{\partial x} = \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0}$$

$a$ : constant

N2

# Scalar Conservation Laws

## Examples

### Inviscid Burgers' Equation

Flux function $f(u) = \frac{1}{2}u^2$

Conservation law :

$$\frac{\partial u}{\partial t} + \frac{\partial \frac{1}{2}u^2}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

N3

# Scalar Conservation Laws

## Examples

### Traffic Flow

Let $\rho(x,t)$ denote the density of cars (vehicles/km) and $u(x,t)$ the velocity. Since cars are conserved,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

Assume that $u$ is a function of $\rho$:

$$u(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

where $0 \leq \rho \leq \rho_{\text{max}}$ and $u_{\text{max}}$ is some maximum speed (the speed limit?).

N4

# Scalar Conservation Laws

## Examples

### Buckley-Leverett Equation

Consider a two phase (oil and water) fluid flow in porous medium. Let $0 \le u(x,t) \le 1$ represent the saturation of water.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$f(u) = \frac{u^2}{u^2 + a(1-u)^2}$$

$a$: constant $\sim 1$

## Smooth Solutions

# Total Derivative

Recall the primitive form of the conservation law

$$\frac{\partial u}{\partial t} + a(u)\,\frac{\partial u}{\partial x} = 0$$

The total time variation of $u(x,t)$, on an arbitrary curve $x = x(t)$, in the $x - t$ plane, is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\,\frac{\partial u}{\partial x}$$

# Smooth Solutions

## Characteristics

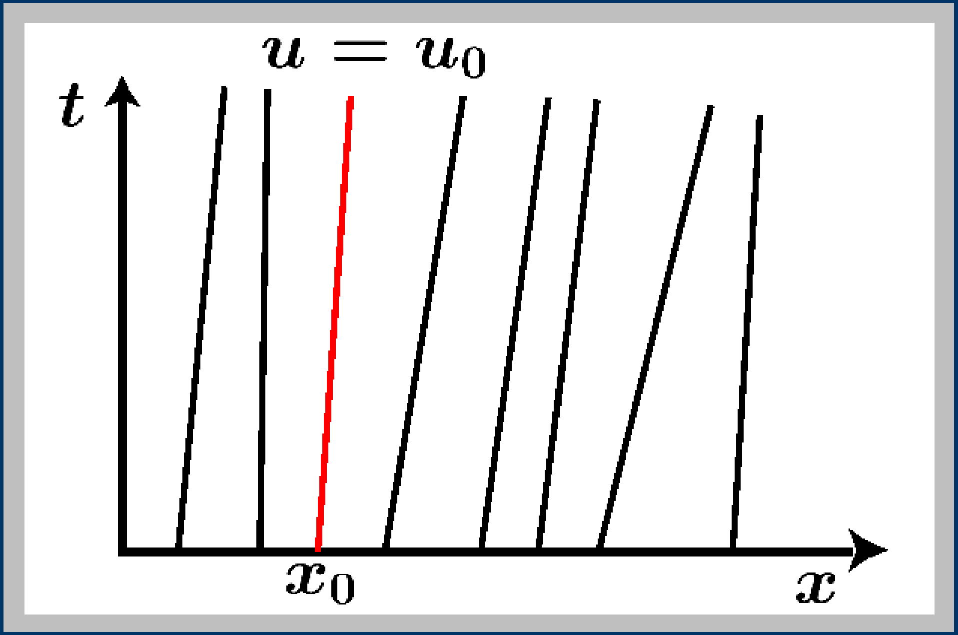
If $\dfrac{dx}{dt} = a(u) \quad \Rightarrow \quad \dfrac{du}{dt} = 0 \quad \Rightarrow \quad u = u_0$ (constant)

The curves $x = x(t)$, such that $\dfrac{dx}{dt} = a(u)$ are called **characteristics**

$u$ constant $\Rightarrow a(u)$ constant $\Rightarrow$
**characteristics are straight lines**

# Smooth Solutions

## Characteristics




$$\frac{dx}{dt} = a(u_0) \quad \Rightarrow \quad x = x_0 + a(u_0)\, t$$

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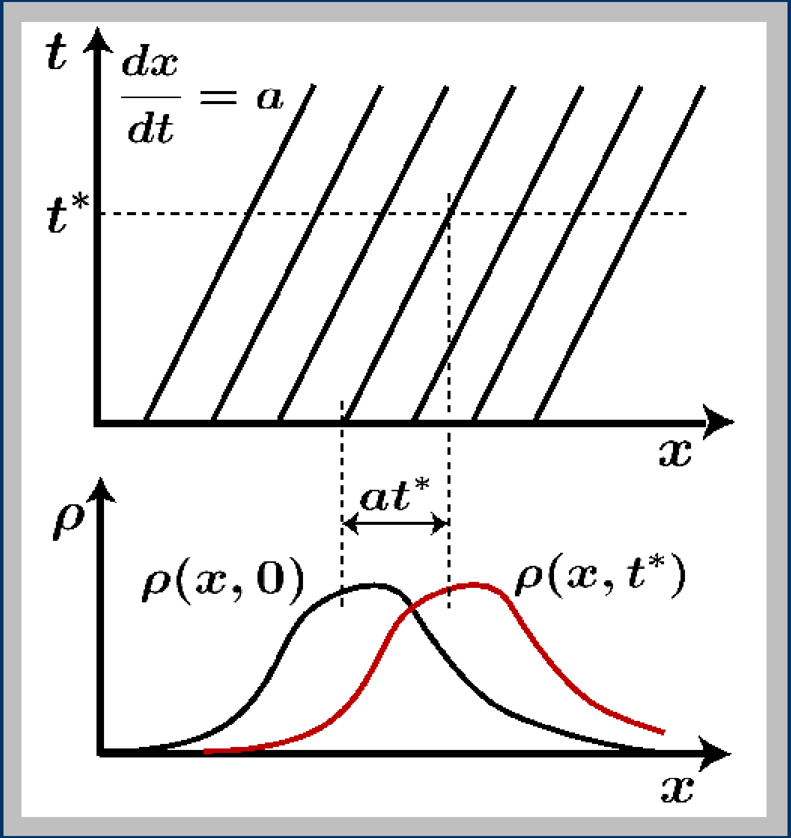
# Smooth Solutions

## Examples

### Linear Advection Equation




Solution

$$\rho(x,t) = \rho_0(x - at)$$

Characteristic lines

$$x = x_0 + at$$

# Smooth Solutions

## Examples

### Burgers' Equation...

Recall $f(u) = \frac{1}{2}u^2$, so $a(u) = u$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Solution : $u(x,t) = u_0(x - ut)$

The solution is constant along the characteristic lines defined by $x - ut = x_0$.

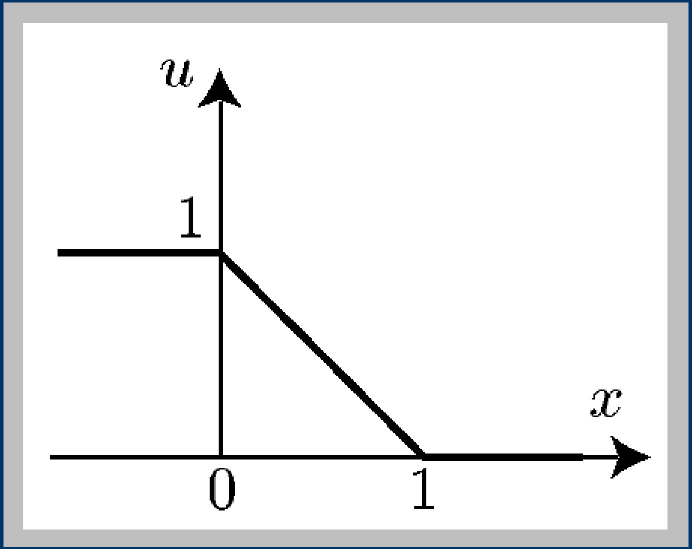
# Smooth Solutions

## Examples

### ...Burgers' Equation...

Consider the initial data

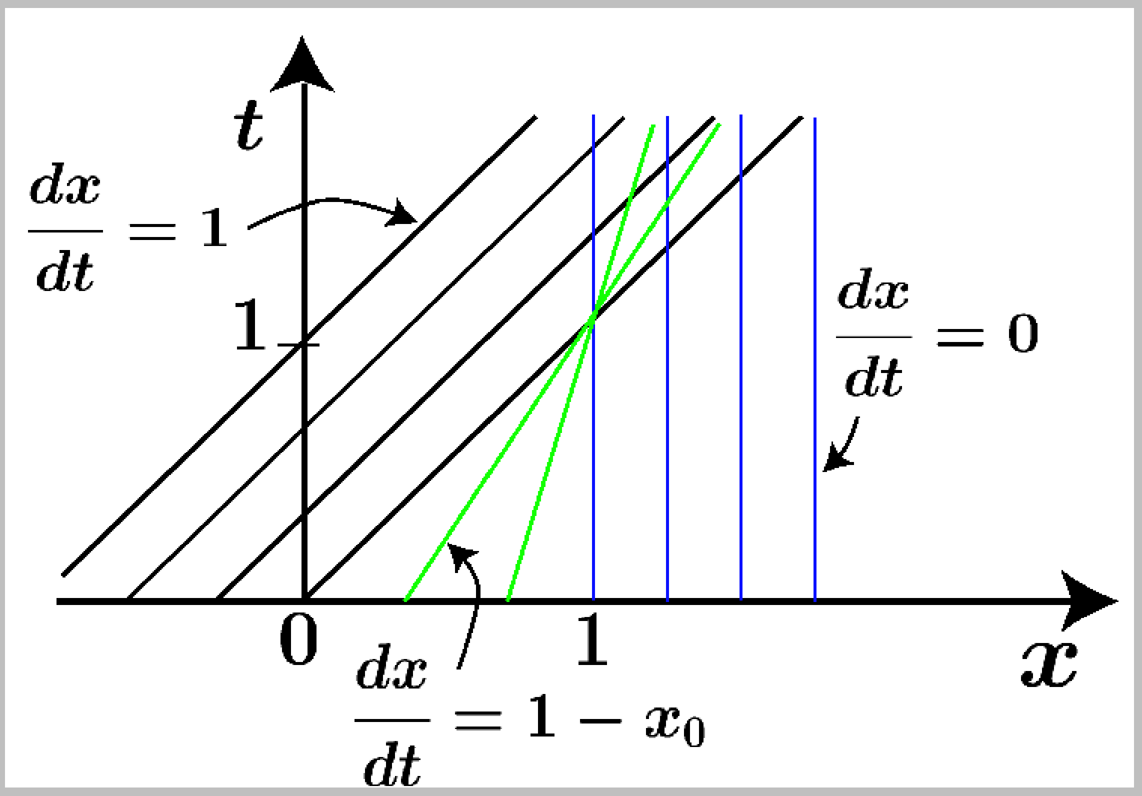
$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$



# Smooth Solutions

## Examples

### ...Burgers' Equation...



# Smooth Solutions

## Examples

### ...Burgers' Equation...

For $t \leq 1$

For $x \leq t$ : $\dfrac{dx}{dt} = 1 \rightarrow x = t + x_0 \rightarrow \boxed{u(x,t) = 1}$

For $t < x < 1$ : $\dfrac{dx}{dt} = 1 - x_0 \rightarrow x = (1 - x_0)t + x_0$

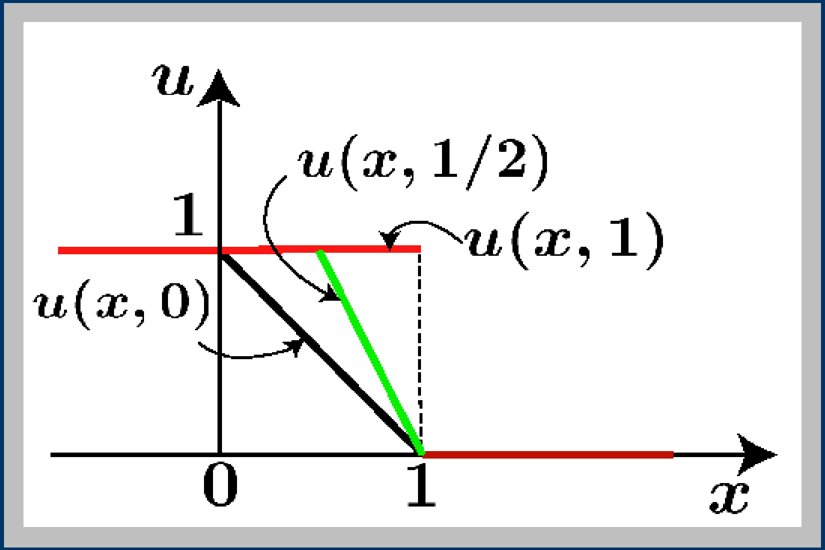
$$\boxed{u(x,t) = 1 - x_0 = \frac{1-x}{1-t}}$$

For $x \geq 1$ : $\dfrac{dx}{dt} = 0 \rightarrow x = x_0 \rightarrow \boxed{u(x,t) = 0}$

# Smooth Solutions

## Examples

### ...Burgers' Equation



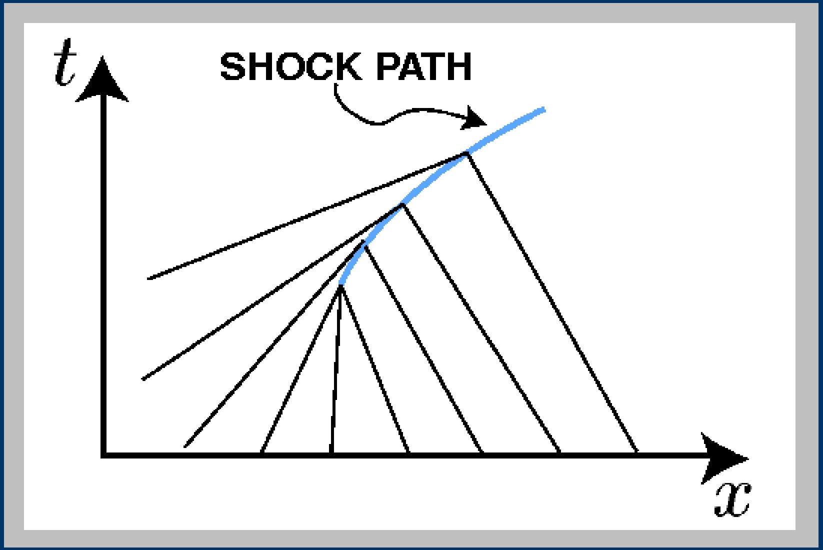

For $t = 1$

$$u(x,t) = \begin{cases} 1 & x < 1 \\ 0 & x > 1 \end{cases}$$

The procedure breaks down for $t > 1$

# Discontinuous Solution

## Shock Formation



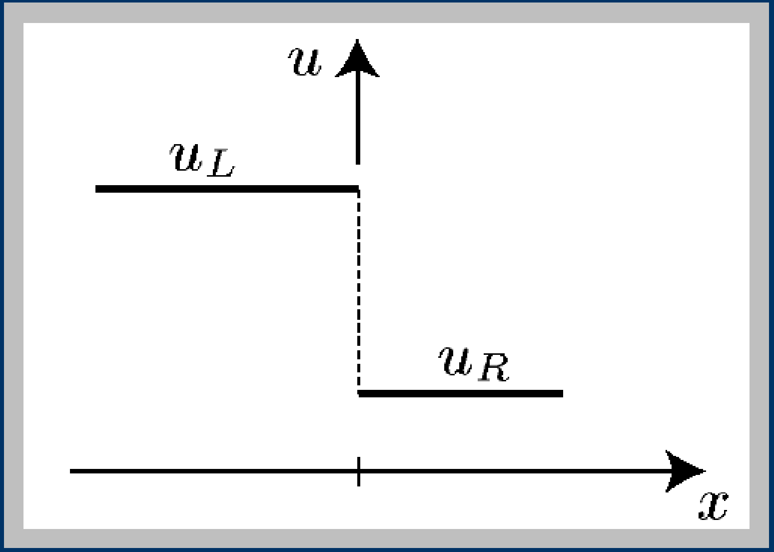

When the characteristics cross, the function $u(x, t)$ has an infinite slope. A discontinuity or **shock** forms, and the differential equation is no longer valid. N7 E1

# Discontinuous Solution

## The Riemann Problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

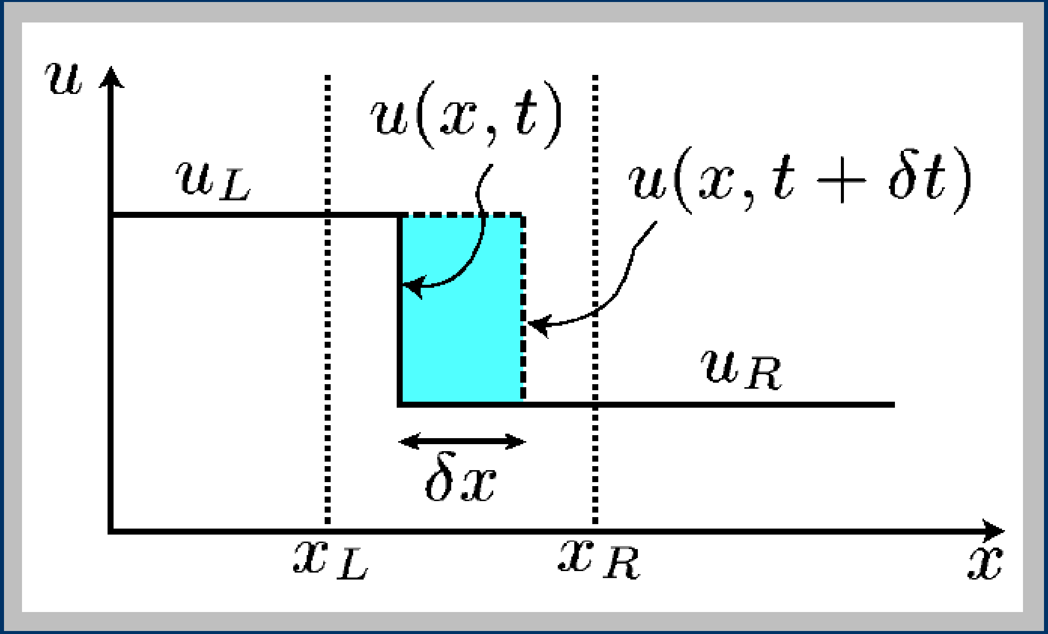
$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$



# Discontinuous Solution

## Shock Path

$$\frac{\partial}{\partial t}\int_{x_L}^{x_R} u\,dx = -\left(f(u_R) - f(u_L)\right)$$



# Discontinuous Solution

## Shock Path

$$-\frac{1}{\delta t}(u_R - u_L)\delta x = -\left(f(u_R) - f(u_L)\right)$$

Shock speed

$$s = \frac{\delta x}{\delta t} = \frac{f(u_R) - f(u_L)}{u_R - u_L} \equiv \frac{[f]}{[u]}$$

**Rankine-Hugoniot** jump condition

N8

# Discontinuous Solution

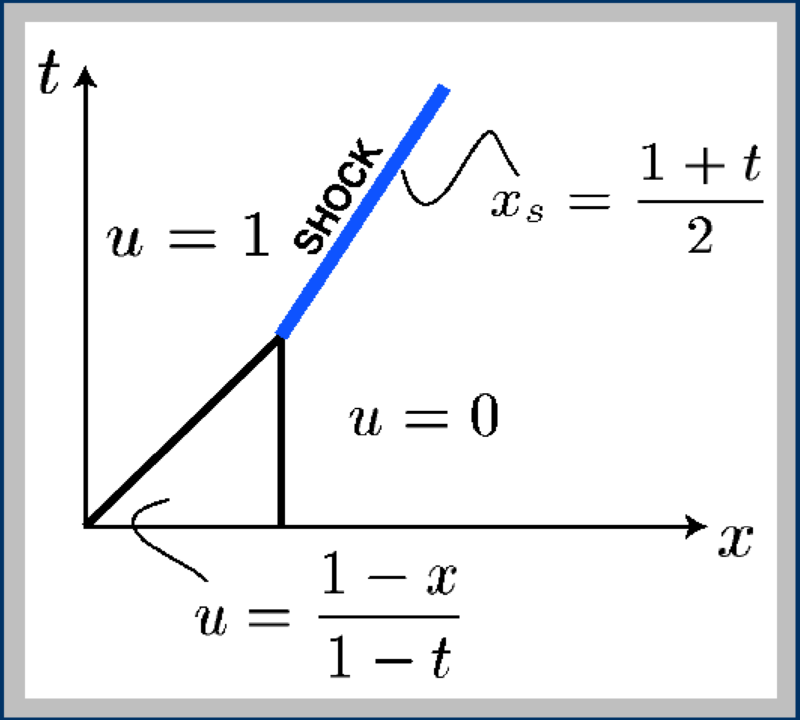
## Shock Path

### Example

For our example,

$$s = \frac{[f]}{[u]} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}$$

$$x_s = \frac{1 + t}{2}$$



# Discontinuous Solution

## Shock Path

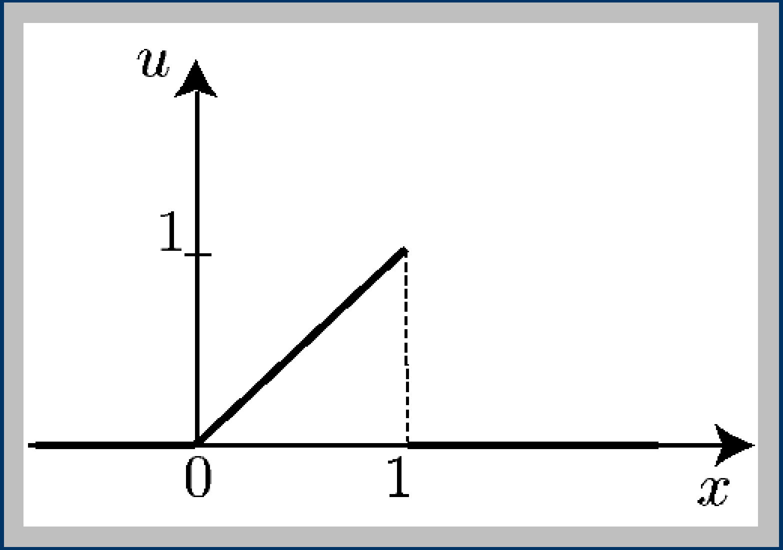
### Variable States...

If $u$ is not constant at both sides of the shock, the jump condition still applies **locally** $\Rightarrow$ shock path is **curved**.

Example:

Burgers' equation with

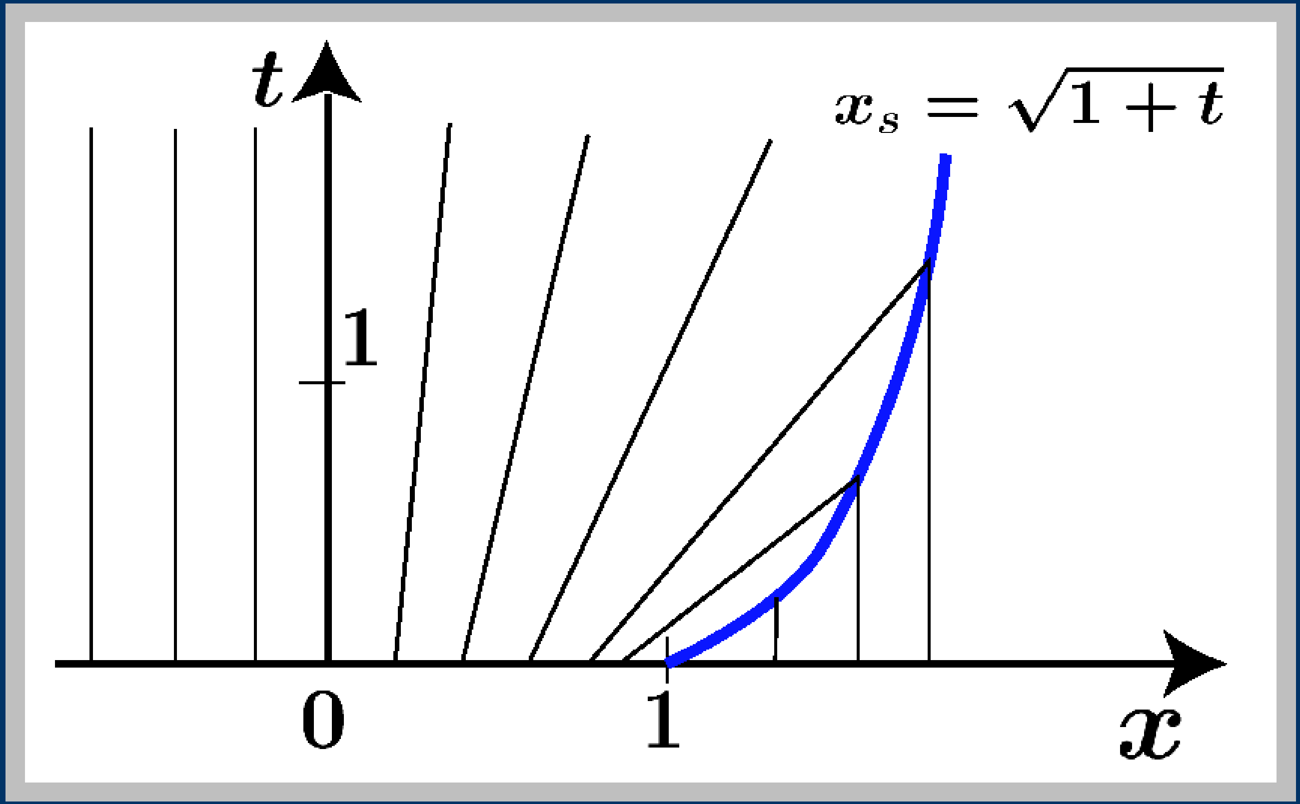
$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$



# Discontinuous Solution

## Shock Path

### ...Variable States



## Discontinuous Solution

### Shock Path

#### Manipulating the Conservation Law...

Consider

$$\frac{\partial u}{\partial t} + \frac{\partial \frac{1}{2}u^2}{\partial x} = 0 \qquad (A)$$

Multiply by $u$

$$u\frac{\partial u}{\partial t} + u\frac{\partial \frac{1}{2}u^2}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \frac{1}{2}u^2}{\partial t} + \frac{\partial \frac{1}{3}u^3}{\partial x} = 0$$

Let $v = \frac{1}{2}u^2$, then we can write

$$\frac{\partial v}{\partial t} + \frac{\partial \frac{\sqrt{8}}{3} \; v^{\frac{3}{2}}}{\partial x} = 0 \qquad (B)$$

# Discontinuous Solution

## Shock Path

### ...Manipulating the Conservation Law

Characteristics:

**A** $$\frac{dx}{dt} = a(u) = u$$

**B** $$\frac{dx}{dt} = a(v) = \frac{\sqrt{8}}{3}\frac{3}{2}\sqrt{v} = \sqrt{2v} = u$$

Shock speed:

**A** $$s_A = \frac{1}{2}(u_R + u_L)$$

**B** $$s_B = \frac{2}{3}\frac{u_R^3 - u_L^3}{u_R^2 - u_L^2} \neq s_A$$

N9

**Use conservation law for physically conserved quantity**

# Weak Solutions

Multiply $u_t + f_x = 0$ by $\phi(x,t) \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$

$$\int_0^\infty \int_{-\infty}^\infty \phi(u_t + f_x)\, dx dt = 0$$

Integrating by parts

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f]\, dx dt + \int_{-\infty}^\infty \phi(x,0) u(x,0) dx = 0$$

# Weak Solutions

If above statement is satisfied for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$ then $u(x,t)$ is a **weak solution**

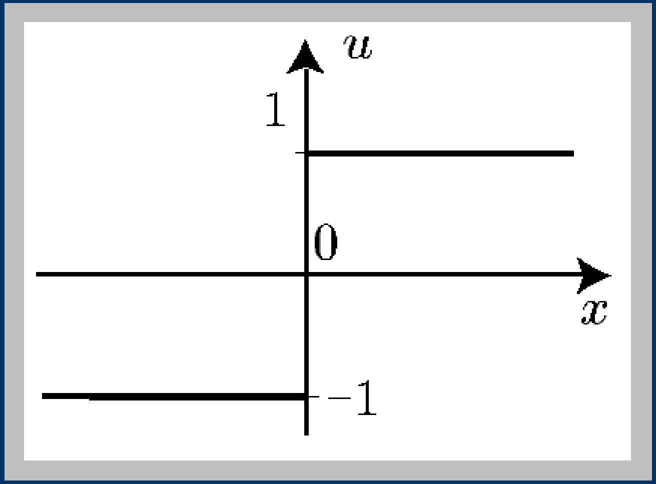
**Weak solutions** to conservation laws **are** often **non unique**.

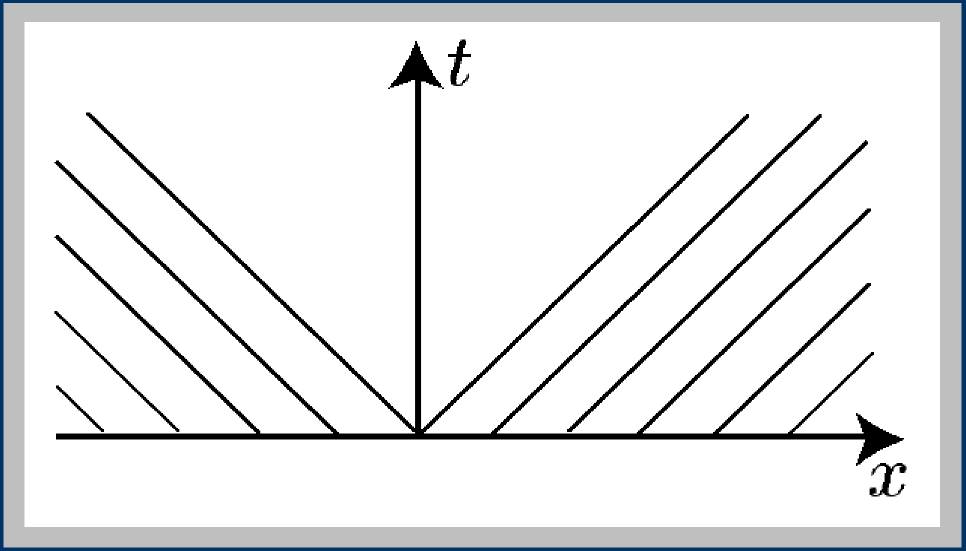
# Weak Solutions

## Non-uniqueness

### Example : Burgers' equation...

$$u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$



# Weak Solutions

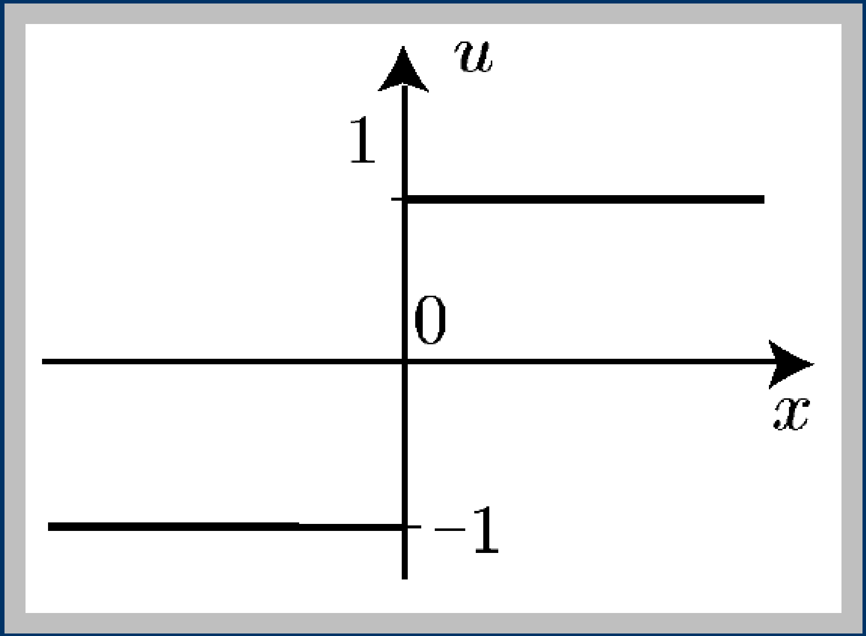
## Non-uniqueness

### ...Example : Burgers' equation...

Shock wave (Solution **A**)

$$s = \frac{u_L + u_R}{2} = 0$$

$$u(x,t) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$



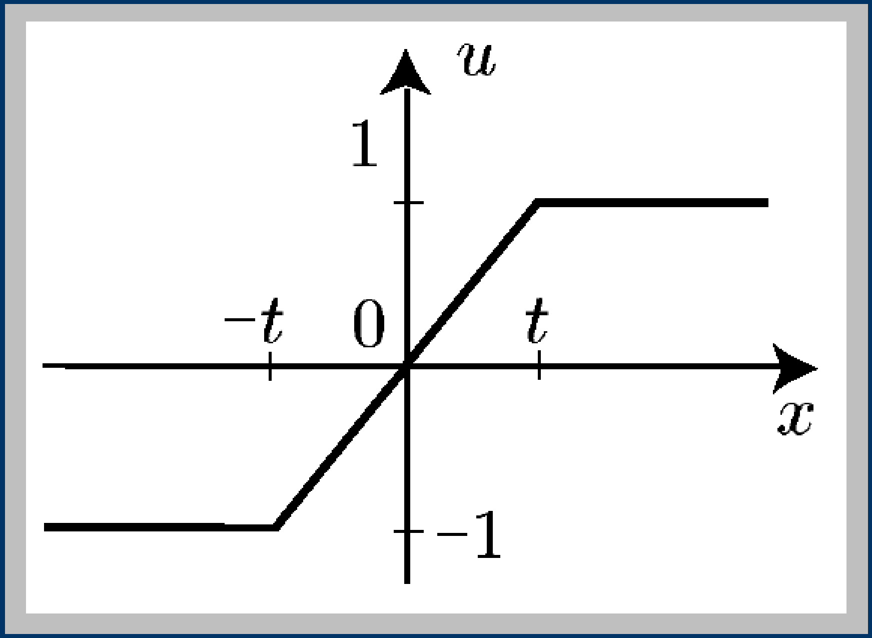
# Weak Solutions

## Non-uniqueness

### ...Example : Burgers' equation...

Rarefaction wave (Solution **B**)

$$u(x,t) = \begin{cases} -1 & x < -t \\ \frac{x}{t} & -t \leq x \leq t \\ 1 & x > t \end{cases}$$

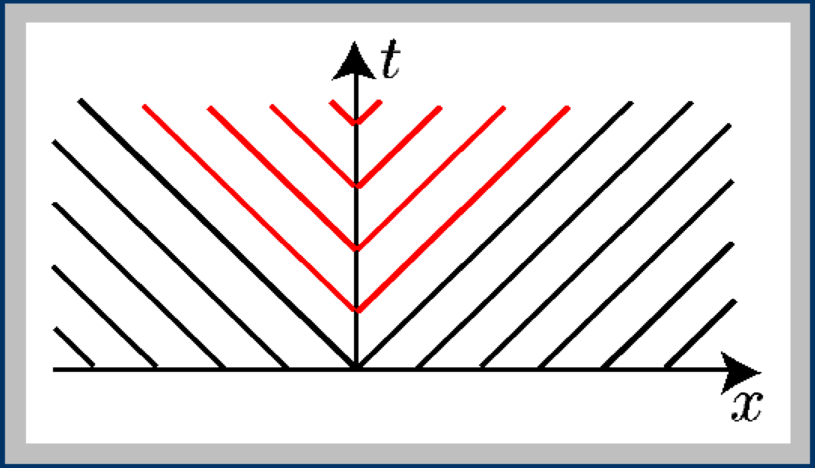


# Weak Solutions

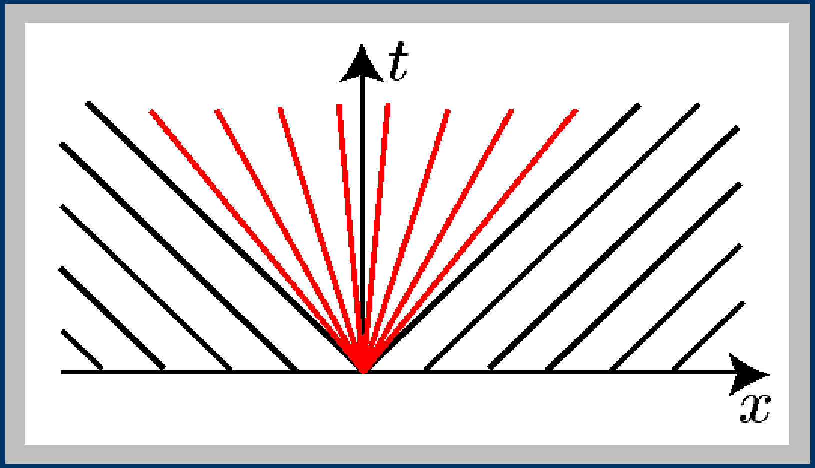
## Non-uniqueness

### ...Example : Burgers' equation

Solution **A**




Solution **B**



# Weak Solutions

## Entropy Condition

**Which is the physically relevant solution?**

Criterion : the physical solution satisfies

$$\lim_{\epsilon \to 0} \left( \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} \right)$$

# Weak Solutions

## Entropy Condition

### Convex (concave) fluxes...

When $f(u)$ is convex i.e. $f''(u) \geq 0$ ( or concave i.e. $f''(u) \leq 0$) for all $u$, the entropy condition can be written as

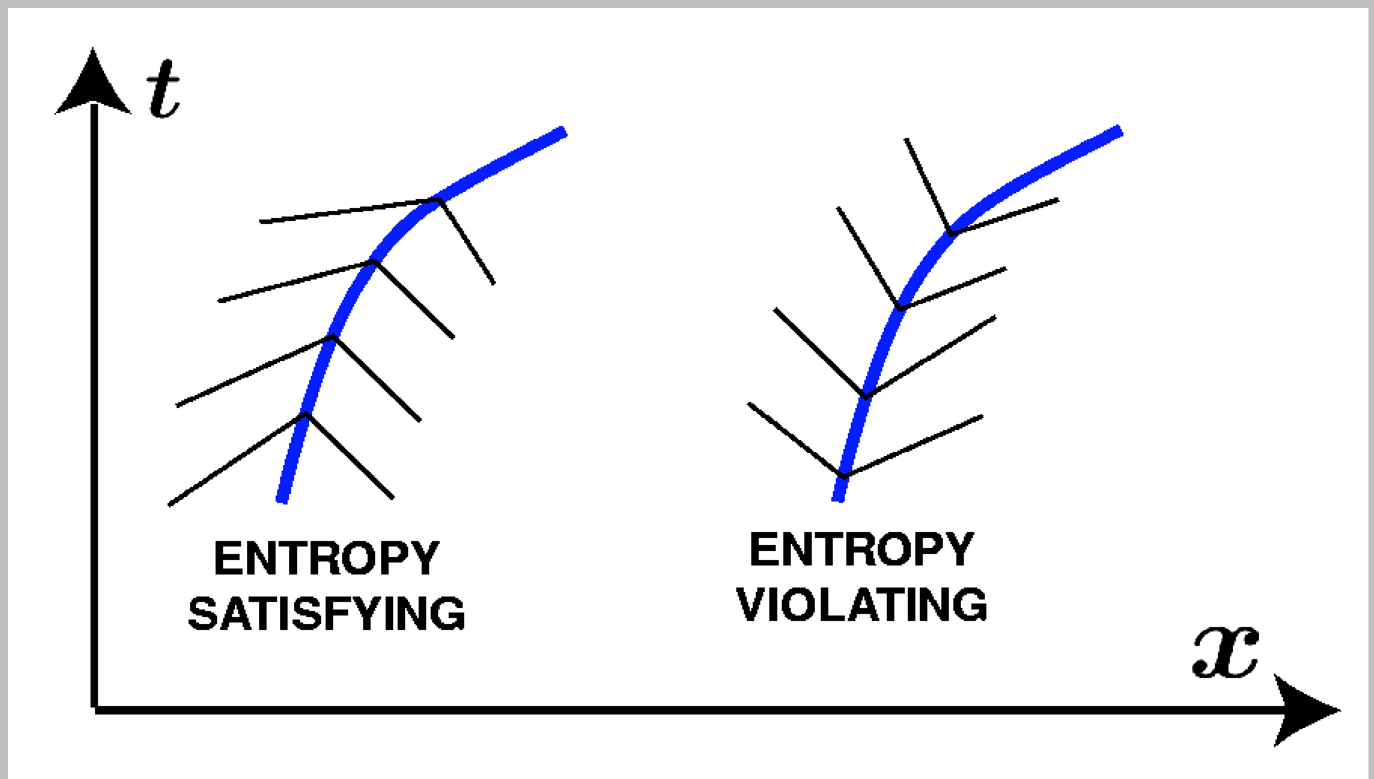
$$\boxed{a(u_L) = f'(u_L) > s > f'(u_R) = a(u_R)}$$

Characteristics must run **into** the shock for increasing $t$, not emerge from it.

# Weak Solutions

## Entropy Condition

### ...Convex (concave) fluxes



# Weak Solutions

## Entropy Condition

### Example

Solution **A**

$$a(u_L) = u_L = -1$$
$$s = \frac{1}{2}(u_R + u_L) = 0$$
$$a(u_R) = u_R = 1$$

Entropy condition is violated - characteristics emerge from the shock

Solution **B** No shock $\Rightarrow$ OK

# Weak Solutions

## Entropy Condition

### Oleinik's Condition

Applicable to general flux functions

$u(x,t)$ is the entropy satisfying solution if all discontinuities satisfy the property that

$$\frac{f(u) - f(u_L)}{u - u_L} \geq s \geq \frac{f(u) - f(u_R)}{u - u_R}$$

for all $u$ between $u_L$ and $u_R$.

## Weak Solutions

### Entropy Condition

#### Entropy Functions...

$U(u)$ is an **entropy function** if it is **positive**, **convex**, and there exists a corresponding **entropy flux** such that

$$F'(u) = U'(u) f'(u)$$

For smooth solutions

$$\Rightarrow \quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

## Weak Solutions

### Entropy Condition

#### ...Entropy Functions

The function $u(x,t)$ is the entropy satisfying solution of the governing equations if, for all convex entropy functions $U(u)$ and corresponding entropy fluxes $F(u)$, the inequality

$$\boxed{\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0}$$

is satisfied in the weak sense.

N10

If $f(u)$ is **convex** we only need to check for **one** $U(u)$

# Weak Solutions

## Entropy Condition

### Example

Burgers' equation $f(u) = \frac{1}{2}u^2$

Take $U(u) = u^2$ and $F(u) = \frac{2}{3}u^3$

Entropy inequality

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x \leq 0$$

# Application Examples

## Traffic Flow

Governing equation : $\rho_t + f(\rho)_x = 0$

$$f(\rho) = \rho u = \rho\, u_{\max} \left(1 - \frac{\rho}{\rho_{\max}}\right)$$

$$f'(\rho) = u_{\max} \left(1 - \frac{2\rho}{\rho_{\max}}\right), \quad f''(\rho) = -2\frac{u_{\max}}{\rho_{\max}} < 0$$

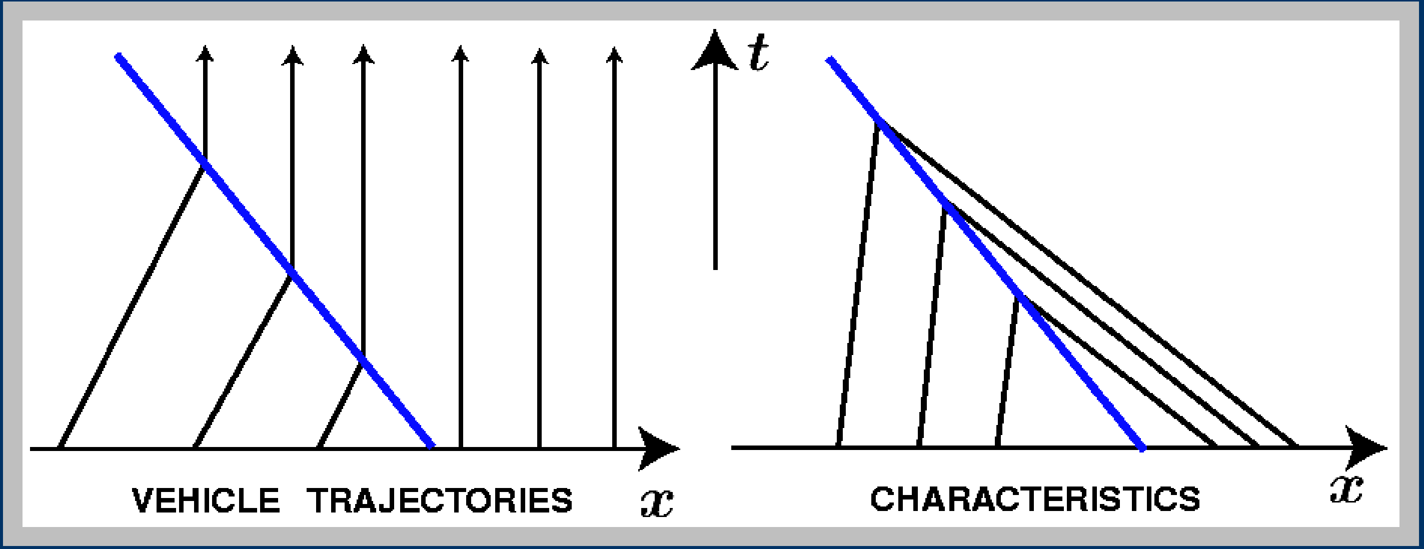
$$s = \frac{[f]}{[\rho]} = u_{\max} \left(1 - \frac{\rho_L + \rho_R}{\rho_{\max}}\right)$$

# Application Examples

## Traffic Flow

### "Traffic Jam"

Consider $\rho_R = \rho_{\max}$ and $\rho_L < \rho_{\max} \Rightarrow$ **Shock**




$s < 0$ and the shock propagates to the left

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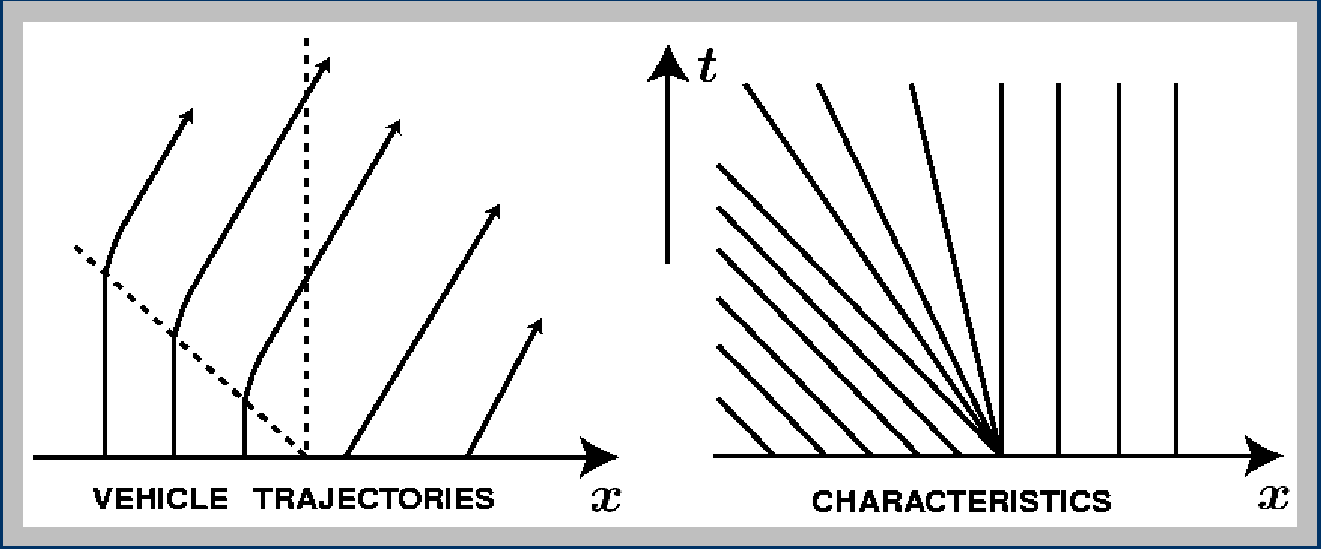
# Application Examples

## Traffic Flow

### "Green Light"

Consider $0 < \rho_R < \rho_L < \rho_{\max}$ $\Rightarrow$ **Rarefaction**.




$\rho_L = \rho_{\max}, \quad \rho_R = \frac{1}{2}\rho_{\max}$

**N12**

# Application Examples

## Traffic Flow

### "Sound Speed"...

For smooth solutions information travels with speed $f'(\rho)$

$$\rho_t + f'(\rho)\rho_x = 0$$

Consider a nearly constant solution

$$\rho(x,t) = \rho_0 + \varepsilon\rho_1(x,t)$$

If $\varepsilon$ is small we can model $\rho_1(x,t)$ with the linear equation

$$\rho_{1t} + f'(\rho_0)\rho_{1x} = 0$$

# Application Examples

## Traffic Flow

### ..."Sound Speed"

$$f'(\rho_0) = u_{\max}\left(1 - \frac{2\rho_0}{\rho_{\max}}\right), \quad u_0 = u_{\max}\left(1 - \frac{\rho_0}{\rho_{\max}}\right)$$

$f'(\rho_0) \le u_0$ cars travel ahead of disturbances

But . . .

$\rho_0 < \frac{1}{2}\rho_{\max}$ $\quad f'(\rho_0) > 0$ $\quad$ disturbances move forward

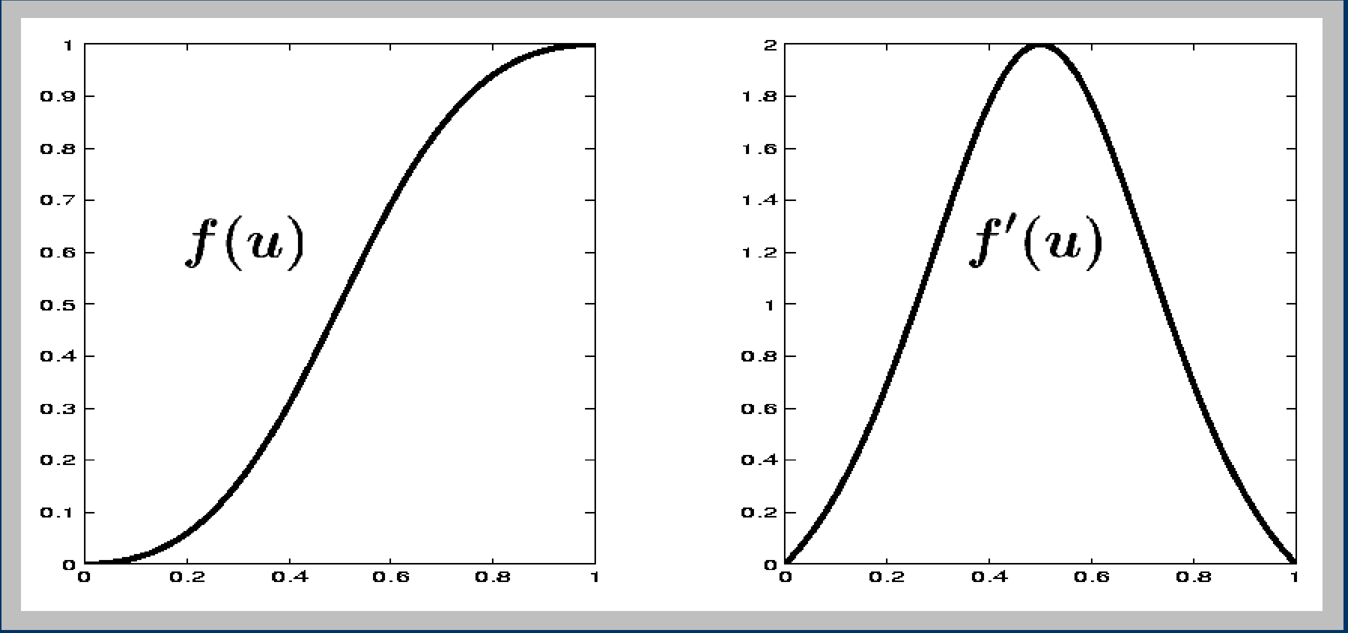
$\rho_0 > \frac{1}{2}\rho_{\max}$ $\quad f'(\rho_0) < 0$ $\quad$ disturbances move backward

$\rho_0 = \frac{1}{2}\rho_{\max}$ : "sonic point"

# Application Examples

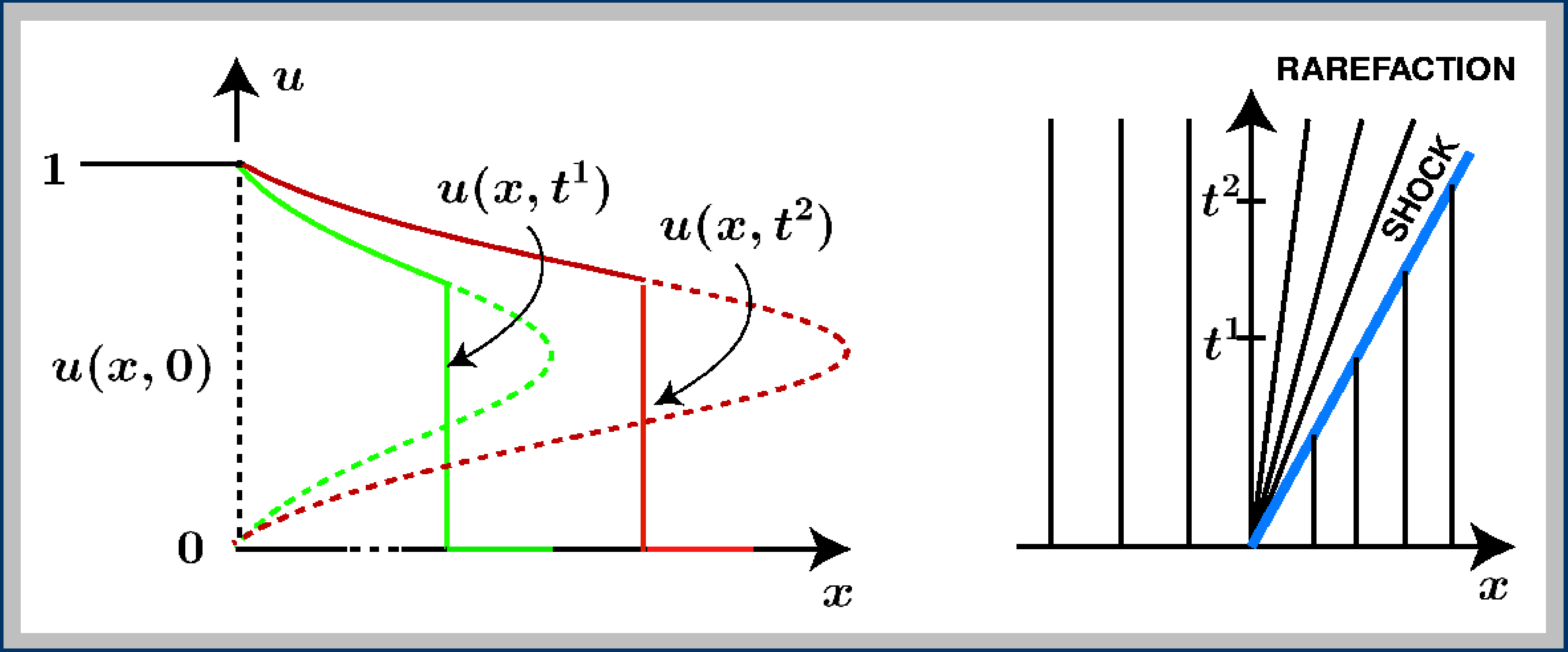
## Buckley-Leverett Equation

$$f(u) = \frac{u^2}{u^2 + a(1-u)^2}, \quad a = 1$$



# Application Examples

## Buckley-Leverett Equation

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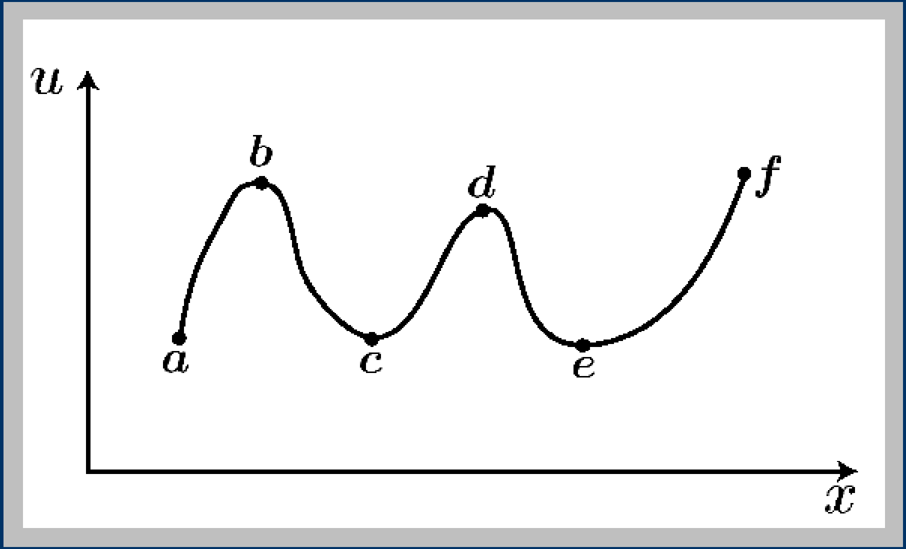
# Total Variation

## Definition

$$TV(u) = \int \left| \frac{\partial u}{\partial x} \right| dx$$

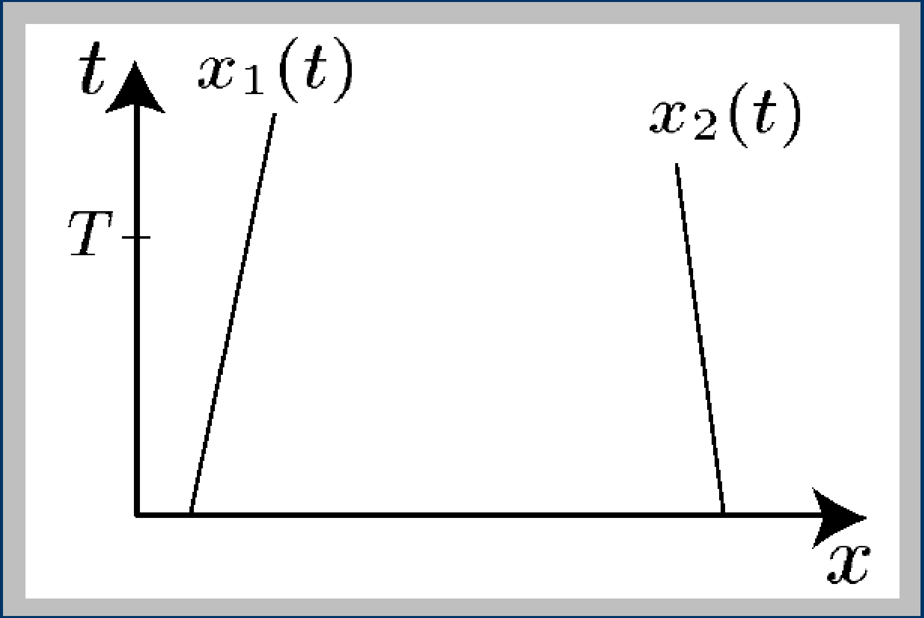



$$TV(u) = |u_b - u_a| + |u_c - u_b| + |u_d - u_c| + |u_e - u_d| + |u_f - u_e|$$

# Total Variation

## Continuous Case

Consider the total variation of $u(x,t)$ between two points $x_1(t)$ and $x_2(t)$ lying on two characteristics.



## Total Variation

### Continuous Case

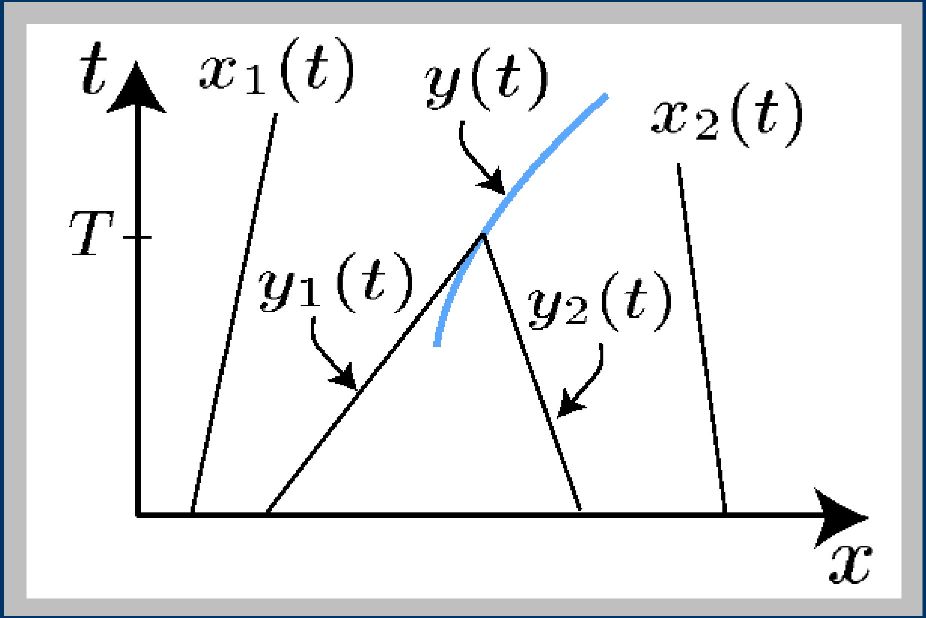
If there are **no shocks** between $x_1(t)$ and $x_2(t)$, then the extrema will not change, and the **total variation will stay constant with time**.

$$TV(u(x,t)) = TV(u(x,0))$$

# Total Variation

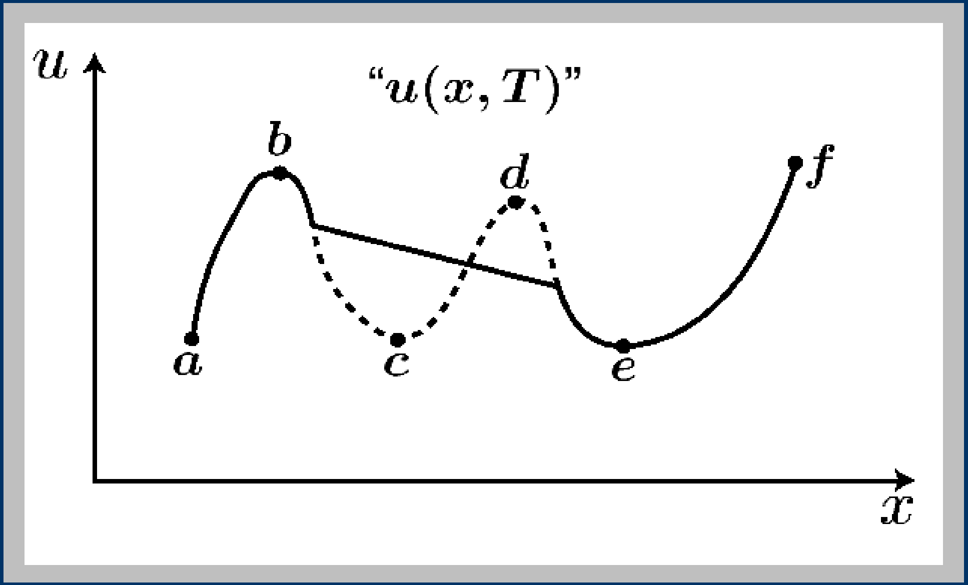
## Discontinuous Case

The solution at a shock is determined by $y_1(t)$ and $y_2(t)$ provided the shock is entropy satisfying.



# Total Variation

## Discontinuous Case




$$TV(u(x,T)) = |u_b - u_a| + |u_c - u_e| + |u_f - u_e|$$

## Total Variation

### Discontinuous Case

If there are shocks: $TV(u(x,T)) \leq TV(u(x,0))$

In general,

$$\frac{d}{dt}\left(\int \left|\frac{\partial u}{\partial x}\right| dx\right) \leq 0$$

**The total variation of an entropy satisfying solution is a non-increasing function of time.**