6.207/14.15: Networks
Lecture 12: Applications of Game Theory to Networks

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Outline

- Traffic equilibrium: the Pigou example
- General formulation with single origin-destination pair
- Multi-origin-destination traffic equilibria
- Congestion games and atomic traffic equilibria
- Potential functions and potential games
- Network cost-sharing
- Strategic network formation.

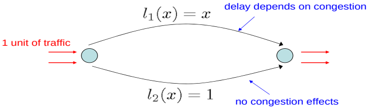
- Reading:
- EK, Chapter 8.
- Jackson, Chapter 6.

Motivation

- Many games are played over networks, in the sense that players interact with others linked to them through a network-like structure.
- Alternatively, in several important games, the actions of players correspond to a path in a given network.
  - The most important examples are choosing a route in a traffic problem or in a data routing problem.
  - Other examples are cost sharing in network-like structures.
- Finally, the formation of networks is typically a game-theoretic (strategic) problem.
- In this lecture, we take a first look at some of these problems, focusing on traffic equilibria and formation of networks.

The Pigou Example of Traffic Equilibrium

- Recall the following simple example from lecture 9, where a unit mass of traffic is to be routed over a network:



- System optimum (minimizing aggregate delay) is to split traffic equally between the two routes, giving

$$\min_{x_1+x_2\leq 1} C_{\text{system}}(x^S) = \sum_i l_i(x_i^S)x_i^S = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

- Instead, the Nash equilibrium of this large (non-atomic) game, also referred to as **Wardrop equilibrium**, is $x_1 = 1$ and $x_2 = 0$ (since for any $x_1 < 1$, $l_1(x_1) < 1 = l_2(1 - x_1)$), giving an aggregate delay of

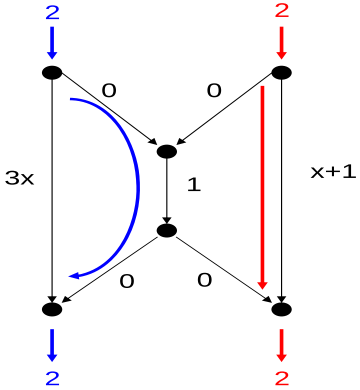
$$C_{\text{eq}}(x^{WE}) = \sum_i l_i(x_i^{WE})x_i^{WE} = 1 + 0 = 1 > \frac{3}{4}.$$

The Wardrop Equilibrium

- Why the Wardrop equilibrium?
- It is nothing but a Nash equilibrium in this game, in view of the fact that it is non-atomic—each player is infinitesimal. Thus, taking the strategies of others as given is equivalent to taking aggregates, here total traffic on different routes, as given.
- Therefore, the Wardrop equilibrium (or the Nash equilibrium of a large game) is a convenient modeling tool when each participant in the game is small.
- A small technical detail: so far we often took the set of players, $\mathcal{I}$, to be a finite set. But in fact nothing depends on this, and in non-atomic games, $\mathcal{I}$ is typically taken to be some interval in $\mathbb{R}$, e.g., $[0, 1]$.

More General Traffic Model

- Let us now generalize the Pigou example. In the general model, there are several origin-destination pairs and multiple paths linking these pairs



Cost = 2x1 + 2x3=8

More General Traffic Model: Notation

- Let us start with a single origin-destination pair.
- Directed network $N = (V, E)$.
- $\mathcal{P}$ denotes the set of paths between origin and destination.
- x_p denotes the flow on path $p \in \mathcal{P}$.
- Each link $i \in E$ has a latency function $l_i(x_i)$, where

$$x_i = \sum_{\{p \in \mathcal{P} | i \in p\}} x_p.$$

- Here the notation $p \in \mathcal{P} | i \in p$ denotes the paths p that traverse link $i \in E$.
- The latency function captures congestion effects. Let us assume for simplicity that $l_i(x_i)$ is nonnegative, differentiable, and nondecreasing.
- We normalize total traffic to 1 and in the context of the game theoretic formulation here, $\mathcal{I} = [0, 1]$. We also assume that all traffic is homogeneous. Each motorist wishes to minimize delay.

More General Traffic Model (continued)

- We denote a routing pattern by the vector $\mathbf{x}$. If it satisfies the two constraints above, it is a feasible routing pattern.
- The total delay (latency) cost of a routing pattern $\mathbf{x}$ is:

$$C(\mathbf{x}) = \sum_{i \in E} x_i l_i(x_i),$$

that is, it is the sum of latencies $l_i(x_i)$ for each link $i \in E$ multiplied by the flow over this link, x_i, summed over all links E.

Socially Optimal Routing

- Socially optimal routing, defined as the routing pattern minimizing aggregate delay, is given by x^S that is a solution to the following problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in E} x_i l_i(x_i) \\ \text{subject to} \quad & \sum_{\{p \in \mathcal{P} | i \in p\}} x_p = x_i, \ \text{for all } i \in E, \\ & \sum_{p \in \mathcal{P}} x_p = 1 \text{ and } x_p \geq 0 \text{ for all } \ p \in \mathcal{P}. \end{aligned}$$

Wardrop Equilibrium

- What is a Wardrop equilibrium?
- Since it is a Nash equilibrium, it has to be the case that for each motorist their routing choice must be optimal.
- This implies that if a motorist $k \in \mathcal{I}$ is using path p, then **there does not exist** path p' such that

$$\sum_{i \in p} l_i(x_i) > \sum_{i \in p'} l_i(x_i).$$

- Put differently, $\mathbf{x}$ must be such that

$$(1) \text{ For all } p, p' \in \mathcal{P} \text{ with } x_p, x_{p'} > 0, \sum_{i \in p'} l_i(x_i) = \sum_{i \in p} l_i(x_i).$$

$$(2) \text{ For all } p, p' \in \mathcal{P} \text{ with } x_p > 0 \text{ and } x_{p'} = 0,$$

$$\sum_{i \in p'} l_i(x_i) \geq \sum_{i \in p} l_i(x_i).$$

Characterizing Wardrop Equilibria

Theorem

A feasible routing pattern $\mathbf{x}^{WE}$ *is a Wardrop equilibrium if and only if it is a solution to*

$$\begin{aligned} \textit{minimize} \quad & \sum_{i \in E} \int_0^{x_i} l_i(z)\, dz \\ \textit{subject to} \quad & \sum_{\{p \in \mathcal{P} \mid i \in p\}} x_p = x_i, \ \textit{for all} \ \ i \in E, \\ & \sum_{p \in \mathcal{P}} x_p = 1 \ \textit{and} \ x_p \geq 0 \ \textit{for all} \ p \in \mathcal{P}. \end{aligned}$$

Moreover, if each l_i *is strictly increasing, then* $\mathbf{x}^{WE}$ *is unique.*

- Note that by Weierstrass's Theorem, a solution exists, and thus a Wardrop equilibrium always exists.

Proof

- Rewrite the minimization problem as

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in E} \int_0^{\sum_{i \in p} x_p} l_i(z)\, dz \\ \text{subject to} \quad & \sum_{p \in \mathcal{P}} x_p = 1 \text{ and } x_p \geq 0 \text{ for all } p \in \mathcal{P}. \end{aligned}$$

- Since each l_i is nondecreasing, this is a convex program. Therefore, first-order conditions are necessary and sufficient.
- First-order conditions with respect to x_p are

$$\sum_{i \in p} l_i \left(x_i^{WE} \right) \geq \lambda$$

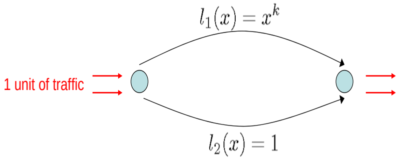
with the complementary slackness, i.e., with equality whenever $x_p^{WE} > 0$.

Proof (continued)

- Here λ is the Lagrange multiplier on the constraint $\sum_{p\in\mathcal{P}} x_p = 1$.
- The Lagrange multiplier will be equal to the lowest cost path, which then implies the result that for all $p, p' \in \mathcal{P}$ with $x_p^{WE}, x_{p'}^{WE} > 0$, $\sum_{i\in p'} l_i(x_i^{WE}) = \sum_{i\in p} l_i(x_i^{WE})$. And clearly, for paths with $x_p^{WE} = 0$, the cost can be higher.
- Finally, if each l_i is strictly increasing, then the set of equalities $\sum_{i\in p'} l_i(x_i^{WE}) = \sum_{i\in p} l_i(x_i^{WE})$ admits unique solution, establishing uniqueness.

Inefficiency of the Equilibrium

- We saw from the Pigou example that the Wardrop equilibrium fails to minimize total delay—hence it is inefficient.
- In fact, it can be arbitrarily inefficient. Consider the following figure, which is the same as the Pigou example, except with a different latency on path 1.



Inefficiency of the Equilibrium (continued)

- In this example, socially optimal routing again involves

$$\begin{aligned} l_1(x_1) + x_1 l_1'(x_1) &= l_2(1-x_1) + (1-x_1)\, l_2'(1-x_1) \\ x_1^k + k x_1^k &= 1 \end{aligned}$$

- Therefore, the system optimum sets $x_1 = (1+k)^{-1/k}$ and $x_2 = 1 - (1+k)^{-1/k}$, so that

$$\min_{x_1 + x_2 \leq 1} C_{\text{system}}(x^S) = \sum_i l_i(x_i^S) x_i^S = (1+k)^{-\frac{k+1}{k}} + 1 - (1+k)^{-1/k}.$$

Inefficiency of the Equilibrium (continued)

- The Wardrop equilibrium again has $x_1 = 1$ and $x_2 = 0$ (since once again for any $x_1 < 1$, $l_1(x_1) < 1 = l_2(1 - x_1)$). Thus

$$C_{\text{eq}}(x^{WE}) = \sum_i l_i(x_i^{WE})x_i^{WE} = 1 + 0 = 1.$$

- Therefore, the **Price of anarchy** is now

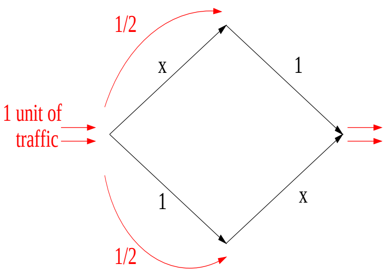
$$\frac{C_{\text{system}}(x^S)}{C_{\text{eq}}(x^{WE})} = (1+k)^{-\frac{k+1}{k}} + 1 - (1+k)^{-1/k}.$$

- This limits to 0 as $k \to \infty$ (the first term tends to zero, while the last term limits to 1).
- Thus the equilibrium can be "arbitrarily" inefficient relative to the social optimum.

Further Paradoxes of Decentralized Equilibrium: Braess's Paradox

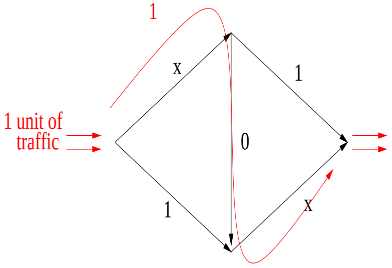
- Idea: Addition of an intuitively helpful route negatively impacts network users.
- Paradoxical, since the addition of another route should help traffic. In fact, the addition of a link can never increase aggregate delay in the social optimum.
- But the situation is different in a Wardrop equilibrium.
- This idea was first introduced in transportation networks by Braess. Hence the name of the paradox.

Further Paradoxes of Decentralized Equilibrium: Braess's Paradox (continued)



$C_{eq} = 1/2\,(1/2+1) + 1/2\,(1/2+1) = 3/2$

$C_{sys} = 3/2$



$C_{eq} = 1 + 1 = 2$

$C_{sys} = 3/2$

Multiple Origin-Destination Pairs

- The above model is straightforward to generalize to multiple origin-destination pairs.
- Suppose that there are K such pairs (some of them having possibly the same origin or destination).
- Origin-destination pair j has total traffic r_j.
- Let us denote the set of paths for origin-destination pair j by $\mathcal{P}_j$, and now $\mathcal{P} = \cup_j \mathcal{P}_j$.
- Then the socially optimal routing pattern is a solution to

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in E} x_i l_i(x_i) \\ \text{subject to} \quad & \sum_{\{p \in \mathcal{P} | i \in p\}} x_p = x_i, \quad i \in E, \\ & \sum_{p \in \mathcal{P}_j} x_p = r_j, \; j = 1, \ldots, k, \; \text{and } x_p \geq 0 \text{ for all } \; p \in \mathcal{P}. \end{aligned}$$

Wardrop Equilibrium with Multiple Origin-Destination Pairs

- Essentially the same characterization theorem for Wardrop equilibrium applies with multiple origin-destination pairs.

Theorem

A feasible routing pattern $\mathbf{x}^{WE}$ is a Wardrop equilibrium if and only if it is a solution to

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in E} \int_0^{x_i} l_i(z)\, dz \\ \text{subject to} \quad & \sum_{\{p \in \mathcal{P} | i \in p\}} x_p = x_i, \quad i \in E, \\ & \sum_{p \in \mathcal{P}_j} x_p = r_j, \; j = 1, \ldots, k, \text{ and } x_p \geq 0 \text{ for all } p \in \mathcal{P}. \end{aligned}$$

Moreover, if each l_i is strictly increasing, then $\mathbf{x}^{WE}$ is uniquely defined.

Congestion Games

- A Wardrop equilibrium presumes that there are so many players that they all take aggregates as given.
- This "non-atomic" player assumption is a good approximation for many situations. But in others, players may be large and may thus naturally take into account their effect on the total amount of traffic on a particular path.
  - For example, United Airlines would naturally take into account the congestion implications of using Washington Dulles as a hub.
- This would be a special case of a **congestion game**.

Congestion Games (continued)

- Congestion Model: $C = \langle \mathcal{I}, \mathcal{M}, (S_i)_{i \in \mathcal{I}}, (c^j)_{j \in \mathcal{M}} \rangle$ where
- $\mathcal{I} = \{1, 2, \cdots, I\}$ is the set of players.
- $\mathcal{M} = \{1, 2, \cdots, m\}$ is the set of resources.
- $S_i \subset \mathcal{M}$ is the set of resource combinations (e.g., links or common resources) that player i can take/use. A strategy for player i is $s_i \in S_i$, corresponding to the resources that this player is using.
- $c^j(k)$ is the benefit for the negative of the cost to each user who uses resource j if k users are using it.
- Define congestion game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j),$$

where k_j is the number of users of resource j under strategies s.
- How do we analyze congestion games? To do so, we will introduce a more general class of games called potential games.

Potential Games

- A finite game (or a game with a finite number of players but with infinite strategy spaces) is a **potential game** [ordinal potential game, exact potential game] if there exists a function $\Phi : S \rightarrow \mathbb{R}$ such that $\Phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$ for each $i \in \mathcal{I}$.
- If so, Φ is referred to as the **potential function.**
- The potential function has a natural analogy to "energy" in physical systems. It will be useful both for locating pure strategy Nash equilibria and also for the analysis of "myopic" dynamics in the next lecture.

Potentials

- A function $\Phi : S \to \mathbb{R}$ is called an **ordinal potential function** for the game G if for each $i \in \mathcal{I}$ and all $s_{-i} \in S_{-i}$,

  $u_i(x, s_{-i}) - u_i(z, s_{-i}) \geq 0$ iff $\Phi(x, s_{-i}) - \Phi(z, s_{-i}) \geq 0$, for all $x, z \in S_i$,

  and

  $u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0$ iff $\Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0$, for all $x, z \in S_i$.

- A function $\Phi : S \to \mathbb{R}$ is called an **exact potential function** for the game G if for each $i \in \mathcal{I}$ and all $s_{-i} \in S_{-i}$,

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}), \text{ for all } x, z \in S_i.$$

Potential Games

- A finite game G is called an ordinal (exact) potential game if it admits an ordinal (exact) potential.
- In what follows, we refer to ordinal potential games as **potential games**, and only add the "exact" qualifier when this is necessary.
- A game G with infinite strategy space and finite number of players is a potential game if it admits a continuous potential function.

Pure Strategy Nash Equilibria in Potential Games

Theorem

Every potential game has at least one pure strategy Nash equilibrium.

- **Proof:** The global maximum of an ordinal potential function is a pure strategy Nash equilibrium. To see this, suppose that s^* corresponds to the global maximum. Then, for any $i \in \mathcal{I}$, we have, by definition, $\Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0$ for all $s \in S_i$. But since Φ is a potential function,

$$u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0 \quad \text{iff} \quad \Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0, \text{ for all} s \in S_i.$$

  Therefore, $u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0$ for all $s \in S_i$ and for all $i \in \mathcal{I}$. Hence s^* is a pure strategy Nash equilibrium.
- Note, however, that there may also be other pure strategy Nash equilibria corresponding to local maxima.

Examples of Ordinal Potential Games

- Example: Cournot competition.
- I firms choose quantity $q_i \in (0, \infty)$
- The payoff function for player i given by $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$.
- We define the function $\Phi(q_1, \cdots, q_I) = \left(\prod_{i=1}^{I} q_i\right)(P(Q) - c)$.
- Note that for all i and all q_{-i},

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) > 0 \text{ iff } \Phi(q_i, q_{-i}) - \Phi(q_i', q_{-i}) > 0 \text{ for all } q_i, q_i' >$$

- Φ is therefore an ordinal potential function for this game.

Examples of Exact Potential Games

- Example: Cournot competition (again).
- Suppose now that $P(Q) = a - bQ$ and costs $c_i(q_i)$ are arbitrary.
- We define the function

$$\Phi^*(q_1, \cdots, q_n) = a \sum_{i=1}^{I} q_i - b \sum_{i=1}^{I} q_i^2 - b \sum_{1 \leq i < l \leq I}^{I} q_i q_l - \sum_{i=1}^{I} c_i(q_i).$$

- It can be shown that for all i and all q_{-i},

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi^*(q_i, q_{-i}) - \Phi^*(q_i, q_{-i}'), \text{ for all } q_i, q_i' > 0.$$

- Φ is an exact potential function for this game.

Congestion and Potential Games

Theorem

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

- **Proof**: For each j define $\bar{k}_j^i$ as the usage of resource j excluding player i, i.e.,

$$\bar{k}_j^i = \sum_{i' \neq i} \mathbf{I}\left[j \in s_{i'}\right],$$

where $\mathbf{I}\left[j \in s_{i'}\right]$ is the indicator for the event that $j \in s_{i'}$.

- With this notation, the utility difference of player i from two strategies s_i and s_i' (when others are using the strategy profile s_{-i}) is

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1).$$

Proof Continued

- Now consider the function

$$\Phi(s) = \sum_{j \in \bigcup_{i' \in \mathcal{I}} s_{i'}} \left[\sum_{k=1}^{k_j} c^j(k) \right].$$

- We can also write

$$\Phi(s_i, s_{-i}) = \sum_{j \in \bigcup\limits_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1).$$

Proof Continued

- Therefore:

$$
\begin{aligned}
\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) &= \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) \\
&\quad - \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).
\end{aligned}
$$

Network Cost Sharing

- Consider the problem of sharing the cost of some network resources among participants.
- Directed graph $N = (V, E)$ with edge cost $c_e \geq 0$, k players.
- Each player i has a set of nodes T_i he wants to connect.
- A strategy of player i set of edges $S_i \subset E$ such that S_i connects to all nodes in T_i.
- **Cost sharing mechanism:** All players using an edge split the cost equally
- Given a vector of player's strategies $S = (S_1, \ldots, S_k)$, the cost to agent i is $C_i(S) = \sum_{e \in S_i} (c_e / x_e)$, where x_e is the number of agents whose strategy contains edge e.

Network Cost Sharing (continued)

- The game here involves each player simultaneously choosing $S_i \subset E$.
- This immediately implies:

Proposition

The network cost-sharing game is a potential game and thus has a pure strategy Nash equilibrium.

- In fact, network cost sharing games typically have several equilibria.
- As a trivial example, in a network consisting of two links with equal (or similar) costs and I players who could all use either link, there are two pure strategy equilibria: all players using link 1 or all players using link 2.

A Simple Game of Network Formation

- The model of network cost sharing can be viewed as a specific instance of "endogenous network formation". However, the most interesting aspect of network formation, the benefits as well as the cost of forming links, are absent from this model.
- Let us now consider a simple game of network formation.
- There are I symmetric players. The set of possible graphs (networks) is denoted by $\mathcal{G}$, and corresponds to all possible configurations of links between the I players.
- The utility function of player i is

$$u_i : \mathcal{G} \to \mathbb{R},$$

and assigns a utility level to every possible network configuration.

A Simple Game of Network Formation (continued)

- More specifically, suppose that for any network $g \in \mathcal{G}$, we have

$$u_i(g) = \sum_{j \neq i} b(\ell_{ij}(g)) - d_i(g)\, c,$$

where:

  - c is the cost of a direct connection;
  - $d_i(g)$ is the degree of player i in the network g;
  - $\ell_{ij}(g)$ is the distance between player i and player j in the network g, with the convention that $\ell_{ij}(g) = \infty$ if the two players are not connected;
  - $b : \mathbb{N} \to \mathbb{R}$ is a benefit function depending on the distance; we assume that $b(\cdot)$ is strictly decreasing and $b(\infty) = 0$.

- A network $g \in \mathcal{G}$ is "efficient" if there does not exist $g' \in \mathcal{G}$ such that

$$U' = \sum_{i=1}^{I} \left[\sum_{j \neq i} b(\ell_{ij}(g')) - d_i(g')\, c \right] > U = \sum_{i=1}^{I} \left[\sum_{j \neq i} b(\ell_{ij}(g)) - d_i(g)\, c \right].$$

Pairwise Stability

- We can next study the equilibrium networks that will emerge in this game. For example, as in the network cost sharing game, we could look for the simultaneous move games and study the pure strategy Nash equilibria. However, as shown in that discussion, there are many such equilibria, and multiplicity problem becomes worse when the formation of a link requires "agreement" from two parties.
- An alternative proposed by Jackson and Wolinsky (1996) "A Strategic Model of Social and Economic Networks" is to look at the concept of pairwise stability.

Pairwise Stability (continued)

- Pairwise stability requires that there are no profitable deviations by a pair of agents by either adding a new link or by removing an existing link.
- More formally, a network g is **pairwise stable** if
  - For all $\{i,j\} \in g$, $u_i(g) \geq u_i(g - \{i,j\})$ and $u_j(g) \geq u_j(g - \{i,j\})$; and
  - For all $\{i,j\} \notin g$, if $u_i(g + \{i,j\}) > u_i(g)$, then $u_j(g + \{i,j\}) < u_j(g)$.

Pairwise Stability and Efficiency in Network Formation

- Suppose that
$$b(1) < c < b(1) + (I-2)\,b(2)\,.$$
- Then, the efficient network is a **star network**.
- To see this, note that in a star network all nodes except the star have utility
$$u_i(g) = b(1) - c + (I-2)\,b(2)\,,$$
since they are connected only to the star and are distance equal to 2 from all other nodes (of which there are $I-2$).
- The utility of the star node is
$$u_i(g) = (I-1)\,[b(1) - c]\,.$$

Pairwise Stability and Efficiency in Network Formation (continued)

- Thus total utility is

$$U = (I-1)\left\{[b(1) - c + (I-2)\,b(2)] + [b(1) - c]\right\}.$$

- Given symmetry, it is sufficient to check that total utility cannot be increased by adding or subtracting one link.
- Removing one link would lead to a new network with total utility

$$U' = (I-2)\left\{[b(1) - c + (I-3)\,b(2)] + [b(1) - c]\right\}.$$

- Therefore

$$U' - U = -2\,[b(1) - c] - 2\,(I-2)\,b(2) < 0,$$

since, by assumption, $c < b(1) + (I-2)\,b(2)$.

Pairwise Stability and Efficiency in Network Formation (continued)

- Now imagine adding one more link. This would not change the distance for all but two players and thus lead to total utility

$$\begin{aligned} U'' &= (I-3)\left[b(1)-c+(I-2)\,b(2)\right]+(I-1)\left[b(1)-c\right] \\ &\quad +2\left\{2\left[b(1)-c\right]+(I-3)\,b(2)\right\}. \end{aligned}$$

- Then

$$\begin{aligned} U''-U &= 2\left\{2\left[b(1)-c\right]+(I-3)\,b(2)\right\} \\ &\quad -2\left[b(1)-c+(I-2)\,b(2)\right] \\ &= 2\left(b(1)-c\right)-2b(2)<0, \end{aligned}$$

in view of the fact that $b(1)<c$.

Pairwise Stability and Efficiency in Network Formation (continued)

- However, the star network is not pairwise stable, since the utility of the star is
$$u_i(g) = (I-1)[b(1)-c] < 0,$$
again in view of the fact that $b(1) < c$.
- This example shows how efficient network formation is difficult to achieve in general.
- Intuitively, in this case, forming a link creates a **positive externality** on others because it reduces the distance between other players.
- This externality is not internalized and thus equilibrium networks will tend to have too few links. Here the equilibrium notion, captured by the pairwise stability concept, reflects this intuition.