Q-Parameterization[1]

This lecture introduces the so-called "Q-Parameterization" (sometimes referred to as the "Youla Parameterization").

10.1 Introduction

Not every stable transfer matrix $w \to z$ can be obtained by closing a stabilizing LTI feedback loop $u = K(s)y$ in canonical feedback design setup shown on Figure 10.1. In
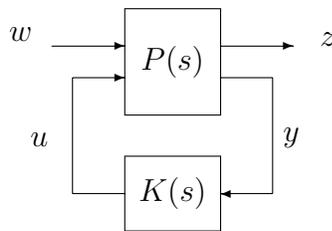
Figure 10.1: LTI feedback

this lecture, we investigate the limits of LTI feedback's ability to change the closed loop transfer matrix. More precisely, it will be shown that the set of all closed loop transfer matrices is affine. In addition, the role of unstable poles and zeros of P in limiting LTI feedback designs will be investigated.

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Q-parameterization gives a very simple *affine* description of the set of all achievable closed loop transfer matrices as function of the so-called "Q" parameter $Q = Q(s)$, which is an arbitrary stable proper transfer matrix of size $m \times k$, where m is the total number of actuator inputs, and k is the total number of sensors in the system. For design purposes, the result is extremely important: instead of thinking in terms of the controller transfer matrix $K = K(s)$, one will be much better off by designing $Q(s)$.

10.2   The Q-parameterization theorem

We consider LTI systems P from Figure 10.1 (where u and y are vectors of size m and k respectively) given in the state space format

$$P(s) := \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right). \tag{10.1}$$

An LTI feedback controller $u = -Kx$, where

$$K := \left(\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array} \right) \tag{10.2}$$

is said to *stabilize* the feedback system if all eigenvalues of the matrix

$$A_{cl} = \left[\begin{array}{cc} A + B_2 D_f C_2 & B_2 C_f \\ B_f C_2 & A_f \end{array} \right] \tag{10.3}$$

have negative real part. Let $T = T(s)$ be the transfer matrix of the resulting *closed loop system* with input w and output z:

$$T := \left(\begin{array}{cc|c} A + B_2 D_f C_2 & B_2 C_f & B_1 + B_2 D_f D_{21} \\ B_f C_2 & A_f & B_f D_{21} \\ \hline C_1 + D_{12} D_f C_2 & D_{12} C_f & D_{11} + D_{12} D_f D_{21} \end{array} \right). \tag{10.4}$$

Note that the description of T given in (10.4) is very inconvenient for design purposes, because the design parameters A_f, B_f, C_f, D_f enter the transfer matrix defined by (10.4) in a very non-linear way, and are themselves constrained nonlinearly by the condition that A_{cl} in (10.3) must be a Hurwitz matrix. The following result gives a much more useful parameterization of all $T = T(s)$.

Theorem 10.1 *Let F, L be constant gains such that matrices $A + B_2 F$ and $A + LC_2$ are Hurwitz. Then a given transfer matrix $T = T(s)$ can be made equal to the transfer matrix of the closed loop system (10.4) by an appropriate selection of a stabilizing feedback controller (10.2) if and only if*

$$T(s) = T_0(s) + T_1(s)Q(s)T_2(s) \tag{10.5}$$

for some stable proper rational transfer matrix $Q = Q(s)$, where T_0, T_1, T_2 are the transfer matrices of systems

$$T_0 = \left(\begin{array}{cc|c} A & B_2 F & B_1 \\ -LC_2 & A + B_2 F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array} \right), \tag{10.6}$$

$$T_1 = \left(\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array} \right), \tag{10.7}$$

$$T_2 = \left(\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right). \tag{10.8}$$

Moreover, $T(s)$ can be achieved by using a strictly proper stabilizing controller if and only if representation (10.5) with a stable strictly proper rational Q is possible.

Theorem 10.1 is a remarkable result, which, in particular, ensures that any *convex* feedback optimization problem formulated in terms of the closed loop transfer matrices can be solved relatively easily. In (10.5), T_0 represents *some* admissible closed loop design, and the structure of T_1, T_2 imposes limitations on the closed loop transfer matrix.

10.3 Derivation of Youla parameterization

Let F, L be defined as in Theorem 10.1, i.e. $A + B_2 F$ and $A + LC_2$ are Hurwitz matrices. For any $u(\cdot)$ let $\hat{x}$ be defined by

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C_2 \hat{x} - y).$$

Then for $e = x - \hat{x}$ we have

$$\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w.$$

Let

$$\theta = y - C_2 \hat{x} = C_2 e + D_{21} w.$$

Note that θ is the output of system T_2 with input w. Now the equations for $\hat{x}, x_f, u$ can be re-written in the form

$$
\begin{aligned}
\dot{\hat{x}} &= (A + B_2 D_f C_2)\hat{x} + B_2 C_f x_f + (B_2 D_f - L)\theta, \\
\dot{x}_f &= B_f C_2 \hat{x} + A_f x_f + B_f \theta, \\
u - F\hat{x} &= (D_f C_2 - K)\hat{x} + C_f x_f + D_f \theta.
\end{aligned}
$$

Hence, for a stabilizing controller (10.2), $u - F\hat{x}$ is the output of the *stable* system

$$
Q = \left(
\begin{array}{c|c}
A_{cl} & \begin{bmatrix} B_2 D_f - L \\ B_f \end{bmatrix} \\
\hline
\begin{bmatrix} D_f C_2 - F & C_f \end{bmatrix} & D_f
\end{array}
\right)
$$

with input θ, where A_{cl} is defined by (10.3). Since, in addition, the equations for $x, u - F\hat{x}, w, e, z$ can be written as

$$
\begin{aligned}
\dot{x} &= (A + B_2 F)x + B_2(u - F\hat{x}) + (B_1 w - B_2 Fe) \\
z &= (C_1 + D_{12}F)x + D_{12}(u - F\hat{x}) + (D_{11}w - D_{12}Fe),
\end{aligned}
$$

z equals the output of T_1 with input $u - F\hat{x}$ plus the output of T_0 with input w. This proves that the closed loop transfer matrix from w to z has the form (10.5) with some stable Q. Conversely, if Q is stable, using u which is a sum of $F\hat{x}$ and the output of Q with input θ defines a stabilizin feedback controller $u = Ky$.

Figure 10.2 gives a simple interpretation of Q-parameterization: in order to describe a general form of a stabilizing LTI controller, one has to find a special stabilizing controller first, then use a copy of the stabilized system as an estimator of the sensor output, and feed a stable LTI transformation $Q = Q(s)$ of the difference between the actual and the estimated sensor values back into the control input.

10.4 Open loop zeros

The restrictions on the closed loop transfer function imposed by the Youla parameterization can be understood in terms of the so-called *open loop zeros*.

To define zeros of MIMO systems, consider state space models

$$
\dot{x}_1 = ax_1 + bu_1, \quad y_1 = cx_1 + du_1,
$$

where the dimensions of x_1, u_1 and y_1 are n, m and k respectively. Consider the complex matrix

$$
M(s) = \begin{bmatrix} a - sI & b \\ c & d \end{bmatrix}
$$
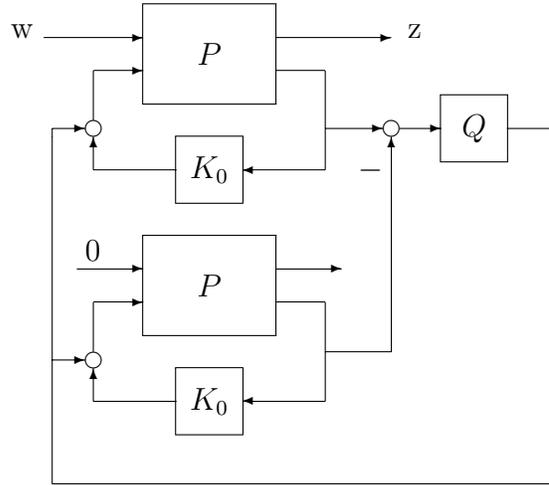
Figure 10.2: Youla parameterization

where $s \in \mathbf{C}$ is a scalar complex variable. The system is said to have a *right zero* at a point s if $\ker M(s) \neq \{0\}$, i.e. if $M(s)$ is not left invertible, or, equivalently, if there exists a non-zero pair (X_1, U_1) of complex vectors such that

$$sX_1 = aX_1 + bU_1, \quad 0 = cX_1 + dU_1.$$

Similarly, the system is said to have a *left zero* at s if the range of $M(s)$ is not the whole vector space $\mathbf{C}^{n+k}$, i.e. if $M(s)$ is not right invertible, or, equivalently, if there exists a non-zero pair (p_1, q_1) of complex vectors such that

$$sp_1 = p_1 a + q_1 c, \quad 0 = p_1 b + q_1 d.$$

As it can be seen from the formulae for T_1 and T_2, the restrictions on the closed loop transfer matrix are caused by the unstable *right zeros* of the system

$$\dot{x} = Ax + B_1 w, \quad y = C_2 x + D_{21} w, \tag{10.9}$$

and by the unstable *left zeros* of the system

$$\dot{x} = Ax + B_2 u, \quad z = C_1 x + D_{12} u. \tag{10.10}$$

It can be seen immediately that the right zeros of (10.9) cause problems by obstructing the *observation process*, while the left zeros of (10.10) describe problems which a control action will expecrience even in the case of a complete knowledge of w and x.

10.5 Time-Domain Q-Parameterization

The following statement re-states Theorem 10.1 in a time-domain format.

Theorem 10.2 *A matrix-valued function* $Z = Z(t)$ *is a closed-loop impulse response from* w *to* z *generated in the closed loop system (10.4) by an appropriate selection of a stabilizing feedback controller (10.2) if and only if there exists a function* $Q = Q(t)$, *elements of which are finite linear combinations of Dirac deltas* $\delta = \delta(t)$ *and generalized one-sided exponents* $t^k e^{st} u(t)$ *with* $\mathrm{Re}(s) < 0$, *such that*

$$Z(t) = (C + D_{12}F)X(t) + D_{12}V(t) + D_{11}\delta(t),$$

where

$$V(t) = \Psi(t)(B_1 + LD_{21}) + Q(t)D_{21},$$

$$\dot\Psi(t) = \Psi(t)(A + LC_2) + Q(t)C_2,$$

$$\dot X(t) = (A + B_2F)X(t) + B_2V(t) + B_1\delta(t),$$

and $X(\cdot), \Psi(\cdot)$ *are zero for* $t < 0$.