Final Exam

Problem 1. [25 points]   The Final Breakdown

Suppose the 6.042 final consists of:

- 36 true/false questions worth 1 point each.

- 1 induction problem worth 15 points.

- 1 giant problem that combines everything from the semester, worth 49 points.

Grading goes as follows:

- The TAs choose to grade the easy true/false questions. For each individual point, they flip a fair coin. If it comes up heads, the student gets the point.

- Marten and Brooke split the task of grading the induction problem.

    - With 1/3 probability, Marten grades the problem. His grading policy is as follows: Either he gets exasperated by the improper use of math symbols and gives 0 points (which happens with 2/5 probability), or he finds the answer satisfactory and gives 15 points (which happens with 3/5 probability).

    - With 2/3 probability, Brooke grades the problem. Her grading policy is as follows: She selects a random integer point value from the range from 0 to 15, inclusive, with uniform probability.

- Finally, Tom grades the giant problem. He rolls two fair **seven**-sided dice (which have values from 1 to 7, inclusive), takes their product, and subtracts it from 49 to determine the score. (Example: Tom rolls a 3 and a 4. The score is then $49 - 3 \cdot 4 = 37$.)

Assume all random choices during the grading process are mutually independent.

The problem parts start on the next page. Show your work to receive partial credit.

 (a) [7 pts] What is the expected score on the exam?

Solution.

$$36/2 + (1/3)15(3/5) + (2/3)15/2 + (49 - (4*4)) = 18 + 3 + 5 + 33 = 59$$

The expected score on the exam is the sum of the expected scores on the individual problems.

$$\text{Ex}\,(\text{test score}) = \text{Ex}\,(\text{mc score}) + \text{Ex}\,(\text{ind score}) + \text{Ex}\,(\text{giant score})$$

- The expected multiple choice score is just the sum of the expectations on 36 coin tosses. Since the coin is fair, the expected number of heads on each flip is $1/2$. Therefore:
$$\text{Ex}\,(\text{mc score}) = 36 \cdot 1/2 = 18$$

- The expected induction score is a weighted sum of the expectations on problems graded by Marten and Brooke. Let M be the event that Marten graded the problem, and B be the event that Brooke graded the problem. Therefore:

$$\text{Ex}\,(\text{ind score} \mid M) = 0 \cdot 2/5 + 15 \cdot 3/5 = 9$$

$$\text{Ex}\,(\text{ind score} \mid B) = \sum_{i=0}^{15}(i \cdot 1/16) = 15/2$$

$$\text{Ex}\,(\text{ind score}) = \text{Ex}\,(\text{ind score} \mid M)\Pr\{M\} + \text{Ex}\,(\text{ind score} \mid B)\Pr\{B\}$$
$$= 9 \cdot 1/3 + 15/2 \cdot 2/3 = 8$$

- The expected giant problem score is the expectation on 49 minus the product of rolls of two fair seven-sided dice. We can pull out the constant 49, and (since the dice are independent) the expectation on the product of the rolls becomes the product of the expectations on the rolls, which is 4 in this case (average of numbers 1 to 7). Therefore, if R is the random variable representing the roll of a single die:

$$\text{Ex}\,(\text{giant score}) = \text{Ex}\left(49 - R^2\right) = 49 - \text{Ex}\,(R)^2 = 49 - (4)^2 = 49 - 16 = 33$$

Therefore, the overall expectation is:

$$\text{Ex}\,(\text{test score}) = 18 + 8 + 33 = 59$$

■

(b) [5 pts] What is the variance on the 36 true/false questions?

Solution. Since the coin flips are independent, we can sum the variances of each flip.

$$\text{Var}\,[\text{mc score}] = 36 \cdot 1/2 \cdot 1/2 = 9$$

■

(c) [5 pts] What is the variance on the induction score, given that Marten graded the problem?

Solution. Using the equation $\text{Var}\,[X] = \text{Ex}\,(X^2) - \text{Ex}\,(X)^2$:

$$\text{Var}\,[\text{ind score}] = (15^2 \cdot 3/5) - (9)^2 = 135 - 81 = 54$$

∎

(d) [3 pts] Argue why the Markov bound can be used to determine an upper bound on the probability that the score on the exam is ≥ 80. You do not need to compute the actual bound.

Solution. The Markov bound can be used if there is a lower bound on the possible values of the random variable. In this case, all test scores are ≥ 0. ∎

(e) [5 pts] Use the Chebyshev bound to determine an upper bound on the probability that the score on the true/false questions is ≥ 24.

Solution. If C be the random variable representing the multiple choice score:

$$
\begin{aligned}
\Pr\{C \geq 24\} &\leq \Pr\{|\,C - 18\,| \geq 6\} \\
&= \Pr\{|\,C - \text{Ex}\,(C)\,| \geq 6\} \\
&\leq \frac{\text{Var}\,[C]}{6^2} \\
&= \frac{9}{36} = \frac{1}{4}
\end{aligned}
$$

∎

Problem 2. [25 points] Woodchucks Chucking Wood

All woodchucks can chuck wood, but only some can do it well.

- 1/3 of all woodchucks like to chuck wood.

- 2/3 of all woodchucks can chuck wood well.

- 1/2 of those that like chucking wood can do it well.

- The expected amount of wood chucked by a woodchuck (randomly chosen with uniform probability) is 7 kg/day.

- The expected amount of wood chucked by a woodchuck that likes chucking wood but can't do it well is 1 kg/day.

- A woodchuck that does not like chucking wood does not chuck any wood at all, regardless of its wood-chucking skillz or lack thereof.

(a) [10 pts] What is the probability that a woodchuck (randomly chosen with uniform probability) likes chucking wood, given that it can do it well?

Solution. Let L be the event that a woodchuck likes chucking wood and C be the event that a woodchuck can chuck wood well. We are given:

$$Pr(L) = 1/3$$
$$Pr(C) = 2/3$$
$$Pr(C|L) = 1/2$$

We wish to find $Pr(L|C)$. Using Bayes' rule, we have

$$Pr(L|C) = \frac{Pr(C|L)Pr(L)}{Pr(C)}$$
$$= \frac{(1/2)(1/3)}{(2/3)}$$
$$= 1/4$$

∎

(b) [15 pts] On average, how much wood would a woodchuck chuck if the woodchuck could chuck wood well?

Solution. Let W be a random variable representing the amount of wood the woodchuck chucks. We are given (in units of kg/day):

$$E(W) = 7$$
$$E(W|(L \cap \bar{U})) = 1$$

Using the law of total expectation, we can partition the sample space into three events: (1) the woodchuck can chuck wood well; (2) the woodchuck can't chuck wood well but likes chucking wood; (3) the woodchuck can't chuck wood well and doesn't like chucking wood.

$$E(W) = E(W|C)Pr(C) + E(W|(\bar{C} \cap L))Pr(\bar{C} \cap L) + E(W|(\bar{C} \cap \bar{L}))Pr(\bar{C} \cap \bar{L})$$
$$7 = E(W|C)(2/3) + 1 \cdot Pr(\bar{C} \cap L) + E(W|(\bar{C} \cap \bar{L}))Pr(\bar{C} \cap \bar{L})$$

Since a woodchuck that doesn't like chucking wood does not chuck any wood, whether or not it can do so, $E(W|\bar{C} \cap \bar{L}) = 0$, so the last product in the sum vanishes:

$$7 = E(W|C)(2/3) + Pr(\bar{C} \cap L)$$

Finally, from the definition of conditional expection, we have

$$Pr(\bar{C} \cap L) = P(\bar{C}|L)P(L)$$
$$= (1 - P(C|L))P(L)$$
$$= (1 - 1/2)(1/3)$$
$$= 1/6$$

We wish to find $E(W|C)$, the expected amount of wood chucked by a woodchuck that can chuck wood well:

$$7 = E(W|C)(2/3) + 1/6$$
$$41/6 = E(W|C)(2/3)$$
$$41/4 = E(W|C)$$

So $E(W|C) = 41/4 = 10.25$.

∎

Problem 3. [25 points] Cardsharing☆Revolution

Three 6.042 students—Kirari, Noelle, and Cobeni—are playing a game of Tan Tan Taan!. During each round of Tan Tan Taan!, each player is dealt 4 cards of their own, and one additional card is shared among all players, so that each player has 5 cards that they can use (the 4 cards of their own along with the single shared card). Cards are uniformly distributed from a 52-card deck. If you get four of a kind (for example, four aces or four 2's), you can continue playing in the next round. If you don't get four of a kind, you must quit and return to doing your 6.042 homework. Cards from round to round are mutually independent. This game is so fun that even if two of the three players must quit and return to their 6.042 homework, the third player will continue playing alone as long as they are able to.

(a) [5 pts] What is the probability that Kirari has four aces in the first round?

Solution. The total number of hands that Kirari can possibly get is $\binom{52}{5}$. Now we count how many ways they can make quad aces. There is only one way to get all four aces, and $52 - 4 = 48$ choices for the last card in the hand. So there are 48 hands that correspond to quad aces, and the probability of making quad aces is

$$\frac{48}{\binom{52}{5}}$$

∎

(b) [5 pts] What is the probability that Kirari doesn't get four of a kind in the first round (and must quit playing)?

Solution. The different four-of-a-card hands do not overlap at all (no one hand is both four-of-a-kind for two different numbers), so the probability of getting any four-of-a-kind hand is

$$Pr[\text{ any four-card hand }] = \sum_{i=\text{Ace}}^{\text{King}} Pr[\text{ four-of-a-kind of card } i]$$

By symmetry, all these probabilities are the same, so the probability of getting four-of-a-kind is $13 * \frac{48}{\binom{52}{5}}$. The event of not getting a four-of-a-kind is the complement of this set, and so has probability

$$1 - \frac{624}{\binom{52}{5}}$$

∎

(c) [5 pts] What is the expected number of rounds that Kirari will play?

Solution. This is a mean-time-to-failure problem. Imagine flipping a coin that has "heads" with probability

$$p = 1 - \frac{624}{\binom{52}{5}}$$

. We flip the coin until we get heads. From results in our class (or through doing the summation ourselves), we know that the result is $1/p$. ∎

(d) [10 pts] What is the probability that all three can play a second round?

Solution. There are two problems in play here. First, we have to figure out the total number of hands that can be assigned to everyone: second, we have to figure out the number of ways that everyone can get four-of-a-kind.

First, consider the problem of finding the total number of hands that can be given out. Note that we can represent each dealing of hands as an ordered list of 13 cards chosen from the 52 cards in the deck, where the first 4 cards belong to Kirari, the second 4 belong to Noelle, the third four belong to Cobeni, and the final card is the communal card. There are $52!/(52 - 13)!$ ways to do this. However, we must remember that the ordering of cards in Kirari, Noelle, and Cobeni's hands do not matter - so we must "remove" the ordering in each of those four-card hands. This is done by dividing out by 4! three times. So the total number of ways we can deal cards is:

$$52!/39!4!4!4!$$

Now, we must count the number of ways to have all three people make four-of-a-kind hands. There are two cases: when the communal card is part of a four-of-a-kind, and when it is not.

- First consider when it is not part of a four-of-a-kind. Choose first the four-of-a-kinds that could happen: there are $13 * 12 * 11$ ways to assign numbers to the three people. After these numbers have been assigned, there are $52 - 12 = 40$ choices for the communal card. So there are $13 * 12 * 11 * 40$ ways to make a four-of-a-kind this way.

- Now say the communal card *is* part of a four-of-a-kind. Say it was part of Kirari's four-of-a-kind (by symmetry, the counting is the same if it were part of the other two's four-of-a-kinds). Once again, there are 13 choices for Kirari, 12 choices for
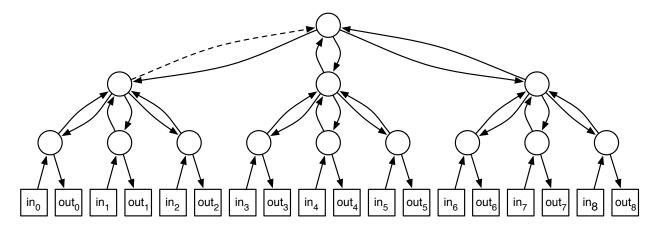
Noelle, and 11 choices for Cobeni. And once again, there are 40 choices for the extra card in Kirari's hand. So in total, there are $13 * 12 * 11 * 40 * 3$ ways for four-of-a-kind to happen in this case.

In total, there are $4(13 * 12 * 11 * 40)$ ways to make four-of-a-kind hands in this format. So the probability of everyone getting a four-of-a-kind is

$$\frac{4 * 13 * 12 * 11 * 40 * (39!4!4!4!)}{52!}$$

∎

Problem 4. [15 points]   Packet Racket!

Consider the complete ternary-tree network with 9 inputs and 9 outputs shown below where packets are routed randomly. The route each packet takes is the shortest path between input and output. Let I_0, I_1, and I_2 be indicator random variables for the events that a packet originating at in_0, in_1, and in_2, respectively, crosses the dashed edge in the figure. Let $T = I_0 + I_1 + I_2$ be a random variable for the number of packets passing through the dashed edge.

(a) [10 pts] Suppose that each input sends a single packet to an output selected uniformly at random; the packet destinations are mutually independent. (Note that outputs may receive packets from multiple inputs including their corresponding input.)

What are the expectation and variance of T?

Solution. A packet will pass through the dashed edge if it originates in inputs 0–2 and is destined for outputs 3–8. For $j \in \{0, 1, 2\}$ Let I_j be an indicator random variable for the event that a packet leaving input j passes through the dashed edge. The probability of this event is $\frac{2}{3}$. It follows that:

$$T = I_1 + I_2 + I_3$$
$$Ex[T] = Ex[I_1 + I_2 + I_3]$$
$$= Ex[I_1] + Ex[I_2] + Ex[I_3]$$
$$= 3 \cdot \frac{2}{3}$$
$$= 2$$

Similarity, but the linearity of variance for independent random variables:

$$T = I_1 + I_2 + I_3$$
$$Var[T] = Var[I_1 + I_2 + I_3]$$
$$= Var[I_1] + Var[I_2] + Var[I_3]$$
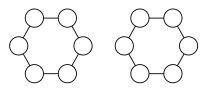$$= 3 \cdot \frac{2}{3} \cdot \left(1 - \frac{2}{3}\right)$$
$$= \frac{2}{3}$$

∎

(b) [5 pts] Now consider the situation where a permutation of inputs to outputs is chosen uniformly at random; each input sends a packet to a distinct output. What is the expected value of T? Briefly justify your answer.

Solution. Once again, $T = I_1 + I_2 + I_3$. By linearity of expectation, $E[T] = E[I_1] + E[I_2] + E[I_3]$. We know that $E[I_i] = 2/3$ still, because each input is equally likely routed to any of the outputs (even when we only restrict ourselves to permutations). Thus, $E[T] = 3 * 2/3 = 2$. ∎

Problem 5. [15 points]   Connected or Not? That Is the Question

Suppose we have a simple, undirected graph G with $2n$ vertices and $2n$ edges, where $n \geq 3$. The graph consists of two disjoint cycles with n edges each. For example, if $n = 6$, the graph would look like this:

(a) [5 pts] A pair of vertices u and v from G is selected uniformly at random from the pairs of distinct vertices with no edge between them. A new graph G' is constructed to be the same as G, except that there is an edge between u and v. What is the probability that G' is connected?

Solution. G' is connected if and only if u and v come from different cycles. There are n^2 pairs of vertices consisting of vertices in different cycles. In all, there are $\binom{2n}{2} - 2n$ pairs of vertices with no edge between them, since there are $\binom{2n}{2}$ pairs of vertices and $2n$ of these pairs have an edge between them. The desired probability p can be computed as follows:

$$
\begin{aligned}
p &= \frac{n^2}{\binom{2n}{2} - 2n} \\
&= \frac{n^2}{\frac{2n(2n-1)}{2} - 2n} \\
&= \frac{n^2}{2n^2 - n - 2n} \\
&= \frac{n}{2n - 3}
\end{aligned}
$$

∎

(b) [10 pts] k pairs of vertices from G are selected uniformly at random from the pairs of distinct vertices with no edge between them. Repetition is allowed; it is possible, for example, that the same pair appears multiple times in the set of k pairs. A new graph G'' is constructed to be the same as G, except that there are k new edges: the edges that correspond to the k selected pairs. What is the probability that G'' is **not** connected?

(*Hint: For $k = 1$, the sum of your answers to part (a) and part (b) should equal 1.*)

Solution. Note that the probability of not connecting the graph in one sampling of two nonadjacent vertices is

$$
p = 1 - \frac{n}{2n - 3}
$$

Because we are able to choose the same pair many times, we are simply taking independent samples of two nonadjacent vertices. Furthermore, in k samples, the graph is not connected if and only if none of the pairs chosen have connected the graph. The probability of this happening is

$$p^k = (1 - \frac{n}{2n-3})^k$$

∎

Problem 6. [15 points]   6.042: The Ultimate Showdown

There are 100 homework problems in 6.042 throughout the term. Let T_i, $1 \leq i \leq 100$, be the random variable indicating the fraction of a day that is needed by a student to solve the ith problem of 6.042.

The distribution for each T_i is different and unknown. We only know that the T_i are mutually independent and that for all i, $0 \leq T_i \leq 1$ and $\text{Ex}[T_i] = 0.3$.

Let T be the sum of all T_i's; T represents the total number of days needed by a student to complete all homework problems for 6.042. Prove that the probability that T is greater than $30e$ is exceedingly small by deriving the best bound you can on this probability. (*Hint: We do not consider* $1/e$ *to be exceedingly small.*)

  Solution. We know that $T = \sum_i T_i$. Thus, we know that $\text{Ex}[T] = \text{Ex}[\sum_i T_i] = \sum_i \text{Ex}[T_i] = 100(.3) = 30$

  Now, from the Chernoff bound (which we can use because the T_i are mutually independent), we have that

$$Pr\{T \geq 30e\} \leq e^{-(e-e+1)\cdot 30} = e^{-30}$$

  Which is quite a small bound.

∎

Problem 7. [25 points]   Gotta Count 'Em All!

An unusual species inhabits the forest surrounding Functional City. Each member of the species can take one of three possible forms, called *Schemander*, *Haskeleon*, and *Camlizard*.

In January of every year, each individual undergoes "evolution"—a process by which the individual splits into two individuals, whose forms depend on the form of the original:

- A Schemander splits into a Schemander and a Haskeleon.

- A Haskeleon splits into a Schemander and a Camlizard.

- A Camlizard splits into a Schemander and a Haskeleon.

We are investigating the distribution of forms within a large population of this species over time. It is known that in June of year 0, the population consisted of a single Schemander. Assume that no individual ever dies and that all individuals successfully undergo evolution exactly once every January.

(a) [3 pts] Let S_n, H_n, and C_n be the number of Schemanders, Haskeleons, and Camlizards, respectively, in June of year n. Express S_n, H_n, and C_n in terms of S_{n-1}, H_{n-1}, and C_{n-1}, for $n > 0$.

Solution. For each form, we look at what forms can be possible parents for it. A Schemander can have any of the three forms as a parent, so the number of Schemanders at time n is the number of Schemanders, Haskeleons, and Camlizards at time $n - 1$. Likewise, a Haskeleon can have either a Schemander or a Camlizard as a parent, and a Camlizard can have only a Haskeleon as a parent.

Therefore, we can the recurrence equations as

$$S_n = S_{n-1} + H_{n-1} + C_{n-1}$$
$$H_n = S_{n-1} + C_{n-1}$$
$$C_n = H_{n-1}$$

■

(b) [5 pts] Let $T_n = S_n + H_n + C_n$ be the total number of individuals in June of year n. Use induction to prove that $T_n = 2^n$ for all $n \geq 0$.

Solution. We expand each of the three terms using the recurrence:

$$
\begin{aligned}
T_n &= S_n + H_n + C_n \\
&= (S_{n-1} + H_{n-1} + C_{n-1}) + (S_{n-1} + C_{n-1}) + (H_{n-1}) \\
&= 2(S_{n-1} + H_{n-1} + C_{n-1}) \\
&= 2(T_{n-1})
\end{aligned}
$$

We can see that the population doubles every year. Since $T_0 = 1$ (just a single individual), $T_n = 2^n$. ■

(c) [2 pts] Show that $H_n = T_{n-1} - H_{n-1}$ for $n > 0$.

Solution. We use the expression for H_n from the recurrence:

$$
\begin{aligned}
H_n &= S_{n-1} + C_{n-1} \\
&= (S_{n-1} + H_{n-1} + C_{n-1}) - H_{n-1} \\
&= T_{n-1} - H_{n-1}
\end{aligned}
$$

■

(d) [15 pts] Give a closed-form expression for H_n. You may use, without proof, the fact stated in part (b) and the recurrence given in part (c).

Solution. From parts (b) and (c), we have

$$\begin{aligned} H_n &= T_{n-1} - H_{n-1} \\ &= 2^{n-1} - H_{n-1} \\ &= (1/2)2^n - H_{n-1} \\ H_n + H_{n-1} &= (1/2)2^n \end{aligned}$$

This is a linear recurrence. We first solve for the particular solution. Since the $(1/2)2^n$ term is an exponential, we try $f(n) = a2^n$. Plugging this into the recurrence gives us

$$\begin{aligned} a2^n + a2^{n-1} &= (1/2)2^n \\ a2^n + (1/2)a2^n &= (1/2)2^n \\ (3/2)a &= 1/2 \\ a &= 1/3 \end{aligned}$$

Next we solve for the homogeneous solution:

$$\begin{aligned} H_n + H_{n-1} &= 0 \\ r + 1 &= 0 \\ r &= -1 \end{aligned}$$

So our expression for H_n is of the form

$$H_n = A(-1)^n + (1/3)2^n$$

We solve for A by using the initial condition $H_0 = 0$, since there are no Haskeleons in year 0.

$$\begin{aligned} 0 &= A(-1)^0 + (1/3)2^0 \\ &= A + 1/3 \\ A &= -1/3 \end{aligned}$$

The final expression for H_n is thus

$$H_n = (-1/3)(-1)^n + (1/3)2^n$$

∎

Problem 8. [15 points]   Asymptotic Awesomeness

For each row in the following table, determine whether there exist functions f and g that satisfy all the properties marked **Yes** and do *not* satisfy the properties marked **No**. You do not have to provide examples.

Solution.   **(a)** No. $f = \Theta(g)$ implies that $f = \Omega(g)$.

(b) Yes. Example: $f(n) = n^2$; $g(n) = n$.

(c) No. $f = o(g)$ implies that $f = O(g)$.

(d) Yes. Example: $f(n) = n$; $g(n) = 2n$.

(e) Yes. Example: $f(n) = n^2$; $g(n) = n^{(-1)^n + 1}$.

(f) Yes. Example: $f(n) = n^{(-1)^{n+1}+1}$; $g(n) = n^{(-1)^n + 1}$.

■

Problem 9. [20 points] Yet Another Graph Proof

Prove that in a finite directed graph, if every node has at least one outgoing edge, then the graph has a cycle.

(*Hint: Consider the longest path.*)

Solution. Suppose that every node has at least one outgoing edge. Since the digraph is finite, there exists a longest path $v_1 \to v_2 \to \dots \to v_h$. Node v_h has an outgoing edge $v_h \to v$. If $v \notin \{v_1, v_2, \dots, v_h\}$, then $v_1 \to v_2 \to \dots \to v_h \to v$ is a longer path of length $h + 1$. Therefore, $v \in \{v_1, v_2, \dots, v_h\}$, that is, $v = v_i$ for some $1 \le i \le h$. This means that the graph has a cycle $v_i \to \dots \to v_h \to v_i$. ■

Problem 10. [20 points] Revenge of the Slipped Disc Puzzle™: The Curse of 6.042

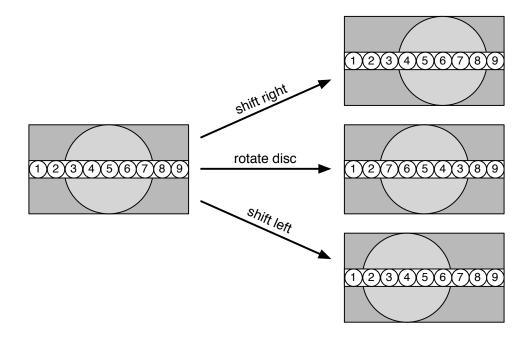(This problem is similar to the Slipped Disc Puzzle™ of Quiz 1, but here we rotate 5 tiles instead of 4.)

	$f = \Theta(g)$	$f = O(g)$	$f = o(g)$	$f = \Omega(g)$	$f = \omega(g)$	Do f, g exist?
(a)	Yes	Yes	Yes	No	No	**No**
(b)	No	No	No	Yes	Yes	**Yes**
(c)	No	No	Yes	No	No	**No**
(d)	Yes	Yes	No	Yes	No	**Yes**
(e)	No	Yes	No	No	No	**Yes**
(f)	No	No	No	No	No	**Yes**

The Super Awesome Extreme zomgroflolwut Spifftastic-to-the-Max Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* **five** tiles. As shown below, there are three ways to manipulate the puzzle:

Shift Right: The center disc is moved one unit to the right (if there is space).

Rotate Disc: The **five** tiles in the center disc are reversed.

Shift Left: The center disc is moved one unit to the left (if there is space).

Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

Write your proof on the next page...

Solution. Order the tiles from left to right in the puzzle. Define an *inversion* to be a pair of tiles that is out of their natural order (e.g. 4 appearing to the left of 3).

Lemma. *Starting from the initial state there is an odd number of inversions after any number of transitions.*

Proof. The proof is by induction. Let $P(n)$ be the proposition that starting from the initial state there is an odd number of inversions after n transitions.

Base case: After 0 transitions, there is one inversion, so $P(0)$ holds.

Inductive step: Assume $P(n)$ is true. Say we have a configuration that is reachable after $n + 1$ transitions.

1. Case 1: The last transition was a shift left or shift right

   In this case, the left-to-right order of the discs does not change and thus the number of inversions remains the same as in

2. The last transition was a rotate disc.

   In this case, six pairs of disks switch order. If there were x inversions among these pairs after n transitions, there will be $6 - x$ inversions after the reversal. If x is odd, $6 - x$ is odd, so after $n + 1$ transitions the number of inversions is odd.

   $\square$

Conclusion: Since all reachable states have an odd number of inversions and the goal state has an even number of inversions (specifically 0), the goal state cannot be reached. ∎

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