Notes for Recitation 22

1 Conditional Expectation and Total Expectation

There are conditional expectations, just as there are conditional probabilities. If R is a random variable and E is an event, then the conditional expectation $\mathrm{Ex}(R \mid E)$ is defined by:

$$\mathrm{Ex}(R \mid E) = \sum_{w \in S} R(w) \cdot \Pr(w \mid E)$$

For example, let R be the number that comes up on a roll of a fair die, and let E be the event that the number is even. Let's compute $\mathrm{Ex}(R \mid E)$, the expected value of a die roll, given that the result is even.

$$\begin{aligned}\mathrm{Ex}(R \mid E) &= \sum_{w \in \{1,\ldots,6\}} R(w) \cdot \Pr(w \mid E) \\ &= 1 \cdot 0 + 2 \cdot \frac{1}{3} + 3 \cdot 0 + 4 \cdot \frac{1}{3} + 5 \cdot 0 + 6 \cdot \frac{1}{3} \\ &= 4\end{aligned}$$

It helps to note that the conditional expectation, $\mathrm{Ex}(R \mid E)$ is simply the expectation of R with respect to the probability measure $\Pr_E()$ defined in PSet 10. So it's linear:

$$\mathrm{Ex}(R_1 + R_2 \mid E) = \mathrm{Ex}(R_1 \mid E) + \mathrm{Ex}(R_2 \mid E).$$

Conditional expectation is really useful for breaking down the calculation of an expectation into cases. The breakdown is justified by an analogue to the Total Probability Theorem:

Theorem 1 (Total Expectation). *Let $E_1, \ldots, E_n$ be events that partition the sample space and all have nonzero probabilities. If R is a random variable, then:*

$$\mathrm{Ex}(R) = \mathrm{Ex}(R \mid E_1) \cdot \Pr(E_1) + \cdots + \mathrm{Ex}(R \mid E_n) \cdot \Pr(E_n)$$

For example, let R be the number that comes up on a fair die and E be the event that result is even, as before. Then $\overline{E}$ is the event that the result is odd. So the Total Expectation theorem says:

$$\underbrace{\mathrm{Ex}(R)}_{=7/2} = \underbrace{\mathrm{Ex}(R \mid E)}_{=4} \cdot \underbrace{\Pr(E)}_{=1/2} + \underbrace{\mathrm{Ex}\left(R \mid \overline{E}\right)}_{=?} \cdot \underbrace{\Pr(E)}_{=1/2}$$

The only quantity here that we don't already know is $\text{Ex}\left(R \mid \overline{E}\right)$, which is the expected die roll, given that the result is odd. Solving this equation for this unknown, we conclude that $\text{Ex}\left(R \mid \overline{E}\right) = 3$.

To prove the Total Expectation Theorem, we begin with a Lemma.

Lemma. *Let R be a random variable, E be an event with positive probability, and I_E be the indicator variable for E. Then*

$$\text{Ex}\left(R \mid E\right) = \frac{\text{Ex}\left(R \cdot I_E\right)}{\Pr\left(E\right)} \tag{1}$$

Proof. Note that for any outcome, s, in the sample space,

$$\Pr\left(\{s\} \cap E\right) = \begin{cases} 0 & \text{if } I_E(s) = 0, \\ \Pr\left(s\right) & \text{if } I_E(s) = 1, \end{cases}$$

and so

$$\Pr\left(\{s\} \cap E\right) = I_E(s) \cdot \Pr\left(s\right). \tag{2}$$

Now,

$$\begin{aligned}
\text{Ex}\left(R \mid E\right) &= \sum_{s \in S} R(s) \cdot \Pr\left(\{s\} \mid E\right) && \text{(Def of } \text{Ex}\left(\cdot \mid E\right)\text{)} \\
&= \sum_{s \in S} R(s) \cdot \frac{\Pr\left(\{s\} \cap E\right)}{\Pr\left(E\right)} && \text{(Def of } \Pr\left(\cdot \mid E\right)\text{)} \\
&= \sum_{s \in S} R(s) \cdot \frac{I_E(s) \cdot \Pr\left(s\right)}{\Pr\left(E\right)} && \text{(by (2))} \\
&= \frac{\sum_{s \in S} (R(s) \cdot I_E(s)) \cdot \Pr\left(s\right)}{\Pr\left(E\right)} \\
&= \frac{\text{Ex}\left(R \cdot I_E\right)}{\Pr\left(E\right)} && \text{(Def of } \text{Ex}\left(R \cdot I_E\right)\text{)}
\end{aligned}$$

□

Now we prove the Total Expectation Theorem:

Proof. Since the E_i's partition the sample space,

$$R = \sum_i R \cdot I_{E_i} \tag{3}$$

for any random variable, R. So

$$\begin{aligned}
\text{Ex}(R) &= \text{Ex}\left(\sum_i R \cdot I_{E_i}\right) && \text{(by (3))} \\
&= \sum_i \text{Ex}(R \cdot I_{E_i}) && \text{(linearity of } \text{Ex}()) \\
&= \sum_i \text{Ex}(R \mid E_i) \cdot \Pr(E_i) && \text{(by (1))}
\end{aligned}$$

$\square$

Problem 1. Final exams in 6.042 are graded according to a rigorous procedure:

- With probability $\frac{4}{7}$ the exam is graded by a *recitation instructor*, with probability $\frac{2}{7}$ it is graded by a *lecturer*, and with probability $\frac{1}{7}$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.
- *Recitation instructors* score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability $\frac{3}{4}$, and no credit is given with probability $\frac{1}{4}$.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.
- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a "general impression" score.
  - With probability $\frac{4}{10}$, the general impression score is 40.
  - With probability $\frac{3}{10}$, the general impression score is 50.
  - With probability $\frac{3}{10}$, the general impression score is 60.

Assume all random choices during the grading process are mutually independent.

(a) What is the expected score on an exam graded by a recitation instructor?

Solution. Let X equal the exam score and C be the event that the exam is graded by a recitation instructor. We want to calculate $\text{Ex}(X \mid C)$. By linearity of (conditional) expectation, the expected sum of the problem scores is the sum of the expected problem scores. Therefore, we have:

$$\begin{aligned}
\text{Ex}(X \mid C) &= 10 \cdot \text{Ex}(\text{T/F score} \mid C) + 4 \cdot \text{Ex}(\text{15pt prob score} \mid C) + \text{Ex}(\text{20pt prob score} \mid C) \\
&= 10 \cdot \left(\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0\right) + 4 \cdot \left(2 \cdot \frac{7}{2} + 3\right) + \left(\frac{1}{2} \cdot 12 + \frac{1}{2} \cdot 18\right) \\
&= 10 \cdot \frac{3}{2} + 4 \cdot 10 + 15 = 70
\end{aligned}$$

(b) What is the expected score on an exam graded by a lecturer?

Solution. Now we want $\text{Ex}\left(X \mid \bar{C}\right)$, the expected score a lecturer would give. Employing linearity again, we have:

$$\begin{aligned}
\text{Ex}\left(X \mid \bar{C}\right) &= \text{Ex}\left(\text{product of dice} \mid \bar{C}\right) \\
&\quad + \text{Ex}\left(\text{general impression} \mid \bar{C}\right) \\
&= \left(\frac{7}{2}\right)^2 && \text{(because the dice are independent)} \\
&\quad + \left(\frac{4}{10} \cdot 40 + \frac{3}{10} \cdot 50 + \frac{3}{10} \cdot 60\right) \\
&= \frac{49}{4} + 49 = 61\frac{1}{4}
\end{aligned}$$

(c) What is the expected score on a 6.042 exam?

Solution. Let X equal the true exam score. The Total Expectation Theorem implies:

$$\begin{aligned}
\text{Ex}(X) &= \text{Ex}(X \mid C)\Pr(C) + \text{Ex}\left(X \mid \bar{C}\right)\Pr\left(\bar{C}\right) \\
&= 70 \cdot \frac{4}{7} + \left(\frac{49}{4} + 49\right) \cdot \frac{2}{7} + 84 \cdot \frac{1}{7} \\
&= 40 + \left(\frac{7}{2} + 14\right) + 12 = 69\frac{1}{2}
\end{aligned}$$

Problem 2. Here's yet another fun 6.042 game! You pick a number between 1 and 6. Then you roll three fair, independent dice.

- If your number never comes up, then you lose a dollar.
- If your number comes up once, then you win a dollar.
- If your number comes up twice, then you win two dollars.
- If your number comes up three times, you win *four* dollars!

What is your expected payoff? Is playing this game likely to be profitable for you or not?

Solution. Let the random variable R be the amount of money won or lost by the player in a round. We can compute the expected value of R as follows:

$$\begin{aligned}
\operatorname{Ex}(R) &= -1 \cdot \Pr(\text{0 matches}) + 1 \cdot \Pr(\text{1 match}) + 2 \cdot \Pr(\text{2 matches}) + 4 \cdot \Pr(\text{3 matches}) \\
&= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 + 2 \cdot 3 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) + 4 \cdot \left(\frac{1}{6}\right)^3 \\
&= \frac{-125 + 75 + 30 + 4}{216} \\
&= \frac{-16}{216}
\end{aligned}$$

You can expect to lose $16/216$ of a dollar (about 7.4 cents) in every round. This is a horrible game!

Problem 3. The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.
- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

(a) What is the expected sum of two dice, given that the same number comes up on both?

Solution. There are six equally-probable sums: 2, 4, 6, 8, 10, and 12. Therefore, the expected sum is:

$$\frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 4 + \ldots + \frac{1}{6} \cdot 12 = 7$$

(b) What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

Solution. Let the random variables D_1 and D_2 be the numbers that come up on the two dice. Let E be the event that they are equal. The Total Expectation Theorem says:

$$\text{Ex}\left(D_1 + D_2\right) = \text{Ex}\left(D_1 + D_2 \mid E\right) \cdot \Pr\left(E\right) + \text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) \cdot \Pr\left(\overline{E}\right)$$

Two dice are equal with probability $\Pr\left(E\right) = 1/6$, the expected sum of two independent dice is 7, and we just showed that $\text{Ex}\left(D_1 + D_2 \mid E\right) = 7$. Substituting in these quantities and solving the equation, we find:

$$\begin{aligned} 7 &= 7 \cdot \frac{1}{6} + \text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) \cdot \frac{5}{6} \\ \text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) &= 7 \end{aligned}$$

(c) To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable X_i be the sum of the dice on the i-th roll, and let E_i be the event that the i-th roll is doubles. Write the expected number of squares a piece advances in these terms.

Solution. From the total expectation formula, we get:

$$\begin{aligned}
\text{Ex}\left(\text{advance}\right) = {} & \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left(\overline{E_1}\right) \\
& + \text{Ex}\left(X_1 + X_2 \mid E_1 \cap \overline{E_2}\right) \cdot \Pr\left(E_1 \cap \overline{E_2}\right) \\
& + \text{Ex}\left(X_1 + X_2 + X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right) \cdot \Pr\left(E_1 \cap E_2 \cap \overline{E_3}\right) \\
& + \text{Ex}\left(0 \mid E_1 \cap E_2 \cap E_3\right) \cdot \Pr\left(E_1 \cap E_2 \cap E_3\right)
\end{aligned}$$

Then using linearity of (conditional) expectation, we refine this to

$$\begin{aligned}
& \text{Ex}\left(\text{advance}\right) \\
& = \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left(\overline{E_1}\right) \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1 \cap \overline{E_2}\right) + \text{Ex}\left(X_2 \mid E_1 \cap \overline{E_2}\right)\right) \cdot \Pr\left(E_1 \cap \overline{E_2}\right) \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1 \cap E_2 \cap \overline{E_3}\right) + \text{Ex}\left(X_2 \mid E_1 \cap E_2 \cap \overline{E_3}\right) + \text{Ex}\left(X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right)\right) \\
& \qquad \cdot \Pr\left(E_1 \cap E_2 \cap \overline{E_3}\right) \\
& \quad + 0.
\end{aligned}$$

Using mutual independence of the rolls, we simplify this to

$$\begin{aligned}
& \text{Ex}\left(\text{advance}\right) \\
& = \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left(\overline{E_1}\right) \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1\right) + \text{Ex}\left(X_2 \mid \overline{E_2}\right)\right) \cdot \Pr\left(E_1\right) \cdot \Pr\left(\overline{E_2}\right) \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1\right) + \text{Ex}\left(X_2 \mid E_2\right) + \text{Ex}\left(X_3 \mid \overline{E_3}\right)\right) \cdot \Pr\left(E_1\right) \cdot \Pr\left(E_2\right) \cdot \Pr\left(\overline{E_3}\right)
\end{aligned} \tag{4}$$

(d) What is the expected number of squares that a piece advances in Monopoly?

Solution. We plug the values from parts (a) and (b) into equation (4):

$$\begin{aligned}
\text{Ex}\left(\text{advance}\right) &= 7 \cdot \frac{5}{6} + (7+7) \cdot \frac{1}{6} \cdot \frac{5}{6} + (7+7+7) \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \\
&= 8\frac{19}{72}
\end{aligned}$$