Final Exam

YOUR NAME: ____________________________________

- **This is an open-notes exam. However, calculators are not allowed.**
- You may assume all results from lecture, the notes, problem sets, and recitation.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- GOOD LUCK!

Problem	Points	Grade	Grader
1	15		
2	10		
3	10		
4	10		
5	10		
6	15		
7	10		
8	10		
9	10		
Total	100		

Problem 1. [15 points] Consider the following sequence of predicates:

$$\begin{aligned}
Q_1(x_1) &= x_1 \\
Q_2(x_1, x_2) &= x_1 \Rightarrow x_2 \\
Q_3(x_1, x_2, x_3) &= (x_1 \Rightarrow x_2) \Rightarrow x_3 \\
Q_4(x_1, x_2, x_3, x_4) &= ((x_1 \Rightarrow x_2) \Rightarrow x_3) \Rightarrow x_4 \\
Q_5(x_1, x_2, x_3, x_4, x_5) &= (((x_1 \Rightarrow x_2) \Rightarrow x_3) \Rightarrow x_4) \Rightarrow x_5 \\
\cdots \quad & \quad\quad \cdots
\end{aligned}$$

Let T_n be the number of different true/false settings of the variables $x_1, x_2, \ldots, x_n$ for which $Q_n(x_1, x_2, \ldots, x_n)$ is true. For example, $T_2 = 3$ since $Q_2(x_1, x_2)$ is true for 3 different settings of the variables x_1 and x_2:

x_1	x_2	$Q_2(x_1, x_2)$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Express T_{n+1} in terms of T_n, assuming $n \geq 1$.

Solution. We have:

$$Q_{n+1}(x_1, x_2, \ldots, x_{n+1}) = Q_n(x_1, x_2, \ldots, x_n) \Rightarrow x_{n+1}$$

If x_{n+1} is true, then Q_{n+1} is true for all 2^n settings of the variables $x_1, x_2, \ldots, x_n$. If x_{n+1} is false, then Q_{n+1} is true for all settings of $x_1, x_2, \ldots, x_n$ *except* for the T_n settings that make Q_n true. Thus, altogether we have:

$$T_{n+1} = 2^n + 2^n - T_n = 2^{n+1} - T_n$$

(b) Use induction to prove that $T_n = \frac{1}{3}(2^{n+1} + (-1)^n)$ for $n \geq 1$. You may assume your answer to the previous part without proof.

Solution. The proof is by induction. Let $P(n)$ be the proposition that $T_n = (2^{n+1} + (-1)^n)/3$.

Base case: There is a single setting of x_1 that makes $Q_1(x_1) = x_1$ true, and $T_1 = (2^{1+1} + (-1)^1)/3 = 1$. Therefore, $P(0)$ is true.

Inductive step: For $n \geq 0$, we assume $P(n)$ and reason as follows:

$$\begin{aligned}
T_{n+1} &= 2^{n+1} - T_n \\
&= 2^{n+1} - \left(\frac{2^{n+1} + (-1)^n}{3}\right) \\
&= \frac{2^{n+2} + (-1)^{n+1}}{3}
\end{aligned}$$

The first step uses the result from the previous problem part, the second uses the induction hypothesis $P(n)$, and the third is simplification. This implies that $P(n+1)$ is true. By the principle of induction, $P(n)$ is true for all $n \geq 1$.

Problem 2. [10 points] There is no clock in my kitchen. However:

- The faucet drips every 54 seconds after I shut off the water.
- The toaster pops out toast every 87 seconds after I plug it in.

I'd like to fry an egg for exactly 141 seconds. My plan is to plug in the toaster and shut off the faucet at the same instant. I'll start frying when the faucet drips for the D-th time and stop frying when the toaster pops for the P-th time. What values of D and P make this plan work?

$D = \boxed{}$ $P = \boxed{}$

Reminder: Calculators are not allowed.

Solution. The Pulverizer gives $5 \cdot 87 - 8 \cdot 54 = 3$. Multiplying by 47 gives:

$$\begin{aligned} 235 \cdot 87 - 376 \cdot 54 &= 141 \\ \Rightarrow \quad 235 \cdot 87 &= 141 + 376 \cdot 54 \end{aligned}$$

Thus, I'll start frying after at drip $D = 376$ and stop 141 seconds later at pop $P = 87$.

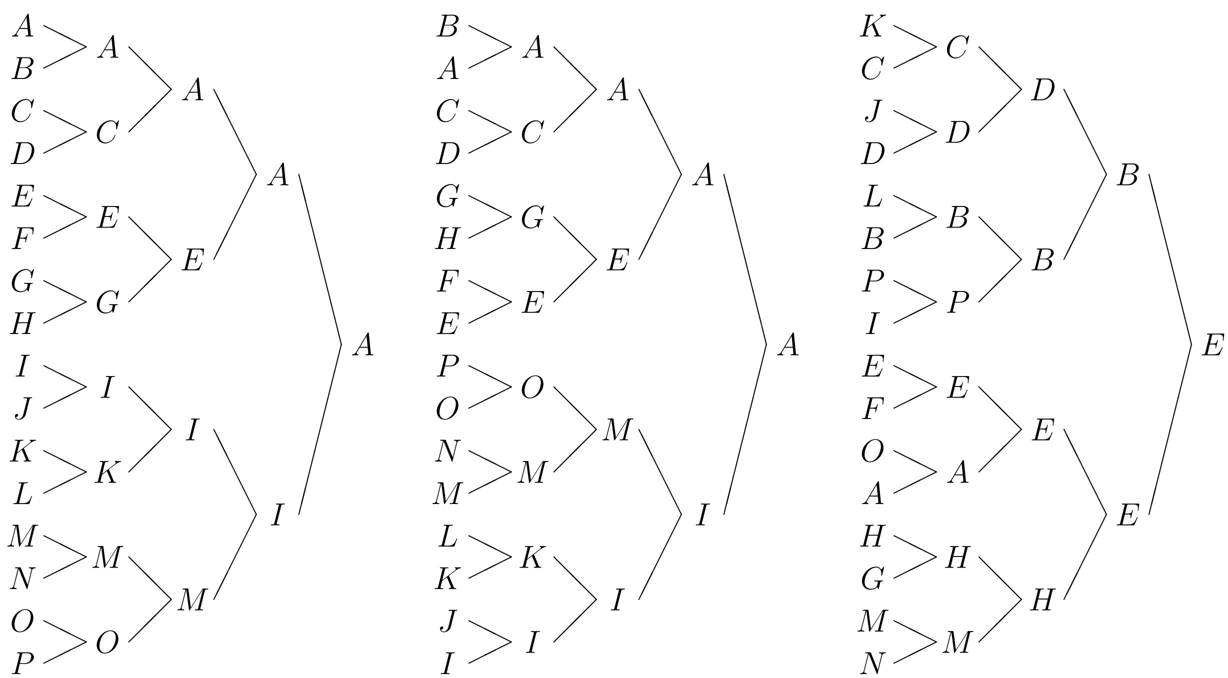
Problem 3. [10 points] Circle either **true** or **false** next to each statement below. Assume graphs are undirected without self-loops or multi-edges.

1. For all $n \geq 3$, the complete graph on n vertices has an Euler tour. **false**
2. If a graph contains no odd-length cycle, then it is bipartite. **true**
3. Every non-bipartite graph contains a 3-cycle as a subgraph. **false**
4. Every graph with a Hamiltonian cycle also has an Euler tour. **false**
5. There exists a graph with 20 vertices, 10 edges, and 5 connected components. **false**
6. Every connected graph has a tree as a subgraph. **true**
7. In every planar embedding of a connected planar graph, the number of vertices plus the number of faces is greater than the number of edges. **true**
8. If every girl likes at least 2 boys, then every girl can be matched with a boy she likes. **false**
9. If every vertex in a graph has degree 3, then the graph is 4-colorable. **true**
10. There exists a six-vertex graph with vertex degrees 0, 1, 2, 3, 4, and 5. **false**

Problem 4. [10 points] In the final round of the World Cup, 16 teams play a *single-elimination tournament*.

- The teams are called A, B, C, $\ldots$, P.
- The tournament consists of a sequence of rounds.
  - In each round, the teams are paired up and play matches.
  - The losers are eliminated, and the winners advance to the next round.
  - The tournament ends when there is only one team left.

For example, three possible single-elimination tournaments are depicted below:



Two tournaments are *the same* if the same matches are played and won by the same teams. For example, the first and second tournaments shown above are the same, but the third is different. How many *different* tournaments are possible?

Solution. Suppose that we draw the tournament so that the winning team in each game is listed *above* the losing team. Then the ordering of teams on the left completely determines all matches and winners. Therefore, there are $16!$ single-elimination tournaments.

Another approach is to use a result from earlier in the course: the number of ways to pair up $2n$ people is $(2n)!/n!\,2^n$. In a single-elimination tournament, we must pair up 16 teams, determine who wins the 8 matches between them, then pair up the 8 winning teams, detrmine who wins the 4 matches, and so forth. The number of ways in which this can be done is:

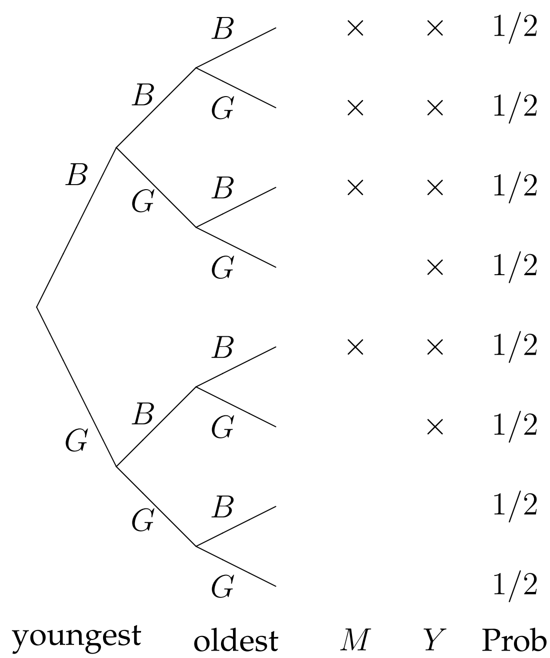
$$\frac{16!}{8!\,2^8} \cdot 2^8 \cdot \frac{8!}{4!\,2^4} \cdot 2^4 \cdot \frac{4!}{2!\,2^2} \cdot 2^2 \cdot \frac{2!}{1!\,2^1} \cdot 2^1 = 16!$$

A final alternative is to use the General Product Rule. The champions can be chosen in 16 ways, the other finalists in 15 ways, the semi-finalist that played the champions in 14 ways, the other semi-finalist in 13 ways, and so forth. In all, this gives $16!$ tournaments again.

Problem 5. [10 points] There are 3 children of different ages. What is the probability that at least two are boys, given that at least one of the two youngest children is a boy?

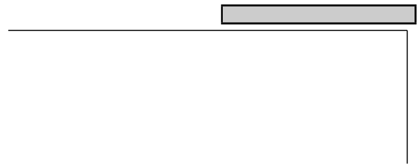
Assume that each child is equally likely to be a boy or a girl and that their genders are mutually independent. A correct answer alone is sufficient. However, to be eligible for partial credit, you must include a clearly-labeled tree diagram.

Solution. Let M be the event that there are at least two boys, and let Y be the event that at least one of the two youngest children is a boy. In the tree diagram below, all edge probabilities are $1/2$.



$$
\begin{aligned}
\Pr(M \mid Y) &= \frac{\Pr(M \cap Y)}{\Pr(Y)} \\
&= \frac{1/2}{3/4} \\
&= 2/3
\end{aligned}
$$

Problem 6. [15 points] On the morning of day 1, I put a gray document on my desk. This creates a stack of height 1:



On each subsequent morning, I insert a white document into the stack at a position selected uniformly at random. For example, the stack might look like this on the evening of day 5:

Then, on the following morning, I would insert a white document at one of the six positions indicated above with equal probability.

Let the random variable B_n be the number of white documents below the gray document on day n.

(a) Express $\Pr(B_{n+1} = 0)$ in terms of $\Pr(B_n = 0)$.

$$\Pr(B_{n+1} = 0) = \boxed{}$$

Solution.

$$\Pr(B_{n+1} = 0) = \frac{n}{n+1} \Pr(B_n = 0)$$

(b) Express $\Pr(B_{n+1} = n)$ in terms of $\Pr(B_n = n - 1)$.

$$\Pr(B_{n+1} = n) = \boxed{}$$

Solution.

$$\Pr(B_{n+1} = n) = \frac{n}{n+1} \Pr(B_n = n - 1)$$

(c) Express $\Pr(B_{n+1} = k)$ in terms of $\Pr(B_n = k)$ and $\Pr(B_n = k-1)$ assuming that $0 < k < n$.

$$\Pr(B_{n+1} = k) = \boxed{}$$

Solution.

$$\Pr(B_{n+1} = k) = \frac{n-k}{n+1} \Pr(B_{n+1} = k) + \frac{k}{n+1} \Pr(B_{n+1} = k-1)$$

(d) Use induction to prove that B_n is uniformly distributed on $\{0, 1, 2, \ldots, n-1\}$. You may assume your answers to the preceding problem parts without justification.

Solution. We use induction. Let $P(n)$ be the proposition that B_n is uniformly distributed on the set $\{0, 1, 2, \ldots, n-1\}$.

Base case. The random variable B_1 is always equal to 0, so it is uniformly distributed on $\{0\}$.

Inductive step. Assume that the random variable B_n is uniformly distributed on the set $\{0, 1, 2, \ldots, n-1\}$ and consider the random variable B_{n+1}. There are three cases:

$$\begin{aligned}
\Pr(B_{n+1} = 0) &= \frac{n}{n+1} \Pr(B_n = 0) \\
&= \frac{n}{n+1}\frac{1}{n} && (*) \\
&= \frac{1}{n+1} \\
\Pr(B_{n+1} = n) &= \frac{n}{n+1} \Pr(B_n = n-1) \\
&= \frac{n}{n+1}\frac{1}{n} && (*) \\
&= \frac{1}{n+1} \\
\Pr(B_{n+1} = k) &= \frac{n-k}{n+1} \Pr(B_{n+1} = k) + \frac{k}{n+1} \Pr(B_{n+1} = k-1) \\
&= \frac{n-k}{n+1}\frac{1}{n} + \frac{k}{n+1}\frac{1}{n} && (*) \\
&= \frac{1}{n+1}
\end{aligned}$$

In each case, the first equation comes from the preceding problem parts. We use the induction hypotheses on the starred lines. The remaining steps are simplfications. This shows that B_{n+1} is uniformly distributed, and the claim follows from the principle of induction.

Problem 7. [10 points] Bubba and Stu are shooting at a road sign. They take shots in this order:

Bubba, Stu, Stu, Bubba, Bubba, Stu, Stu, Bubba, Bubba, Stu, Stu, etc.

With each shot:

- Bubba hits the sign with probability $2/5$.
- Stu hits the sign with probability $1/4$.

What is the probability that Bubba hits the sign before Stu? Assume that hits occur mutually independently. You must give a *closed-form* answer to receive full credit.

Solution.

$$\begin{aligned}
\Pr\left(\text{Bubba hits first}\right) &= \frac{2}{5} + \\
&\frac{3}{5}\left(\frac{3}{4}\right)^2\frac{2}{5} + \frac{3}{5}\left(\frac{3}{4}\right)^2\frac{3}{5}\frac{2}{5} + \\
&\frac{3}{5}\left(\frac{3}{4}\right)^2\left(\frac{3}{5}\right)^2\left(\frac{3}{4}\right)^2\frac{2}{5} + \frac{3}{5}\left(\frac{3}{4}\right)^2\left(\frac{3}{5}\right)^2\left(\frac{3}{4}\right)^2\frac{3}{5}\frac{2}{5} + \\
&\cdots \\
&= \frac{2}{5} + \frac{3}{5}\frac{2}{5}\sum_{i=1}^{\infty}\Big[\left(\frac{3}{4}\right)^{2i}\left(\frac{3}{5}\right)^{2i-2} + \left(\frac{3}{4}\right)^{2i}\left(\frac{3}{5}\right)^{2i-1}\Big] \\
&= \frac{2}{5} + \frac{6}{25}\sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^{2i}\left(\frac{3}{5}\right)^{2i-2}\left(1+\frac{3}{5}\right) \\
&= \frac{2}{5} + \frac{6}{25}\frac{8}{5}\sum_{i=1}^{\infty}\left(\frac{9}{16}\right)^{i}\left(\frac{9}{25}\right)^{i-1} \\
&= \frac{2}{5} + \frac{6}{25}\frac{8}{5}\frac{25}{9}\sum_{i=1}^{\infty}\left(\frac{9}{16}\frac{9}{25}\right)^{i} \\
&= \frac{2}{5} + \frac{16}{15}\sum_{i=1}^{\infty}\left(\frac{81}{400}\right)^{i} \\
&= \frac{2}{5} + \frac{16}{15}\left(\frac{1}{1-81/400} - 1\right) \\
&= \frac{2}{5} + \frac{16}{15}\frac{81}{319} \\
&= \frac{214}{319}
\end{aligned}$$

Problem 8. [10 points] There are three types of men (A, B, and C), and three types of women (D, E, and F). Some couples are *compatible* and others are not, as indicated below:

	A	B	C
D	no	yes	yes
E	no	no	yes
F	yes	no	no

Men and women with the personality types shown below attend a dance.

Men:	A	A	B	B	B	C	C	C	C
Women:	D	D	D	E	F	F	F	F	F

Suppose a pairing of the women and men is selected uniformly at random.

(a) What is the probability that a particular man of type A is paired with a compatible woman?

Solution. $5/9$

(b) What is the expected number of compatible couples?

Solution. Let I_k be an indicator for the event that the k-th man is paired with a compatible woman. Then the total number of compatible couples is:

$$\begin{aligned}
\text{Ex}(I_1 + \ldots + I_9) &= \text{Ex}(I_1) + \ldots + \text{Ex}(I_9) \\
&= \frac{5}{9} + \frac{5}{9} + \frac{3}{9} + \frac{3}{9} + \frac{3}{9} + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} \\
&= \frac{35}{9}
\end{aligned}$$

Problem 9. [10 points] Every Skywalker serves either the *light side* or the *dark side*.

- The first Skywalker serves the dark side.
- For $n \geq 2$, the n-th Skywalker serves the same side as the $(n-1)$-st Skywalker with probability $1/4$, and the opposite side with probability $3/4$.

Let d_n be the probability that the n-th Skywalker serves the dark side.

(a) Express d_n with a recurrence equation and sufficient base cases.

Solution.

$$\begin{aligned} d_1 &= 1 \\ d_{n+1} &= \frac{1}{4} \cdot d_n + \frac{3}{4} \cdot (1 - d_n) \\ &= \frac{3}{4} - \frac{1}{2} d_n \end{aligned}$$

(b) Give a closed-form expression for d_n.

Solution. The characteristic equation is $x - 1/2 = 0$. The only root is $x = -1/2$. Therefore, the homogenous solution has the form $d_n = A \cdot (-1/2)^n$. For a particular solution, we first guess $d_n = c$. This is indeed a solution for $c = 1/2$. Therefore, the complete solution has the form $d_n = 1/2 + A \cdot (-1/2)^n$. Since $d_1 = 1$, we must have $A = -1/2$. Therefore:

$$d_n = \frac{1}{2} + \left(-\frac{1}{2}\right)^{n+1}$$