Problem Set 1 Solutions

Due: Monday, February 7 at 9 PM

Problem 1. The connectives $\wedge$ (and), $\vee$ (or), and $\Rightarrow$ (implies) come often not only in computer programs, but also everyday speech. But devices that compute the *nand* operation are preferable in computer chip designs. Here is the truth table for nand:

P	Q	P nand Q
T	T	F
T	F	T
F	T	T
F	F	T

For each of the following expressions, find an equivalent expression using only nand and $\neg$ (not).

(a) $A \wedge B$

Solution. $\neg(A$ nand $B)$

(b) $A \vee B$

Solution. $(\neg A)$ nand $(\neg B)$

(c) $A \Rightarrow B$

Solution. A nand $(\neg B)$

Problem 2. A self-proclaimed "great logician" has invented a new quantifier, on par with $\exists$ ("there exists") and $\forall$ ("for all"). The new quantifier is symbolized by U and read "there exists a unique". The proposition $Ux\ P(x)$ is true iff there is *exactly one* x for which $P(x)$ is true. The logician has noted, "There used to be two quantifiers, but now there are three! I have extended the whole field of mathematics by 50%!"

(a) Write a proposition equivalent to $Ux\ P(x)$ using only the $\exists$ quantifier, =, and logical connectives.

Solution.

$$\exists x\ (P(x) \wedge \neg(\exists y\ (\neg(x = y) \wedge P(y)))$$

(b) Write a proposition equivalent to $Ux\ P(x)$ using only the $\forall$ quantifier, =, and logical connectives.

Solution.

$$\neg\forall x\ (\neg P(x) \vee \neg\forall y\ (x = y \vee \neg P(y)))$$

Problem 3. A media tycoon has an idea for an all-news television network called LNN: The Logic News Network. Each segment will begin with the definition of some relevant sets and predicates. The day's happenings can then be communicated concisely in logic notation. For example, a broadcast might begin as follows:

> "THIS IS LNN. Let S be the set $\{\text{Bill}, \text{Monica}, \text{Ken}, \text{Linda}, \text{Betty}\}$. Let $D(x)$ be a predicate that is true if x is deceitful. Let $L(x, y)$ be a predicate that is true if x likes y. Let $G(x, y)$ be a predicate that is true if x gave gifts to y."

Complete the broadcast by translating the following statements into logic notation.

(a) If neither Monica nor Linda is deceitful, then Bill and Monica like each other.

Solution.

$$(\neg(D(\text{Monica}) \vee D(\text{Linda}))) \Rightarrow (L(\text{Bill}, \text{Monica}) \wedge L(\text{Monica}, \text{Bill}))$$

(b) Everyone except for Ken likes Betty, and no one except Linda likes Ken.

Solution.

$$\begin{gathered}\forall x \in S\ (x = \text{Ken} \wedge \neg L(x, \text{Betty})) \vee (x \neq \text{Ken} \wedge L(x, \text{Betty})) \wedge \\ \forall x \in S\ (x = \text{Linda} \wedge L(x, \text{Ken})) \vee (x \neq \text{Linda} \wedge \neg L(x, \text{Ken}))\end{gathered}$$

(c) If Ken is not deceitful, then Bill gave gifts to Monica, and Monica gave gifts to someone.

Solution.

$$\neg D(\text{Ken}) \Rightarrow (G(\text{Bill}, \text{Monica}) \wedge \exists x \in S\ G(\text{Monica}, x))$$

(d) Everyone likes someone and dislikes someone else.

Solution.

$$\forall x \in S\ \exists y \in S\ \exists z \in S\ (y \neq z) \wedge L(x, y) \wedge \neg L(x, z)$$

The remaining problems will be graded primarily on the clarity of your proofs— not on whether you have the right idea. In fact, if you can't figure out the right idea, we'll give it to you– just ask your TA!

Problem 4. Let n be a postive integer. Prove that $\log_2 n$ is rational if and only if n is a power of 2. Assume any basic facts about divisibility that you need; just state your assumptions explicitly.

Solution. **Assumption:** If n^b is a power of 2, then n is a power of 2.

Proof. We prove that if n is a power of 2, then $\log_2 n$ is rational and vice-versa.

First, we prove that if n is a power of 2, then $\log_2 n$ is rational. Assume that n is a power of 2. Then $n = 2^k$ for some integer $k \geq 0$. Thus, $\log_2 n = \log_2 2^k = k$, which is a rational number.

Next, we prove that if $\log_2 n$ is rational, then n is a power of 2. Assume that $\log_2 n$ is rational. That means there exist integers a and b such that:

$$\log_2 n = \frac{a}{b}$$

We can rewrite this equation as follows:

$$\begin{aligned} n &= 2^{a/b} && \text{(Take 2 to power of each side.)} \\ n^b &= 2^a && \text{(Take the } b\text{-th power of each side.)} \end{aligned}$$

Thus, n^b is a power of 2. By our assumption, n is a power of 2. □

Problem 5. A *triangle* is a set of three people such that either every pair has shaken hands or no pair has shaken hands. Prove that among every six people there is a triangle. Suggestion: Initially, break the problem into two cases:

1. There exist at least three people who shook hands with person X.
2. There exist at least three people didn't shake hands with X

(Why must at least one of these conditions hold?)

Solution.

Proof. We use case analysis. Let X denote one of the six people. There are two possibilities:

1. There exist three people who shook hands with person X. Now there are two further possibilities:
   (a) Among these three, some pair shook hands. Then these two and X form a triangle.
   (b) Among these three, no pair shook hands. Then these three form a triangle.

2. Otherwise, at most two people shook hands with person X. Thus, there exist three people who didn't shake hands with X. Again, there are two further possibilities:
   (a) Among these three, every pair shook hands. Then these three form a triangle.
   (b) Among these three, some pair didn't shake hands. Then these two and X for a triangle.

□

Problem 6. Let x and y be nonnegative real numbers. The *arithmetic mean* of x and y is defined to be $(x + y)/2$, and the *geometric mean* is defined to be $\sqrt{xy}$. Prove that the arithmetic mean is equal to the geometric mean if and only if $x = y$.

Solution.

Proof. We construct a chain of if-and-only-if implications. The arithmetic mean is equal to the geometric mean if and only if:

$$
\begin{aligned}
\frac{x+y}{2} = \sqrt{xy} \quad &\Leftrightarrow \quad x + y = 2\sqrt{xy} \\
&\Leftrightarrow \quad (x + y)^2 = 4xy \\
&\Leftrightarrow \quad x^2 + 2xy + y^2 = 4xy \\
&\Leftrightarrow \quad x^2 - 2xy + y^2 = 0 \\
&\Leftrightarrow \quad (x - y)^2 = 0 \\
&\Leftrightarrow \quad x - y = 0 \\
&\Leftrightarrow \quad x = y
\end{aligned}
$$

□

Problem 7. Use case analysis to prove that all integral solutions to the equation

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}$$

subject to these constraints

$$m \geq 3 \qquad n \geq 3 \qquad e > 0$$

are in this table:

m	n	e
3	3	6
3	4	12
3	5	30
4	3	12
5	3	30

These equations reveal something fundamental about the geometry of our three-dimensional world; we'll revisit them in about three weeks.

Solution.

Proof. We use case analysis. Since $m \geq 3$, one of four cases must hold:

1. $m = 3$. There are now four subcases:

   (a) $n = 3$. Rewriting the equation in the form

   $$e = \frac{1}{\frac{1}{m} + \frac{1}{n} - \frac{1}{2}}$$

   and subtituting in $m = n = 3$ implies that $e = 6$, which is the first solution.

   (b) $n = 4$. Substituting the values of m and n into the equation show that $e = 12$, which is the second solution.

   (c) $n = 5$. Substituting into the equation shows that $e = 30$, which is the third solution.

   (d) $n \geq 6$. This implies:

   $$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

   Thus, the left side of the equation is strictly less than the right for all $e > 0$, so there are no solutions in this case.

2. $m = 4$. There are two subcases:

   (a) $n = 3$. Subsituting gives $e = 12$, which is the fourth solution.

   (b) $n \geq 4$. This implies:

   $$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

   Again, the left side of the equation is strictly less than the right for all $e > 0$, so there are no solutions in this case.

3. $m = 5$. There are two subcases:

   (a) $n = 3$. Subsituting gives $e = 30$, which is the fifth solution.

   (b) $n \geq 4$. This implies:

   $$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{5} + \frac{1}{4} < \frac{1}{2}$$

   Again, the equation can not hold, so there are no solutions in this case.

4. $m \geq 6$. This implies:

   $$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

   Once more, the equation can not hold, so there are no solutions in this case.

$\square$