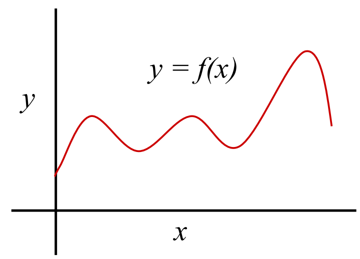
Mathematics for Computer Science

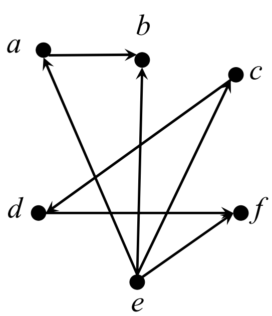
MIT 6.042J/18.062J

DAGs, Partial Orders, Scheduling

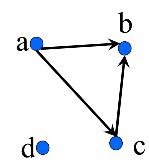
Normal Person's Graph



Computer Scientist's Graph



Relations and Graphs



set of *vertices* V

set of *edges* E, $E \subseteq V \times V$

(*Formally* the same as

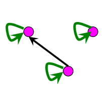
a binary relation on V.)

V={a,b,c,d}

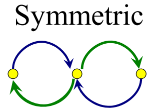
E = {(a,b), (a,c), (c,b)}

Graphical Properties of Relations

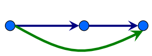
Reflexive



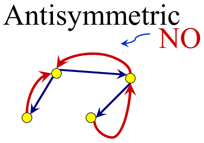
Symmetric



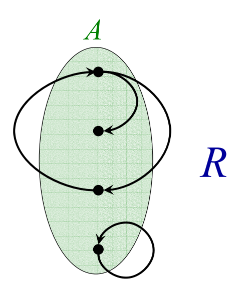
Transitive



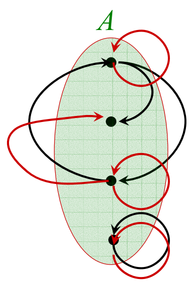
Antisymmetric



Paths



R^2 from A to A



R = paths of length 1

R^2 = paths of length 2

$R^{\le 2}$ = paths of length ≤ 2

Path Relations

$a_1 R^k a_2$ iff

a_1 and a_2 *are* connected by path of length *exactly* k in the graph of R

$a_1 R^{\le k} a_2$ iff

a_1 and a_2 *are* connected by path of length *at most* k in the graph of R

Reflexive Transitive Closure

$a_1 R^* a_2$ iff

a_1 and a_2 are connected by path

Prerequisite Relation on Classes

class c is a prerequisite for class d

$$c \longrightarrow d$$

Some Course 6 Prerequisites

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

Indirect Prerequisites

18.01 → 6.042 → 6.046 → 6.840

18.01 is *indirect* prereq. of 6.840

Classes with no prereqs

18.01 8.01 6.001

"Freshman classes"

d is a Freshman class iff

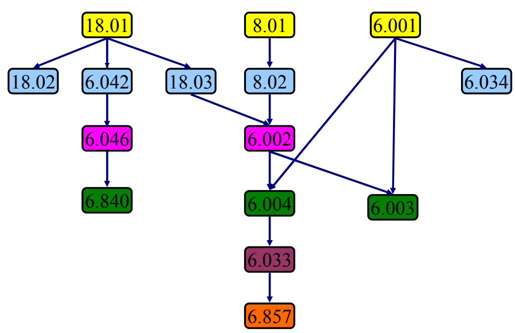
nothing → *d*

Minimal elements

d is *minimal* for →

there is no *c* s.t. *c*→*d*

Prerequisite graph



minim*al* not minim*um*

minim*um* means "smallest"

-- a prereq. for *every* class

no minimum in this example

Team Problem

Problem 1,2

Prerequisite graph

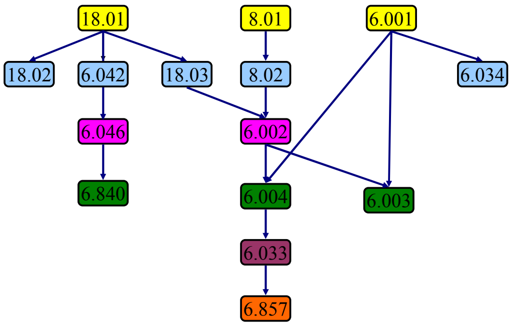
What if there is a cycle in this graph?

-- a path from class *c* to class *d* and back to class *c*?

No one can graduate!

Comm. on Curricula & Registrar are supposed to prevent cycles.

Directed *Acyclic* Graph (DAG)



DAG's = Partial Orders

Theorem:

- The path relation of a DAG is a partial order.
- The graph of a partial order is a DAG.

Constructing the DAG

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
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Identify *Minimal* Elements

Directed Acyclic Graph (DAG)

18.01 8.01 6.001

Constructing the DAG

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
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- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

Remove minimal elements

Constructing the DAG

- 6.042
- 18.02
- 18.03
- 8.02
- 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.002 → 6.004
- 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

Remove minimal elements

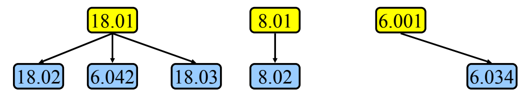
Constructing the DAG

- 6.042
- 18.02
- 18.03
- 8.02
- 6.034
- 6.042 → 6.046

- 18.03, 8.02 → 6.002
- 6.002 → 6.004
- 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

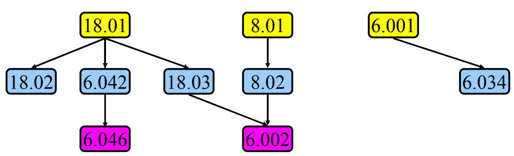
Identify new minimal elements

Directed Acyclic Graph (DAG)

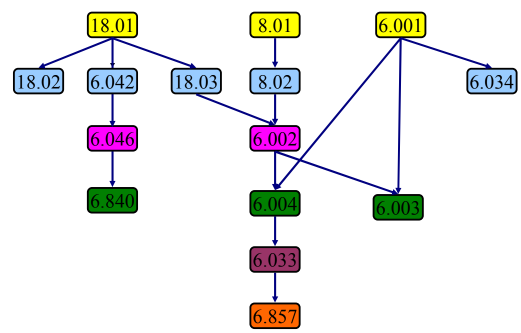


continue in this way…

Directed Acyclic Graph (DAG)



Prerequisite graph

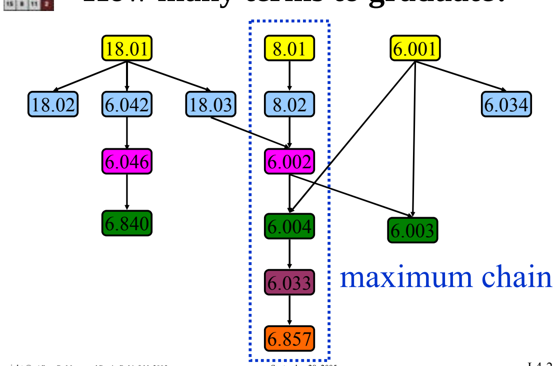


Topological sort

- Is there a way of graduating?
  (in any number of semesters?)

- Yes - take a minimal remaining course each semester

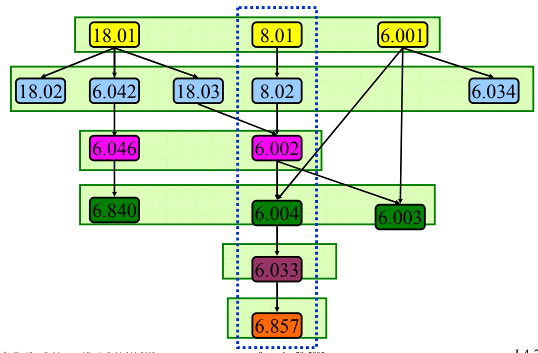
How many terms to graduate?



Parallel Task Scheduling

- 6 terms are **necessary** to complete the curriculum
- *and* **sufficient** (if you can take unlimited courses per term...)

and Sufficient…



Antichains

Set of courses that can be taken in any order:

Any two courses in set are

incomparable

Parallel Task Scheduling

Theorem: If the longest chain has size t, then the elements can be *partitioned* into

t successive *antichains*,

with no element in any block *preceding* anything in a preceding block

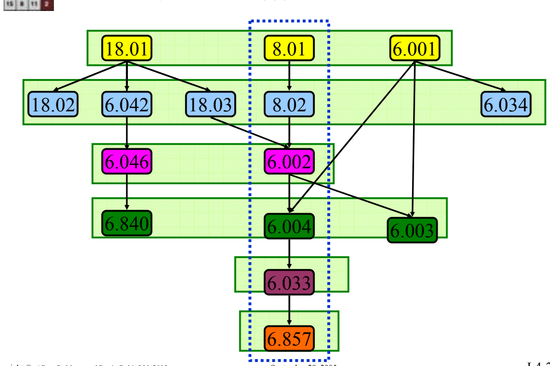
Why sufficient?

Take

$B_i = \{a \mid$ largest chain ending in a is of size $i\}$

If there is a y in B_i such that $x \rightarrow y$ and x not in $B_1 \ldots . B_{i-1}$ then there is a chain of size $>i$ ending in y

and Sufficient…



Minimum "Parallel" Time

parallel time = max chain size.

required # processors

≤ max antichain size.

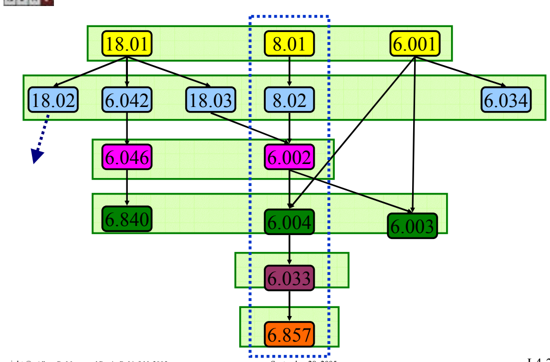
Minimum "Parallel" Time

but 5-course term not *necessary*.
Possible that

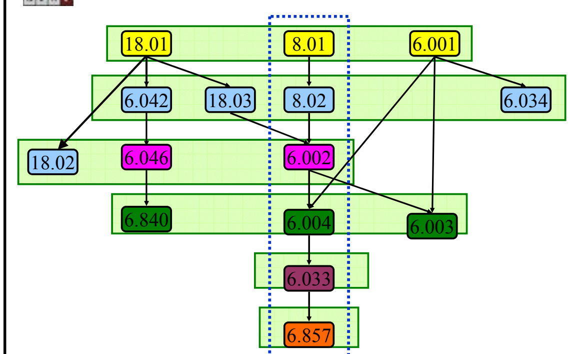
min-time #processors

< max antichain size

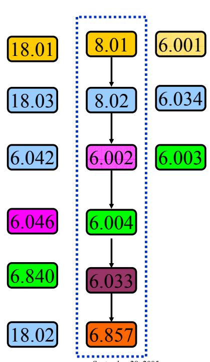
Min-time Schedule



Another Min-time Schedule



3 Subjects per Term Possible



A 3-course term is necessary

- **15** subjects
- max chain size = **6**
- size of *some* block must be

$\geq \lceil 15/6 \rceil = 3.$

∴ to finish in 6 terms, must take ≥3 classes some term

Dilworth's Lemma

A partial order on n items has

- a chain of size $\geq t$, ***or***
- or an antichain of size $\geq \left\lceil \frac{n}{t} \right\rceil$

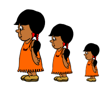
for all $1 \leq t \leq n$.

Height/Birthday Partial Order

Two students are related to each other iff one is shorter *and* younger than the other

$$(s_1, a_1) \preccurlyeq (s_2, a_2) \text{ iff}$$

$$(s_1 \leq s_2) \text{ and } (a_1 \leq a_2)$$



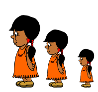
Height/Birthday Partial Order

Chain of students:

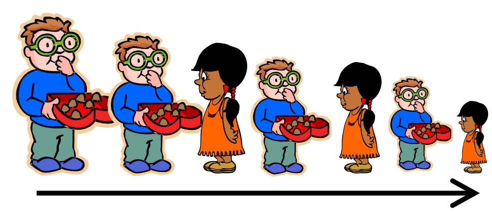
get *older* as they get taller.

AntiChain of students:

get *younger* as they get taller.



Dilworth Demo



Older

Team Problem

Problem 4