In-Class Problems Week 8, Mon.

Problem 1. Given a simple graph G, we apply the following operation to the graph: pick two vertices $u \neq v$ such that either

1. there is an edge of G between u and v and there is also a path from u to v which does *not* include this edge; in this case, delete the edge $\{u, v\}$.
2. or, there is no path from u to v; in which case, add the edge $\{u, v\}$.

We keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.

Assume the vertices of G are the integers $1, 2, \ldots, n$ for some $n \geq 2$. This procedure can be modelled as a state machine whose states are all possible simple graphs with vertices $1, 2, \ldots, n$. The start state is G, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\{1, 2\}, \{3, 4\}\}$$

What are the possible final states reachable from start state G? Draw them.

(b) For any state, G', let e be the number of edges in G', c be the number of connected components it has, and s be the number of simple cycles. For each of the derived variables below, indicate the *strongest* of the properties that it is guaranteed to satisfy, no matter what the starting graph G is. The choices for properties are: *constant*, *strictly increasing*, *strictly decreasing*, *weakly increasing*, *weakly decreasing*, *none of these*. The derived variables are

(i) $e - s$
(ii) $c + e$
(iii) $3c + 2e$
(iv) $c + s$
(v) (c, e), partially ordered coordinatewise (the *product* partial order).

(c) Choose a derived variable from above and prove that it is strictly decreasing in some well-founded partial order. Conclude that the procedure terminates.

(d) Prove that any final state must be a tree on the vertices.

Problem. See if you can come up with a stable marriage assignment given the following preferences. You are not expected to know/remember the Dating Protocol that solves this problem and which is about to be covered in lecture. (And if you do remember the protocol, don't spoil your teammates' fun by telling them.)

boys	*girls*
$1 : CBEAD$	$A : 35214$
$2 : ABECD$	$B : 52143$
$3 : DCBAE$	$C : 43512$
$4 : ACDBE$	$D : 12345$
$5 : ABDEC$	$E : 23415$

Problem 2. Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
David:	HP, Bellcore, AT&T, Draper
Jelani:	AT&T, Bellcore, Draper, HP
Hanson:	HP, Draper, AT&T, Bellcore
Sayan:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Sayan, David, Hanson, Jelani
Bellcore:	Hanson, Jelani, David, Sayan
HP:	Sayan, Hanson, David, Jelani
Draper:	Jelani, Sayan, Hanson, David

(a) Use the Mating Algorithm to find *two* stable assignments of Students to Companies.

Problem 3. Prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Course Notes.)