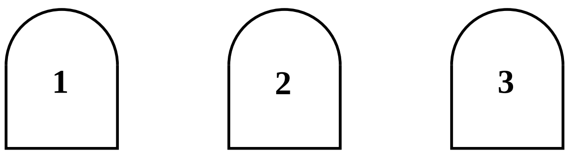
Solutions to In-Class Problems Week 12, Wed.

Problem 1. A Barglesnort makes its lair in one of three caves:



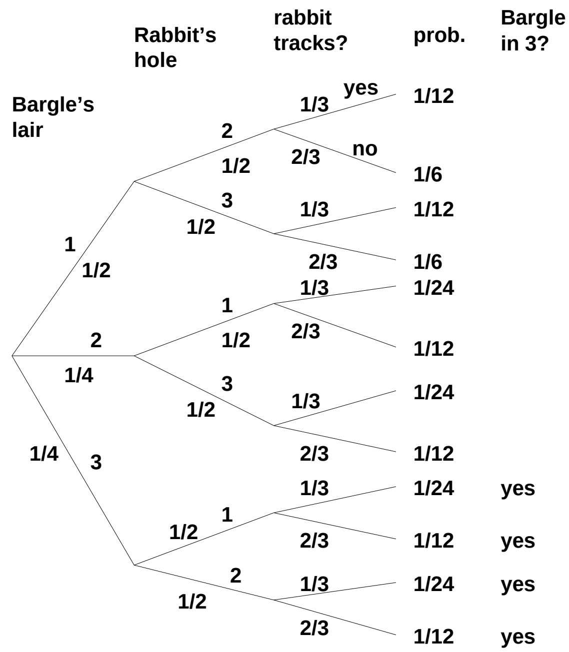
The Barglesnort inhabits cave 1 with probability $1/2$, cave 2 with probability $1/4$, and cave 3 with probability $1/4$. A rabbit subsequently moves into one of the two unoccupied caves, selected with equal probability. With probability $1/3$, the rabbit leaves tracks at the entrance to its cave. (Barglesnorts are much too clever to leave tracks.) What is the probability that the Barglesnort lives in cave 3, given that there are no tracks in front of cave 2?

Use a tree diagram and the four-step method.

Solution. A tree diagram is given below. Let B_3 be the event that the Barglesnort inhabits cave 3, and let T_2 be the event that there are tracks in front of cave 2. Taking data from the tree diagram, we can compute the desired probability as follows:

$$\begin{aligned}\Pr\left\{B_3 \mid \overline{T_2}\right\} &= \frac{\Pr\left\{B_3 \cap \overline{T_2}\right\}}{\Pr\left\{\overline{T_2}\right\}} \\ &= \frac{\frac{1}{24} + \frac{1}{12} + \frac{1}{12}}{1 - \frac{1}{12} - \frac{1}{24}} \\ &= \frac{5}{21}\end{aligned}$$

In the denominator, we apply the formula $\Pr\left\{\overline{T_2}\right\} = 1 - \Pr\left\{T_2\right\}$ for convenience.



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Problem 2. There is a rare and deadly disease called *Nerditosis* which afflicts about 1 person in 1000. One symptom is a compulsion to refer to everything— fields of study, classes, buildings, etc.— using numbers. It's horrible. As victims enter their final, downward spiral, they're awarded a degree from MIT. Two doctors claim that they can diagnose Nerditosis.

(a) Doctor X received his degree from Harvard Medical School. He practices at Massachusetts General Hospital and has access to the latest scanners, lab tests, and research. Suppose you ask Doctor X whether you have the disease.

- If you have Nerditosis, he says "yes" with probability 0.99.
- If you don't have it, he says "no" with probability 0.97.

Let D be the event that you have the disease, and let E be the event that the diagnosis is erroneous. Use the Total Probability Law to compute $\Pr\{E\}$, the probability that Doctor X makes a mistake.

The Total Probability Law is

$$\Pr\{A\} = \Pr\{A \mid E\} \cdot \Pr\{E\} + \Pr\left\{A \mid \overline{E}\right\} \cdot \Pr\left\{\overline{E}\right\}.$$

Solution. By the Total Probability Law:

$$\begin{aligned}\Pr\{E\} &= \Pr\{E \mid D\} \cdot \Pr\{D\} + \Pr\left\{E \mid \overline{D}\right\} \cdot \Pr\left\{\overline{D}\right\} \\ &= 0.01 \cdot 0.001 + 0.03 \cdot 0.999 \\ &= 0.02998\end{aligned}$$

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(b) "Doctor" Y received his genuine degree from a fully-accredited university for $49.95 via a special internet offer. He knows that Nerditosis stikes 1 person in 1000, but is a little shakey on how to interpret this. So if you ask him whether you have the disease, he'll helpfully say "yes" with probability 1 in 1000 regardless of whether you actually do or not.

Let D be the event that you have the disease, and let F be the event that the diagnosis is faulty. Use the Total Probability Law to compute $\Pr\{F\}$, the probability that Doctor Y made a mistake.

Solution. By the Total Probability Law:

$$\begin{aligned}\Pr\{F\} &= \Pr\{F \mid D\} \cdot \Pr\{D\} + \Pr\left\{F \mid \overline{D}\right\} \cdot \Pr\left\{\overline{D}\right\} \\ &= 0.999 \cdot 0.001 + 0.001 \cdot 0.999 \\ &= 0.001998\end{aligned}$$

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(c) Which doctor is more reliable?

Solution. Doctor X makes more than 15 times as many errors as Doctor Y. ■

Problem 3. Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \tag{1}$$

So for example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \dots, n\})$ of subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of the components which fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the sample points.

Solution. There could be a probability p of system failure if the all individual failures occur together. That is, let $\Pr\{\{1, \dots, n\}\} ::= p$, $\Pr\{\emptyset\} ::= 1 - p$, and let the probability of all other outcomes be zero. So $F_i = \{s \in \mathcal{S} \mid i \in s\}$ and $\Pr\{F_i\} = 0 + 0 + \cdots + 0 + \Pr\{\{1, \dots, n\}\} = \Pr\{\{1, \dots, n\}\} = p$. Also, the only outcome with positive probability in F is $\{1, \dots, n\}$, so $\Pr\{F\} = p$, as required. ■

(b) Show that the probability that the system fails could actually could be as large as np by describing appropriate probabilities for the sample points.

Solution. Suppose at most one component ever fails at a time. That is, $\Pr\{\{i\}\} = p$ for $1 \leq i \leq n$, $\Pr\{\emptyset\} = 1 - np$, and probability of all other points is zero. The sum of the probabilities of all the points is one, so this is a well-defined probability space. Also, the only sample point in F_i with positive probability is $\{i\}$, so $\Pr\{F_i\} = \Pr\{\{i\}\} = p$ as required. Finally, $\Pr\{F\} = np$ because $F = \{A \subseteq \{1, \dots, n\} \mid A \neq \emptyset\}$, so F in particular contains all the n outcomes of the form $\{i\}$. ■

(c) Prove the inequality (1).

Solution. $F = \bigcup_{i=1}^{n} F_i$ so

$$\begin{aligned}
p &= \Pr\{F_1\} && \text{(given)} && (2)\\
&\leq \Pr\{F\} = \Pr\left\{\bigcup F_i\right\} && \text{(def. of } F\text{)} && (3)\\
&\leq \sum_{i=1}^{n} \Pr\{F_i\} && \text{(Union bound)} && (4)\\
&= np. && && (5)
\end{aligned}$$

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Problem 4. There were n Immortal Warriors born into our world, but in the end *there can be only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion of probabilistic independence, they opt to give the following protocol a try:

1. The Immortals forge a coin that comes up heads with probability p.
2. Each Immortal flips the coin once.
3. If *exactly one* Immortal flips heads, then he or she is declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

(a) One of the Immortals (Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen *very carefully*. What does your intuition tell you?

Solution. Your intuition tells you that a short nap would be nice right now. As would a couple cookies to dunk in a cold glass of milk. ■

(b) What is the probability that the experiment succeeds as a function of p and n?

Solution. The sample space consists of all possible results of n coin flips, which we can represent by the set $\{H, T\}^n$. Let E be the event that the experiment successfully selects The One. Then E consists of the n outcomes which contain a single head. In general, the probability of an outcome with h heads and $n - h$ tails is:

$$p^h(1-p)^{n-h}$$

Summing the probabilities of the n outcomes in E gives the probability that the procedure succeeds:

$$\Pr\{E\} \;=\; np(1-p)^{n-1}$$

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(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds? (You're going to have to compute a derivative!)

Solution. We compute the derivative of the success probability:

$$\frac{d}{dp}\, np(1-p)^{n-1} \;=\; n(1-p)^{n-1} - np(n-1)(1-p)^{n-2}$$

Now we set the right side equal to zero to find the best probability p:

$$\begin{aligned} n(1-p)^{n-1} &= np(n-1)(1-p)^{n-2} \\ (1-p) &= p(n-1) \\ p &= 1/n \end{aligned}$$

This answer makes sense, since we want the coin to come up heads exactly 1 time in n. ■

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?

Solution. Setting $p = 1/n$ in the formula for the probability that the experiment succeeds gives:

$$\Pr\{E\} \;=\; \left(1 - \frac{1}{n}\right)^{n-1}$$

In the limit, this tends to $1/e$. McLeod is right. ■