Quaternions

① "hyper-complex" $q_0 + i q_x + j q_y + k q_z$

$$\begin{cases} i^2 = j^2 = k^2 = 1 \\ ij = -ji = k \\ jk = -kj = i \\ ki = -ik = j \end{cases}$$

② Four-vector (q_x, q_y, q_z, q_w)

③ Scalar + vector $(s, \underline{v})$

Easy multiplication: not commutative !

- $(p, \underline{p})(q, \underline{q}) = (pq - \underline{p}\cdot\underline{q},\ p\underline{q} + q\underline{p} + \underline{p}\times\underline{q})$
- $\|\mathring{p}\| = \mathring{p}\mathring{p}$
- $\mathring{p}\mathring{p} = 1$ unit quaternion
- conjugate : $\mathring{q}^* = (q, -\underline{q})$
- $\mathring{q}\mathring{q}^* = (\mathring{q}\mathring{q})\mathring{e}$ where $\mathring{e} = (1, \underline{0})$ identity
- $\mathring{b}^{-1} = \mathring{b}^* / (\mathring{b}.\mathring{b})$
- $(\mathring{p}\mathring{q})^* = \mathring{q}^*\mathring{p}^*$
- $(\mathring{p}\mathring{q}).(\mathring{p}\mathring{r}) = (\mathring{p}.\mathring{p})(\mathring{q}\,\mathring{r})$
- $(\mathring{p}\mathring{q}).(\mathring{p}\mathring{q}) = (\mathring{p}.\mathring{p})(\mathring{q}\,\mathring{q})$
- $\mathring{p}\mathring{q} . \mathring{r} = \mathring{p} . \mathring{r}\mathring{q}^*$

<u>Scalars</u>: $s \rightarrow (s, \underline{0})$

<u>Vectors</u>: $\underline{v} \rightarrow (0, \underline{v})$

<u>Rotation</u>: $\mathring{r} = (0, \underline{r})$

$\mathring{r}' = (0, \underline{r}')$

$\mathring{r}' = \mathring{q}\mathring{r}$? $\rightarrow \mathring{r}' = \mathring{q}\mathring{r}\mathring{q}^*$

$\mathring{q}^* \mathring{r}' \mathring{q} = \mathring{r}$ inverse

preserves magnitude $\mathring{r}' \cdot \mathring{r}' = \mathring{r}\mathring{r}$ } rotation ?

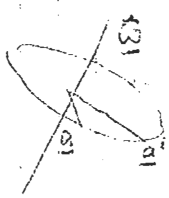
preserves angles $\mathring{r} \cdot \mathring{s}' = \mathring{r} \cdot \mathring{s}$ } reflection ?

handedness preserved

What rotation?

$\mathring{q}\mathring{q}\mathring{q}^* = \mathring{q} \rightarrow \underline{q} \,//\, \underline{\hat{\omega}}$ axis of rotation

angle? $\underline{a} \rightarrow \underline{a}'$ $\underline{a} = \underline{\omega} \times \underline{b}$ ($\underline{b}$ not // axis)



$\underline{a} \cdot \underline{a}' = (q^2 - \underline{q} \cdot \underline{q}) \|\underline{a}\|^2 = \|a\| \cdot \|a'\| \cos\theta$

$q^2 = \frac{1}{2}(1 + \cos\theta)$

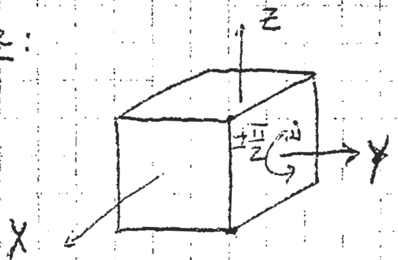
$q = \pm \cos\frac{\theta}{2}$

Since $\mathring{q}$ unit, $q^2 + \underline{q} \cdot \underline{q} = 1 \Rightarrow \underline{q} \cdot \underline{q} = \sin^2 \theta/2$

$\|q\| = \pm \sin\theta/2$

$\pm \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, \underline{\hat{\omega}}\right)$

Example:



$\theta = \frac{\pi}{2}$ $\quad \mathring{q}_1 = \left(\frac{1}{\sqrt{2}};\ \frac{1}{\sqrt{2}}, 0, 0\right) \quad (+x)$

or $\quad \mathring{q}_2 = \left(\frac{1}{\sqrt{2}};\ -\frac{1}{\sqrt{2}}, 0, 0\right) \quad (-x)$

$\mathring{q}_3 = \left(\frac{1}{\sqrt{2}};\ 0, \frac{1}{\sqrt{2}}, 0\right) \quad (+y)$

or $\quad \mathring{q}_4 = \left(\frac{1}{\sqrt{2}};\ 0, -\frac{1}{\sqrt{2}}, 0\right) \quad (-y)$

$\mathring{q}_5 = \left(\frac{1}{\sqrt{2}};\ 0, 0, \frac{1}{\sqrt{2}}\right) \quad (+z)$

or $\quad \mathring{q}_6 = \left(\frac{1}{\sqrt{2}};\ 0, 0, -\frac{1}{\sqrt{2}}\right) \quad (-z)$

$\mathring{q}_1\, \mathring{q}_3 = \left(\frac{1}{2};\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

↪ axis through vertices

angle = 120°

Bridge Quaternion - Rotation Matrix

$$\underline{r}' = (q^2 - \underline{q}\,\underline{q})\, I\, \underline{r} + \ldots$$

$$= \underbrace{\left((q^2 - \underline{q}\cdot\underline{q}) I + 2qQ + 2(\underline{q}\,\underline{q}^T)\right)}_{R} \underline{r}$$

$$R = \begin{pmatrix} q_0 + q_x^2 - q_y^2 - q_z^2 & 2(-q_0 q_z + q_x q_y) & 2(q_0 q_y + q_x q_z) \\ 2(q_0 q_z + q_y q_z) & q_0^2 - q_x^2 + q_y^2 - q_z^2 & 2(-q_0 q_x + q_y q_z) \\ 2(-q_0 q_y + q_z q_x) & 2(q_0 q_x + q_z q_y) & q_0^2 - q_x^2 - q_y^2 + q_z^2 \end{pmatrix}$$

Advantages of quaternions

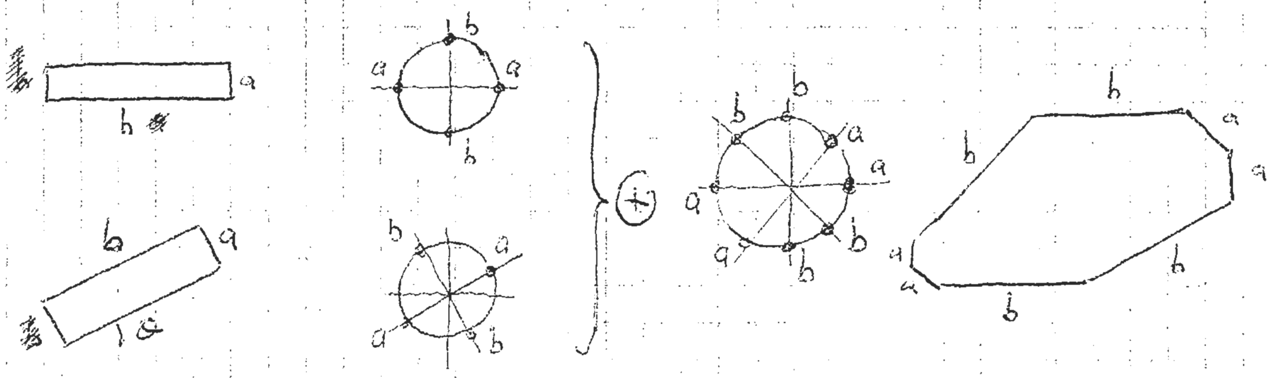
- Space of rotation
  - ↳ averages / integrals
  - ↳ sample space
  - ↳ search
- redundant (hence not singular)
- easy constraint $\mathring{q}\cdot\mathring{q} = 1$
- renormalization easy
- cheap to compose
- finite, rotation groups

Ch 16: operations on EGIs

"Convolution" not easy unless we have a symmetric weighting function.

"Add" . Centroid theorem is preserved.

. hard to visualize but we can make it in 2D



"mixed area" : $A = (a\sin\theta + b + b\cos\theta)(a\cos\theta + a + b\sin\theta) \quad 0 \le \theta \le \frac{\pi}{2}$

$\theta = 0 : A = 4ab \qquad \theta = \frac{\pi}{2} : A = (a+b)^2$

We pick the orientation that minimizes $A(\theta)$