15.081J/6.251J Introduction to Mathematical Programming

Lecture 12: Sensitivity Analysis

1 Motivation

1.1 Questions

Slide 1

$$\begin{aligned} z = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- How does z depend globally on $\boldsymbol{c}$? on $\boldsymbol{b}$?
- How does z change locally if either $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{A}$ change?
- How does z change if we add new constraints, introduce new variables?
- Importance: Insight about LO and practical relevance

2 Outline

Slide 2

1. Global sensitivity analysis
2. Local sensitivity analysis
   (a) Changes in $\boldsymbol{b}$
   (b) Changes in $\boldsymbol{c}$
   (c) A new variable is added
   (d) A new constraint is added
   (e) Changes in $\boldsymbol{A}$
3. Detailed example

3 Global sensitivity analysis

3.1 Dependence on $\boldsymbol{c}$

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$$\begin{aligned} G(\boldsymbol{c}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

$G(\boldsymbol{c}) = \min_{i=1,\ldots,N} \boldsymbol{c}'\boldsymbol{x}^i$ is a concave function of $\boldsymbol{c}$

3.2 Dependence on $\boldsymbol{b}$

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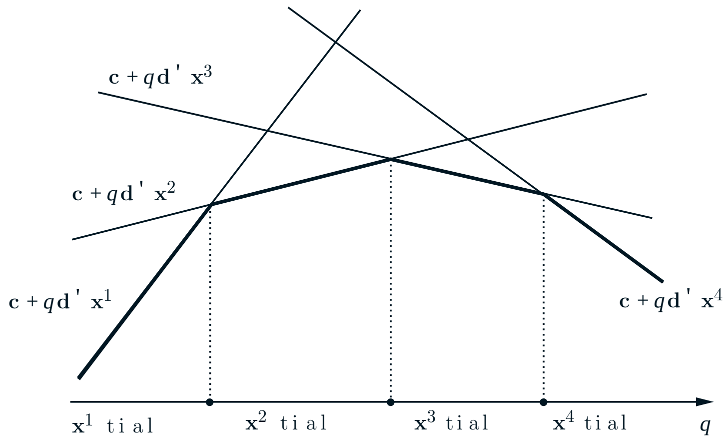
Primal

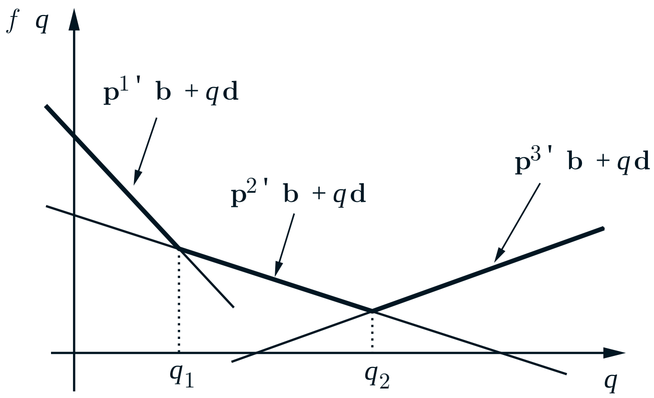
$$\begin{aligned} F(\boldsymbol{b}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual

$$\begin{aligned} F(\boldsymbol{b}) = \max \quad & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{aligned}$$

$F(\boldsymbol{b}) = \max_{i=1,\ldots,N} (\boldsymbol{p}^i)'\boldsymbol{b}$ is a convex function of $\boldsymbol{b}$


c + qd ' x3
c + qd ' x2
c + qd ' x1
c + qd ' x4
x1 tial
x2 tial
x3 tial
x4 tial
q


f q
p1 ' b + qd
p2 ' b + qd
p3 ' b + qd
q1
q2
q

4 Local sensitivity analysis

SLIDE 5

$$\begin{aligned} z = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

What does it mean that a basis $\boldsymbol{B}$ is optimal?

1. Feasibility conditions: $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
2. Optimality conditions: $\boldsymbol{c}' - \boldsymbol{c}'_{\boldsymbol{B}}\boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$

SLIDE 6

- Suppose that there is a change in either $\boldsymbol{b}$ or $\boldsymbol{c}$ for example
- How do we find whether $\boldsymbol{B}$ is still optimal?
- Need to check whether the feasibility and optimality conditions are satisfied

5 Local sensitivity analysis

5.1 Changes in b

SLIDE 7

b_i becomes $b_i + \Delta$, i.e.

$$\begin{aligned} (P) \quad \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \quad \rightarrow \quad \begin{aligned} (P') \quad \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta\boldsymbol{e}_i \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- $\boldsymbol{B}$ optimal basis for (P)
- Is $\boldsymbol{B}$ optimal for (P')?

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Need to check:

1. Feasibility: $\boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta\boldsymbol{e}_i) \geq \boldsymbol{0}$
2. Optimality: $\boldsymbol{c}' - \boldsymbol{c}'_{\boldsymbol{B}}\boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$

Observations:

1. Changes in $\boldsymbol{b}$ affect feasibility
2. Optimality conditions are not affected

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$\boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta\boldsymbol{e}_i) \geq \boldsymbol{0}$
$\beta_{ij} = [\boldsymbol{B}^{-1}]_{ij}$
$\overline{b}_j = [\boldsymbol{B}^{-1}\boldsymbol{b}]_j$
Thus,
$(\boldsymbol{B}^{-1}\boldsymbol{b})_j + \Delta(\boldsymbol{B}^{-1}\boldsymbol{e}_i)_j \geq 0 \Rightarrow \quad \overline{b}_j + \Delta\beta_{ji} \geq 0 \Rightarrow$

$$\max_{\beta_{ji}>0}\left(-\frac{\overline{b}_j}{\beta_{ji}}\right) \leq \Delta \leq \min_{\beta_{ji}<0}\left(-\frac{\overline{b}_j}{\beta_{ji}}\right)$$

SLIDE 10

$$\underline{\Delta} \leq \Delta \leq \overline{\Delta}$$

Within this range

- Current basis $\boldsymbol{B}$ is optimal
- $z = \boldsymbol{c}'_B \boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta \boldsymbol{e}_i) = \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{b} + \Delta p_i$
- What if $\Delta = \overline{\Delta}$?
- What if $\Delta > \overline{\Delta}$?

  Current solution is infeasible, but satisfies optimality conditions $\rightarrow$ use dual simplex method

5.2 Changes in $\boldsymbol{c}$

SLIDE 11

$c_j \rightarrow c_j + \Delta$
Is current basis $\boldsymbol{B}$ optimal?
Need to check:

1. Feasibility: $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$, unaffected
2. Optimality: $\boldsymbol{c}' - \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$, affected

There are two cases:

- x_j basic
- x_j nonbasic

5.2.1 x_j nonbasic

SLIDE 12

$\boldsymbol{c}_B$ unaffected
$(c_j + \Delta) - \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{A}_j \geq 0 \Rightarrow \overline{c}_j + \Delta \geq 0$
Solution optimal if $\Delta \geq -\overline{c}_j$
What if $\Delta = -\overline{c}_j$?
What if $\Delta < -\overline{c}_j$?

5.2.2 x_j basic

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$$\boldsymbol{c}_B \leftarrow \hat{\boldsymbol{c}}_B = \boldsymbol{c}_B + \Delta \boldsymbol{e}_j$$

Then,

$$[\boldsymbol{c}' - \hat{\boldsymbol{c}}'_B \boldsymbol{B}^{-1}\boldsymbol{A}]_i \geq 0 \Rightarrow c_i - [\boldsymbol{c}_B + \Delta \boldsymbol{e}_j]'\boldsymbol{B}^{-1}\boldsymbol{A}_i \geq 0$$

$[\boldsymbol{B}^{-1}\boldsymbol{A}]_{ji} = \overline{a}_{ji}$

$$\overline{c}_i - \Delta \overline{a}_{ji} \geq 0 \Rightarrow \max_{\overline{a}_{ji}<0} \frac{\overline{c}_i}{\overline{a}_{ji}} \leq \Delta \leq \min_{\overline{a}_{ji}>0} \frac{\overline{c}_i}{\overline{a}_{ji}}$$

What if Δ is outside this range? use primal simplex

5.3 A new variable is added

SLIDE 14

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \rightarrow \quad \begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} + c_{n+1}\boldsymbol{x}_{n+1} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}_{n+1}\boldsymbol{x}_{n+1} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

In the new problem is $x_{n+1} = 0$ or $x_{n+1} > 0$? (i.e., is the new activity profitable?)

SLIDE 15

Current basis $\boldsymbol{B}$. Is solution $\boldsymbol{x} = \boldsymbol{B}^{-1}\boldsymbol{b}, x_{n+1} = 0$ optimal?

- Feasibility conditions are satisfied
- Optimality conditions:

$$c_{n+1} - \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{A}_{n+1} \geq 0 \Rightarrow c_{n+1} - \boldsymbol{p}'\boldsymbol{A}_{n+1} \geq 0?$$

- If yes, solution $\boldsymbol{x} = \boldsymbol{B}^{-1}\boldsymbol{b}, x_{n+1} = 0$ optimal
- Otherwise, use primal simplex

5.4 A new constraint is added

SLIDE 16

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \rightarrow \quad \begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{a}'_{m+1}\boldsymbol{x} = b_{m+1} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

If current solution feasible, it is optimal; otherwise, apply dual simplex

5.5 Changes in $\boldsymbol{A}$

SLIDE 17

- Suppose $a_{ij} \leftarrow a_{ij} + \Delta$
- Assume $\boldsymbol{A}_j$ does not belong in the basis
- Feasibility conditions: $\boldsymbol{B^{-1}b} \geq \boldsymbol{0}$, unaffected
- Optimality conditions: $c_l - \boldsymbol{c'_B B^{-1} A}_l \geq 0$, $l \neq j$, unaffected
- Optimality condition: $c_j - \boldsymbol{p}'(\boldsymbol{A}_j + \Delta \boldsymbol{e}_i) \geq 0 \Rightarrow \overline{c}_j - \Delta p_i \geq 0$
- What if $\boldsymbol{A}_j$ is basic? BT, Exer. 5.3

6 Example

6.1 A Furniture company

SLIDE 18

- A furniture company makes desks, tables, chairs
- The production requires wood, finishing labor, carpentry labor

	Desk	Table (ft)	Chair	Avail.
Profit	60	30	20	-
Wood (ft)	8	6	1	48
Finish hrs.	4	2	1.5	20
Carpentry hrs.	2	1.5	0.5	8

6.2 Formulation

SLIDE 19

Decision variables:
$x_1 = \#$ desks, $x_2 = \#$ tables, $x_3 = \#$ chairs

$$
\begin{array}{lll}
\max & 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 & \leq 48 \\
 & 4x_1 + 2x_2 + 1.5x_3 & \leq 20 \\
 & 2x_1 + 1.5x_2 + 0.5x_3 & \leq 8 \\
 & x_1, x_2, x_3 & \geq 0
\end{array}
$$

6.3 Simplex tableaus

SLIDE 20

Initial tableau:

		s_1	s_2	s_3	x_1	x_2	x_3
	0	0	0	0	-60	-30	-20
$s_1=$	48	1			8	6	1
$s_2=$	20		1		4	2	1.5
$s_2=$	8			1	2	1.5	0.5

Final tableau:

		s_1	s_2	s_3	x_1	x_2	x_3
	280	0	10	10	0	5	0
$s_1 =$	24	1	2	-8	0	-2	0
$x_3 =$	8	0	2	-4	0	-2	1
$x_1 =$	2	0	-0.5	1.5	1	1.25	0

6.4 Information in tableaus

SLIDE 21

- What is $\boldsymbol{B}$?

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$$

- What is $\boldsymbol{B}^{-1}$?

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

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- What is the optimal solution?
- What is the optimal solution value?
- Is it a bit surprising?
- What is the optimal dual solution?
- What is the shadow price of the wood constraint?
- What is the shadow price of the finishing hours constraint?
- What is the reduced cost for x_2?

6.5 Shadow prices

SLIDE 23

Why the dual price of the finishing hours constraint is 10?

- Suppose that finishing hours become 21 (from 20).
- Currently only desks (x_1) and chairs (x_3) are produced
- Finishing and carpentry hours constraints are tight
- Does this change leaves current basis optimal?

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New solution:

$$\begin{aligned} 8x_1 + x_3 + s_1 &= 48 \\ 4x_1 + 1.5x_3 &= 21 \\ 2x_1 + 0.5x_3 &= 8 \end{aligned} \quad \Rightarrow$$

New	Previous
$s_1 = 26$	24
$x_1 = 1.5$	2
$x_3 = 10$	8

Solution change:

$$z' - z = (60 * 1.5 + 20 * 10) - (60 * 2 + 20 * 8) = 10$$

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- Suppose you can hire 1h of finishing overtime at \$7. Would you do it?
- Another check

$$\boldsymbol{c}_B'\boldsymbol{B}^{-1} = (0, -20, -60)\begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} =$$

$$(0, -10, -10)$$

6.6 Reduced costs

SLIDE 26

- What does it mean that the reduced cost for x_2 is 5?
- Suppose you are forced to produce $x_2 = 1$ (1 table)
- How much will the profit decrease?

$$\begin{array}{llll} 8x_1 + x_3 + s_1 & + 6 \cdot 1 & = 48 & & s_1 = 26 \\ 4x_1 + 1.5x_3 & + 2 \cdot 1 & = 20 & \Rightarrow & x_1 = 0.75 \\ 2x_1 + 0.5x_3 & + 1.5 \cdot 1 & = 8 & & x_3 = 10 \end{array}$$

$z' - z = (60 * 0.75 + 20 * 10) - (60 * 2 + 20 * 8 + 30 * 1) = -35 + 30 = -5$ SLIDE 27
Another way to calculate the same thing: If $x_2 = 1$

Direct profit from table	$+30$
Decrease wood by -6	$-6 * 0 = 0$
Decrease finishing hours by -2	$-2 * 10 = -20$
Decrease carpentry hours by -1.5	$-1.5 * 10 = -15$
Total Effect	-5

Suppose profit from tables increases from \$30 to \$34. Should it be produced? At \$35? At \$36?

6.7 Cost ranges

SLIDE 28

Suppose profit from desks becomes $60 + \Delta$. For what values of Δ does current basis remain optimal?
Optimality conditions:

$$c_j - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{A}_j \geq 0 \Rightarrow$$

$$\boldsymbol{p}' = \boldsymbol{c}_B'\boldsymbol{B}^{-1} = [0, -20, -(60 + \Delta)]\begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

$$= -[0, \quad 10 - 0.5\Delta, \quad 10 + 1.5\Delta]$$

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s_1, x_3, x_1 are basic
Reduced costs of non-basic variables

$$\overline{c}_2 = c_2 - \boldsymbol{p'A}_2 = -30 + [0, 10 - 0.5\Delta, 10 + 1.5\Delta] \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} = 5 + 1.25\Delta$$

$\overline{c}_{s_2} = 10 - 0.5\Delta$
$\overline{c}_{s_3} = 10 + 1.5\Delta$
Current basis optimal:

$$\left.\begin{array}{l} 5 + 1.25\Delta \geq 0 \\ 10 - 0.5\Delta \geq 0 \\ 10 + 1.5\Delta \geq 0 \end{array}\right\} \boxed{-4 \leq \Delta \leq 20}$$

$\Rightarrow 56 \leq c_1 \leq 80$ solution remains optimal.
If $c_1 < 56$, or $c_1 > 80$ current basis is not optimal.
Suppose $c_1 = 100(\Delta = 40)$ What would you do?

6.8 Rhs ranges

SLIDE 30

Suppose finishing hours change by Δ becoming $(20 + \Delta)$ What happens?

$$\boldsymbol{B}^{-1} \begin{bmatrix} 48 \\ 20 + \Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 + \Delta \\ 8 \end{bmatrix}$$
$$= \begin{bmatrix} 24 + 2\Delta \\ 8 + 2\Delta \\ 2 - 0.5\Delta \end{bmatrix} \geq 0$$

$\Rightarrow -4 \leq \Delta \leq 4$ current basis optimal

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Note that even if current basis is optimal, optimal solution variables change:

$$\begin{aligned} s_1 &= 24 + 2\Delta \\ x_3 &= 8 + 2\Delta \\ x_1 &= 2 - 0.5\Delta \\ z &= 60(2 - 0.5\Delta) + 20(8 + 2\Delta) = 280 + 10\Delta \end{aligned}$$

SLIDE 32

Suppose $\Delta = 10$ then

$$\begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 44 \\ 25 \\ -3 \end{pmatrix} \leftarrow \text{inf. (Use dual simplex)}$$

6.9 New activity

SLIDE 33

Suppose the company has the opportunity to produce stools
Profit \$15; requires 1 ft of wood, 1 finishing hour, 1 carpentry hour
Should the company produce stools?

$$\begin{array}{llllllll} \max & 60x_1 & +30x_2 & +20x_3 & +15x_4 & & & \\ & 8x_1 & +6x_2 & +x_3 & +x_4 & +s_1 & & & = 48 \\ & 4x_1 & +2x_2 & +1.5x_3 & +x_4 & & +s_2 & & = 20 \\ & 2x_1 & +1.5x_2 & +0.5x_3 & +x_4 & & & +s_3 & = 8 \\ & & & & x_i \geq 0 & & & & \end{array}$$

$$c_4 - \boldsymbol{c'_B B^{-1}} A_4 = -15 - (0, -10, -10) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5 \geq 0$$

Current basis still optimal. Do not produce stools

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