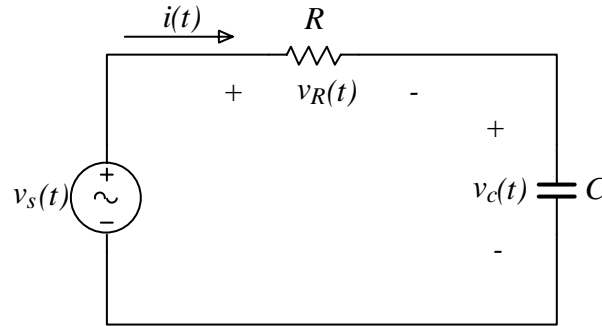


## Sinusoidal Steady State Response of Linear Circuits

The circuit shown on Figure 1 is driven by a sinusoidal voltage source  $v_s(t)$  of the form

$$v_s(t) = v_o \cos(\omega t) \quad (1.1)$$



**Figure 1. Series RC circuit driven by a sinusoidal forcing function**

Our goal is to determine the voltages  $v_c(t)$  and the current  $i(t)$  which will completely characterize the “Steady State” response of the circuit.

The equation that describes the behavior of this circuit is obtained by applying KVL around the mesh.

$$v_R(t) + v_c(t) = v_s(t) \quad (1.2)$$

Using the current voltage relationship of the resistor and the capacitor, Equation (1.2) becomes

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_o \cos(\omega t) \quad (1.3)$$

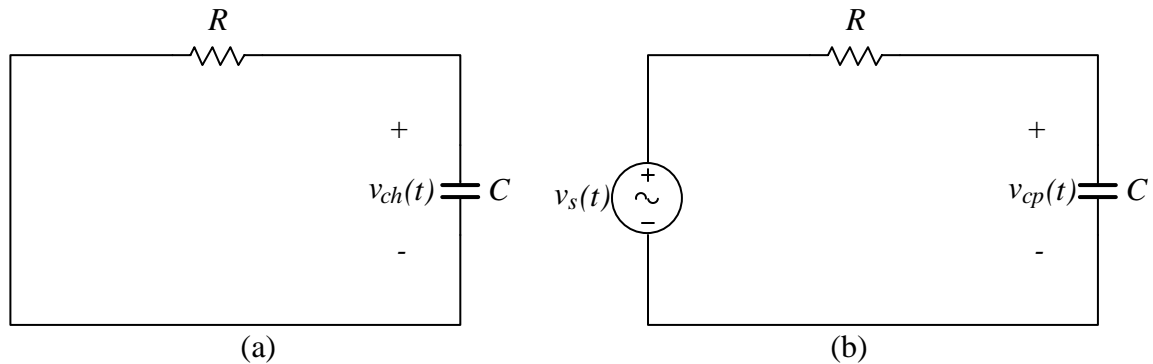
Note that the coefficient  $RC$  has the unit of time. (Ohm)(Farad)  $\rightarrow$  seconds

Before proceeding with the solution of this differential equation let's explore its physical significance.

This linear circuit is driven (forced) by an independent sinusoidal source,  $v_s(t)$ . We may view its response (its effect on the circuit) as the superposition of the response with the source set equal to zero (source-free or natural response ( $v_{ch}(t)$ )) and the forced response ( $v_{cp}(t)$ ).

$$v_c(t) = v_{ch}(t) + v_{cp}(t) \quad (1.4)$$

Schematically the superposition is shown on Figure 2(a) and (b).



**Figure 2**

In mathematical language we call these two responses the homogeneous solution ( $v_{ch}(t)$ ) and the particular solution ( $v_{cp}(t)$ ) of the equation that characterizes the system (Equation (1.3) in our case)

The homogeneous solution corresponds to the differential equation

$$RC \frac{dv_{ch}(t)}{dt} + v_{ch}(t) = 0 \quad (1.5)$$

And the particular solution to the equation

$$RC \frac{dv_{cp}(t)}{dt} + v_{cp}(t) = v_o \cos(\omega t) \quad (1.6)$$

The homogeneous solution (or the natural response of the system) has the form

$$v_{ch}(t) = B \exp\left[\frac{-t}{RC}\right] \quad (1.7)$$

The particular solution (or the forced response of the system) is the cosine function of amplitude  $A$ , frequency  $\omega$ , and phase  $\phi$ .

$$v_{cp}(t) = A \cos(\omega t + \phi) \quad (1.8)$$

And so the general form of the system response is

$$v_c(t) = A \cos(\omega t + \phi) + B \exp\left[\frac{-t}{RC}\right] \quad (1.9)$$

At this time let's recall the problem statement which says that we are interested in obtaining the Steady State response of the system. This is equivalent to saying that the source  $v_s(t)$  was connected to the system long time ago and all transient phenomena are gone. In order to be more precise, if the time  $t \gg RC$  then the exponential term in Equation (1.9) would go to zero. In this case the observable and thus the important response of the system is the Steady State response which is given by

$$v_c(t) = A \cos(\omega t + \phi) \quad (1.10)$$

We may now proceed to determine the details of the solution by calculating the amplitude  $A$  and the phase  $\phi$ . To do this we substitute the form of the solution (Equation (1.10)) into the differential Equation (1.3).

Before we make that substitution let's use the trigonometric identities to expand Equation (1.10).

$$\begin{aligned} v_c(t) &= A \cos(\omega t + \phi) \\ &= A \cos \phi \cos(\omega t) - A \sin \phi \sin(\omega t) \end{aligned} \quad (1.11)$$

And the corresponding derivative

$$\frac{dv_c}{dt} = -A\omega \cos \phi \sin(\omega t) - A\omega \sin \phi \cos(\omega t) \quad (1.12)$$

Substituting Equations (1.11) and (1.12) into Equation (1.3) we have

$$\begin{aligned} -A\omega \cos \phi \sin(\omega t) - A\omega \sin \phi \cos(\omega t) + \\ \frac{1}{RC} [A \cos \phi \cos(\omega t) - A \sin \phi \sin(\omega t)] = \frac{v_0}{RC} \cos(\omega t) \end{aligned} \quad (1.13)$$

Collecting the coefficients of  $\cos(\omega t)$  and  $\sin(\omega t)$  we have the following equations for the unknowns  $A$  and  $\phi$ .

$$-A\omega \cos \phi - \frac{A}{RC} \sin \phi = 0 \quad (1.14)$$

$$-A\omega \sin \phi + \frac{A}{RC} \cos \phi = \frac{v_0}{RC} \quad (1.15)$$

These equations are independent and thus they may be solved simultaneously for  $A$  and  $\phi$ .

From Equation (1.14) we obtain the phase

$$\phi = \arctan(-RC \omega) \quad (1.16)$$

and  $A$  becomes

$$A = \frac{v_o}{\cos \phi - \omega RC \sin \phi} \quad (1.17)$$

Therefore, the Steady State response of the system is

$$v_c(t) = \frac{v_o}{\cos \phi - \omega RC \sin \phi} \cos(\omega t + \phi) \quad (1.18)$$

where  $\phi = \arctan(-\omega RC)$

Figure 3 shows a plot of the phase and the amplitude ratio  $A/v_o$  as a function of the dimensionless parameter  $\omega RC$ . Observe the frequency selectivity of this system. It passes “low” frequencies while it attenuates “higher” frequencies.

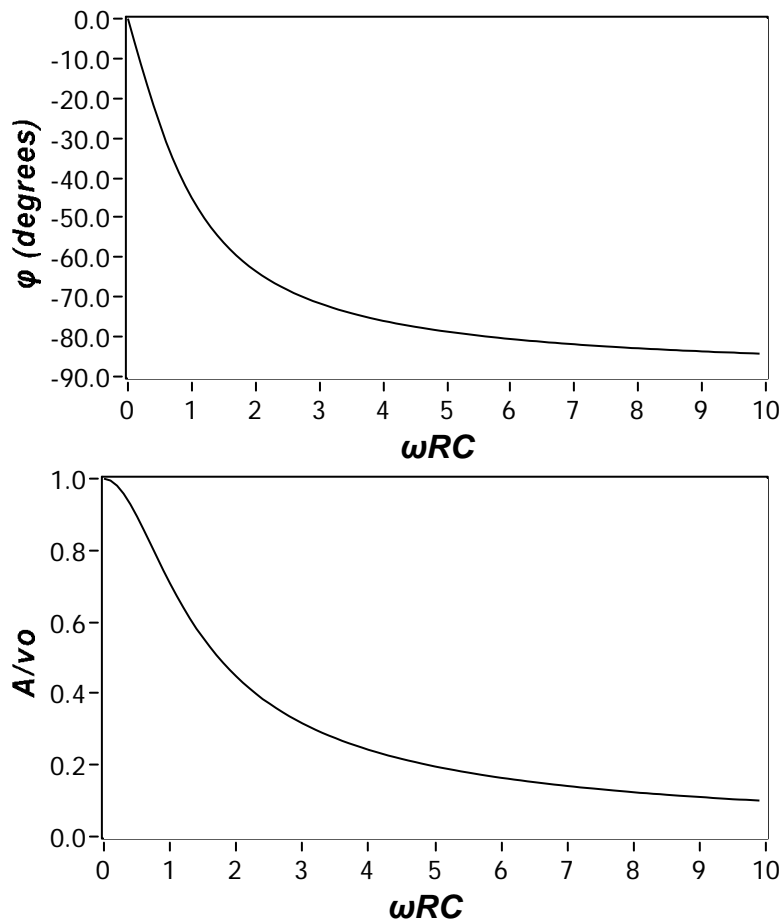


Figure 3

The voltage across the resistor,  $v_R(t)$ , may also be determined by calculating the current  $i(t)$  and multiplying it by  $R$ .

$$i(t) = C \frac{dv_c(t)}{dt} = \frac{v_o C \omega}{\cos \phi - \omega RC \sin \phi} \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \quad (1.19)$$

$$v_R(t) = i(t)R = \frac{v_o \omega RC}{\cos \phi - \omega RC \sin \phi} \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \quad (1.20)$$

Figure 4 shows the plot of amplitude ratio  $v_R / v_o$  as a function of the dimensionless parameter  $\omega RC$ . Here we see the complementary behavior to that shown on Figure 3. Observe again the frequency selectivity of this system. If the output is thus taken across the resistor it passes “high” frequencies while it attenuates “lower” frequencies.

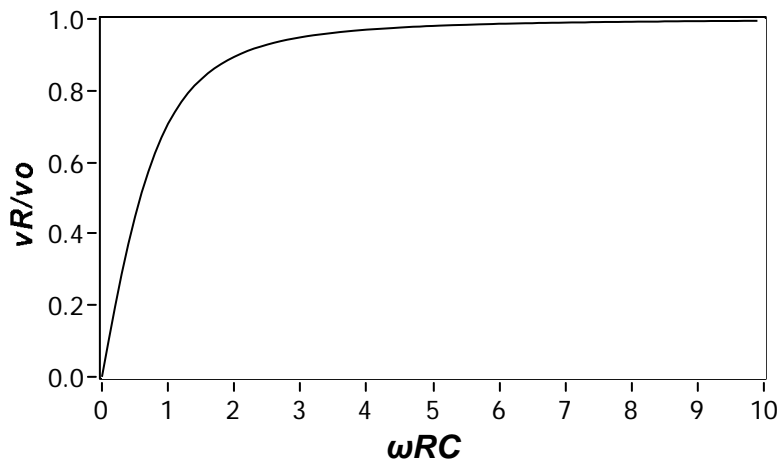


Figure 4.

Now let's look at a few frequencies of interest

1. For  $\omega = 0$  (dc signal) the phase  $\phi = 0$  and the amplitude  $A = v_o$ . The voltage across the capacitor is constant.

No current flows in the circuit

The capacitor behaves as an open circuit.

The circuit equivalent when  $\omega = 0$  is shown on Figure 5.

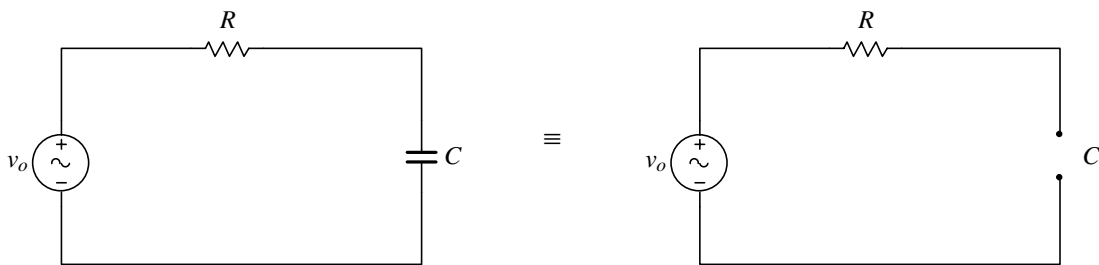
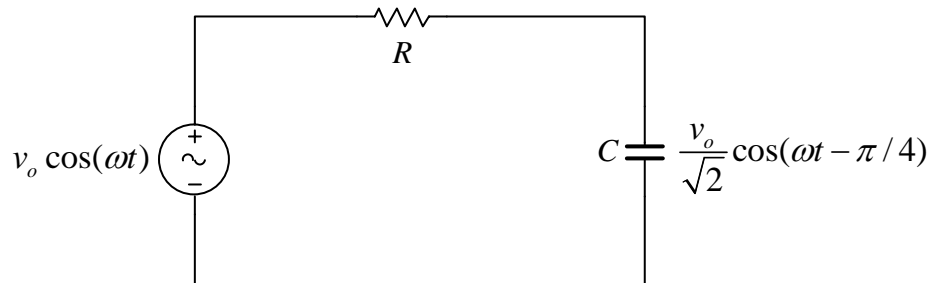


Figure 5

2. For  $\omega RC = 1$

The phase  $\phi = -45^\circ$  and the amplitude  $A = \frac{v_o}{1/\sqrt{2} + 1/\sqrt{2}} = \frac{v_o}{\sqrt{2}}$



Here current is flowing and thus some power is dissipated in  $R$ . Also the capacitor stores some energy.

3. For  $\omega RC \gg 1$

The phase  $\phi = -90^\circ$  and the amplitude  $A = \frac{v_o}{0 - \omega RC(-1)} = \frac{v_o}{\omega RC} \rightarrow 0$

As  $\omega$  increases more power is dissipated in the resistor  $R$ .

When  $\omega \rightarrow \infty$  the capacitor acts as a short circuit and all power is dissipated in  $R$ .

The circuit equivalent when  $\omega \rightarrow \infty$  is shown on Figure 6.

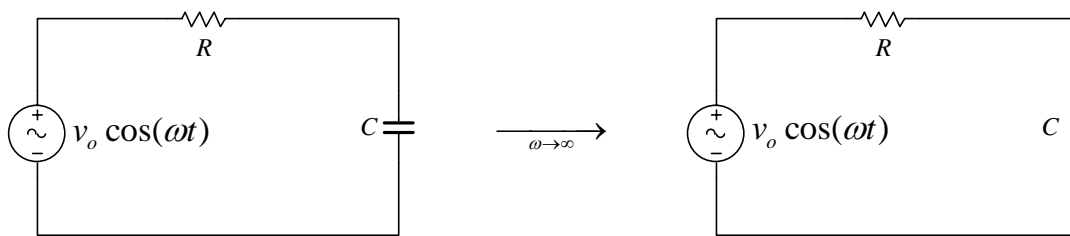
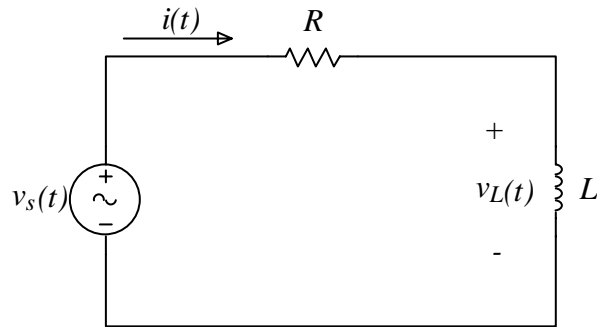


Figure 6

Now let's consider the RL circuit



We would like to characterize this circuit by obtaining the current  $i(t)$  and the voltage  $v_L(t)$ . Apply KVL around the mesh

$$v_s(t) = i(t)R + L \frac{di(t)}{dt} \quad (1.21)$$

With  $v_s(t)$  of the form  $v_s(t) = v_o \cos(\omega t)$ , Equation (1.21) becomes

$$\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{v_o}{L} \cos(\omega t) \quad (1.22)$$

The coefficient  $\frac{L}{R}$  is a time constant.  $\frac{\text{Henry}}{\text{Ohm}} \rightarrow \text{seconds}$

Here again we are interested in the behavior of the system for times that are long compared to the time constant  $\frac{L}{R}$ . In this scenario the only contribution to the solution is again the one forced by the sinusoidal source voltage  $v_s(t) = v_o \cos(\omega t)$ .

And the form of the forced response solution is

$$\begin{aligned} i(t) &= A \cos(\omega t + \phi) \\ &= A \cos \phi \cos(\omega t) - A \sin \phi \sin(\omega t) \end{aligned} \quad (1.23)$$

$$\frac{di(t)}{dt} = -A\omega \cos \phi \sin(\omega t) - A\omega \sin \phi \cos(\omega t) \quad (1.24)$$

Substituting back into the differential equation (1.22) we get



$$\begin{aligned}
 & -A\omega \cos \phi \sin(\omega t) - A\omega \sin \phi \cos(\omega t) + \\
 & \frac{R}{L} [A \cos \phi \cos(\omega t) - A \sin \phi \sin(\omega t)] = \frac{v_0}{L} \cos(\omega t)
 \end{aligned} \tag{1.25}$$

We now separate the cosine and sine functional forms of the solution since they are independent contributions.

The coefficients of the  $\sin(\omega t)$  terms are:

$$-A\omega \cos \phi - \frac{RA}{L} \sin \phi = 0 \tag{1.26}$$

And the coefficients of the  $\cos(\omega t)$  terms are:

$$-A\omega \sin \phi + \frac{RA}{L} \cos \phi = \frac{v_0}{L} \tag{1.27}$$

Equation (1.26) gives us the expression for the phase  $\phi$ .

$$\phi = \tan^{-1} \left( -\frac{\omega L}{R} \right) \tag{1.28}$$

And the amplitude  $A$  becomes

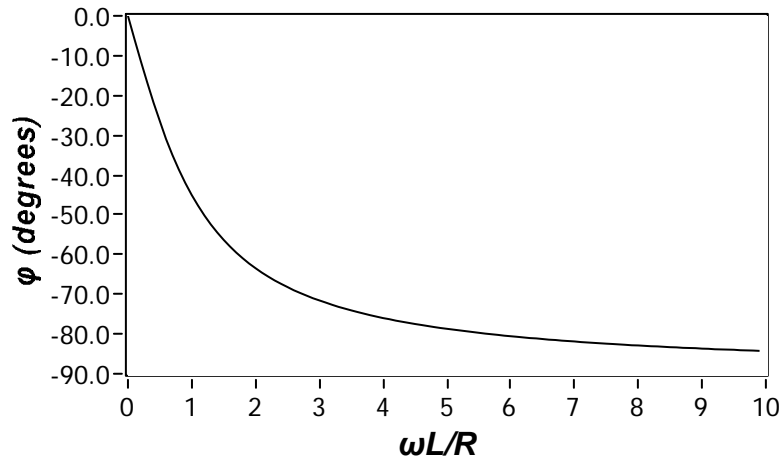
$$A = \frac{v_0 / R}{\cos \phi - \frac{\omega L}{R} \sin \phi} \tag{1.29}$$

And the solution for  $i(t)$  becomes

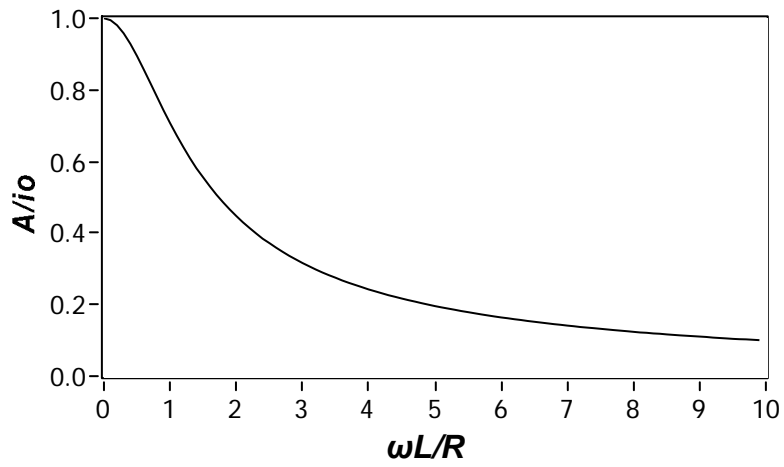
$$i(t) = \frac{v_0 / R}{\cos \phi - \frac{\omega L}{R} \sin \phi} \cos(\omega t + \phi) \tag{1.30}$$

Where the phase  $\phi = \tan^{-1} \left( -\frac{\omega L}{R} \right)$

Figure 7 (a) and (b) show the plots of the phase and the amplitude as a function of the parameter  $\frac{\omega L}{R}$ .



(a)



(b)

Figure 7

Let's look at a few important frequencies as before

1. For  $\omega = 0$  (dc signal) the phase  $\phi = 0$  and the amplitude  $A = v_o / R$ .

The voltage across the inductor is zero and the current flowing in the circuit is  $v_o / R$ .

All power is dissipated in R.

A magnetic field is generated in the inductor but it does not change over time (no change in the magnetic field no voltage)

The inductor behaves as a short circuit as indicated on Figure 8.

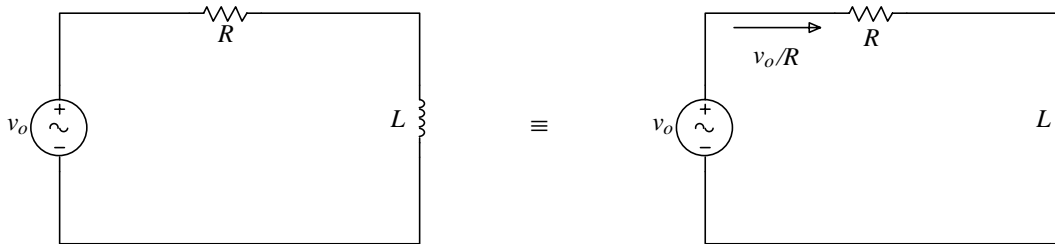
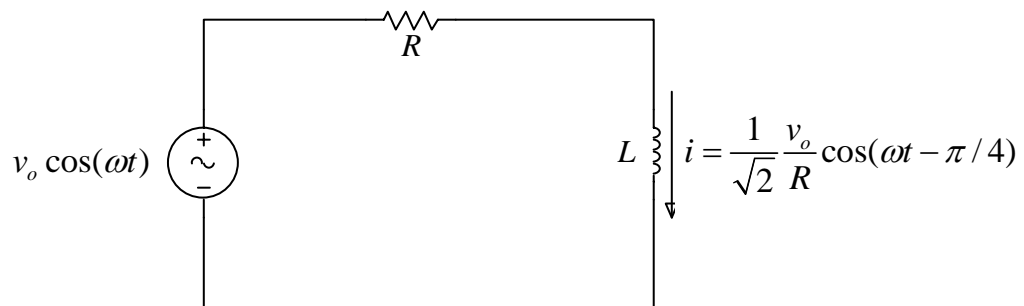


Figure 8

2. For  $\omega L / R = 1$

The phase  $\phi = -45^\circ$  and the amplitude of the current is  $A = \frac{v_o / R}{1/\sqrt{2} + 1/\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{v_o}{R}$

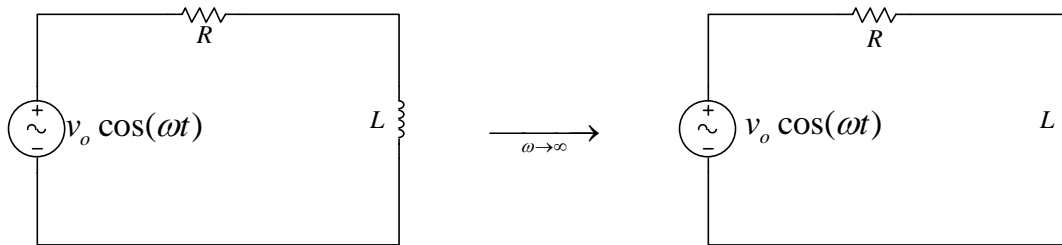


Here current is flowing and thus some power is dissipated in R. Also the inductor stores some energy.

3. For  $\omega L/R \gg 1$

The phase  $\phi = -90^\circ$  and the amplitude  $A = \frac{v_o/R}{0 - \frac{\omega L}{R}(-1)} = \frac{v_o/R}{\frac{\omega L}{R}} \rightarrow 0$

As  $\omega$  increases the current decreases and as  $\omega \rightarrow \infty$  the inductor acts as an open circuit (see Figure 9).



**Figure 9**

## Using the complex forcing function

Our goal is to be able to analyze RC and RL circuits without having to every time employ the differential equation method, which can be cumbersome. If we draw upon our current understanding of RC and RL networks and the fact that they represent linear systems we will be able to considerably simplify the mathematical steps involved in the computation. This simplification will require the use of fundamental complex arithmetic and will in the end reduce the differential equations into simple algebraic equations. (indeed a simplification using complex numbers!)

The linearity of the system implies that if we use a forcing function of the form  $v_o \cos(\omega t + \theta)$  then the output will have the same frequency but with a different phase  $A \cos(\omega t + \phi)$ . Also if we were to scale the source by a factor  $k$  then the output will be scaled by the same factor. Figure 10 graphically demonstrates these two statements.

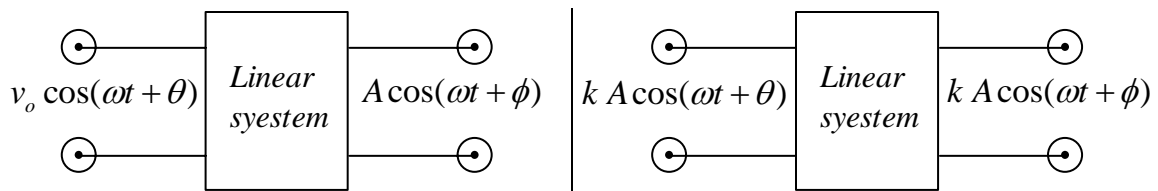


Figure 10

If the factor  $k$  is an imaginary number like  $j = \sqrt{-1}$ , then linearity still holds as we graphically demonstrate on Figure 11.

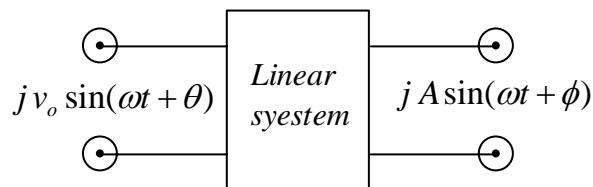


Figure 11

Therefore by superposition a forcing function of the form

$$v_o \cos(\omega t) + j v_o \sin(\omega t) \quad (1.31)$$

Will produce a response of the form

$$A \cos(\omega t + \phi) + j A \sin(\omega t + \phi) \quad (1.32)$$

By using Euler's identity Equations (1.31) and (1.32), the forcing function and the corresponding response, become respectively

$$v_o e^{j(\omega t)} \quad (1.33)$$

and

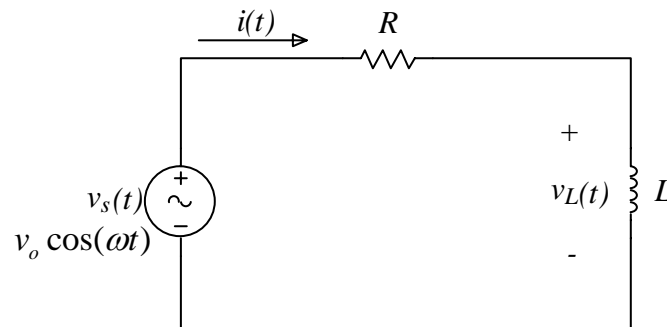
$$A e^{j(\omega t + \phi)} \quad (1.34)$$

Therefore we could employ the complex form of the forcing function, proceed with the development of the solution and then extract the desirable part of the response depending on whether the forcing function was a cosine or a sine function.

So what is the advantage of this method?

1. Very easy to perform algebra with the exponentials
2. Reduces differential equations to algebraic equations.

Let's explore this method by considering again the RL circuit analyzed previously and shown again on Figure 12. The source has the form  $v_s(t) = v_o \cos(\omega t)$ .



**Figure 12**

Since  $v_o \cos(\omega t)$  is the real part of  $v_o e^{j\omega t}$  we will proceed with the analysis and at the end we will simply extract the real part of the solution.

The equation characterizing the system is

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{v_s}{L} \quad (1.35)$$

If we use the complex source

$$v_o e^{j\omega t} \quad (1.36)$$

Then the corresponding complex response is

$$Ae^{j(\omega t + \phi)} \quad (1.37)$$

Equation (1.35) becomes

$$j\omega Ae^{j\phi} + A\frac{R}{L}e^{j\phi} = \frac{v_o}{L} \quad (1.38)$$

which upon simplification becomes

$$Ae^{j\phi}(R + j\omega L) = v_o \quad (1.39)$$

Continue with more simplification we obtain

$$Ae^{j\phi} = \frac{v_o / R}{1 + j\omega L / R} \quad (1.40)$$

And by writing the complex number in polar form we have

$$Ae^{j\phi} = \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} e^{j\left(\tan^{-1}\left(-\frac{\omega L}{R}\right)\right)} \quad (1.41)$$

Where the amplitude of the current is

$$A = \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \quad (1.42)$$

And the phase is

$$\phi = \tan^{-1}\left(-\frac{\omega L}{R}\right) \quad (1.43)$$

The complete complex response of the system is

$$\frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} e^{j\left(\tan^{-1}\left(-\frac{\omega L}{R}\right)\right)} e^{j\omega t} \quad (1.44)$$

Since our forcing source was the cosine function, all we need to do is extract the real part of the function given by Equation (1.44) which is

$$i(t) = \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \cos\left(\omega t + \tan^{-1}\left(-\frac{\omega L}{R}\right)\right) \quad (1.45)$$

Which is the same as Equation (1.30).