Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 2, April 21, 2010

ANSWER BOOKLET

Your Full Name:	SOLUTIONS
Recitation Time :	o'clock

- This quiz is **closed book**, but **3** sheet of notes are allowed. Calculators will not be necessary and are not allowed.
- Check that this **ANSWER BOOKLET** has pages numbered up to 14. The booklet contains spaces for **all** relevant reasoning and answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **4 problems, weighted as shown**. (The points indicated on the following pages for the various subparts of the problems are our best guess for now, but may be modified slightly when we get to grading.)

Problem	Your Score
1 (6 points)	
2 (15 points)	
3 (15 points)	
4 (14 points)	
Total (50 points)	

Problem 1 (6 points)

For each of the following statements, specify whether the statement is **true** or **false** and give a brief (at most a few lines) justification or counterexample.

(1a) (1.5 points) If X and Y are independent random variables, the unconstrained MMSE estimator $\widehat{Y}$ of Y given $X = x$ is $\widehat{Y} = \mu_Y$.

True: The unconstrained MMSE estimator takes the form $E[Y|X] = \int y f_{Y|X} dy$. Since Y and X are independent, we know that $f_{Y|X} = f_Y$. This implies $E[Y|X] = \mu_Y = \int y f_Y dy$.

(1b) (1.5 points) The following random process is strict sense stationary: $x(t) = A$ where A is a continuous random variable with a pdf uniform between ± 1.

True: Intuitively, you can conclude that the process $x(t)$ is strict sense stationary because there is no way to tell where on the time axis you are by looking at any set of samples, no matter at what times they are taken. Mathematically, this intuition is captured by saying that PDF's of all orders are shift invariant, or $f_{x(t_1),x(t_2),\ldots,x(t_N)} = f_{x(t_1+\tau),x_(t_2+\tau),\ldots,x(t_N+\tau)}$ for all N and τ.

(1c) (1.5 points) The following random process is ergodic in the mean: $x(t) = A$ where A is a continuous random variable with a pdf uniform between ± 1.

False: The process is not ergodic in the mean because the ensemble mean does not equal the time-average of a realization of the process $x(t)$. The ensemble mean of the process $x(t)$ is 0. The time-average of a realization of the process $x(t)$ is the particular value of A obtained in that realization

(1d) (1.5 points) If the input to a stable LTI system is WSS then the output is guaranteed to be WSS.

True: If a WSS process $x(t)$ with mean μ_x and autocorrelation function $R_{xx}(\tau)$ is the input to a stable LTI system with frequency response $H(j\omega)$, then the output process has mean $\mu_y = H(j0)\mu_x$ and autocorrelation function $R_{yy}(\tau) = h * \overleftarrow{h} * R_{xx}(\tau)$. Since the output process has a constant mean and autocorrelation function that depends only on the lag τ, the process $y(t)$ is also WSS.

Problem 2 (15 points)

Suppose $x(t)$ and $v(t)$ are two independent WSS random processes with autocorrelation functions respectively $R_{xx}(\tau)$ and $R_{vv}(\tau)$.

(2a) (3 points) Using $x(t)$ and $v(t)$, show how you would construct a random process $g(t)$ whose autocorrelation function $R_{gg}(\tau)$ is that shown in Eq. (2.1). **Be sure to demonstrate that the resulting process $g(t)$ has the desired autocorrelation function**.

$$R_{gg}(\tau) = R_{xx}(\tau)R_{vv}(\tau) \tag{2.1}$$

Solution: Choose $g(t) = x(t)v(t)$. For this choice the autocorrelation function of $g(t)$ is

$$\begin{aligned} R_{gg}(\tau) &= E[x(t+\tau)v(t+\tau)x(t)v(t)] \\ &= E[x(t+\tau)x(t)]E[v(t+\tau)v(t)] \\ &= R_{xx}(\tau)R_{vv}(\tau) \end{aligned}$$

For the remainder of this problem assume that $R_{xx}(\tau)$ and $R_{vv}(\tau)$ are known to be

$$R_{xx}(\tau) = 2e^{-|\tau|} \qquad R_{vv}(\tau) = e^{-3|\tau|} \tag{2.2}$$

You can also invoke the Fourier transform identity

$$e^{-\beta|\tau|} \Longleftrightarrow \frac{2\beta}{\beta^2 + \omega^2} \tag{2.3}$$

(2b) (6 points) Let $w(t)$ denote a third WSS random process with autocorrelation function $R_{ww}(\tau) = \delta(\tau)$. Suppose $w(t)$ is the input to a first-order, stable, causal LTI system for which the linear constant coefficient differential equation is

$$\frac{dx(t)}{dt} + ax(t) = bw(t) \tag{2.4}$$

Determine the values of a and b so that the autocorrelation of the output $x(t)$ is $R_{xx}(\tau)$ as given in Eq. (2.2).

Solution: The transfer function $H(s)$ representing this linear, constant coefficient differential equation is given by

$$H(s) = \frac{b}{s + a}$$

The PSD (function of $j\omega$) and complex PSD (function of s) of the output process $x(t)$ are

$$\begin{aligned} S_{xx}(j\omega) &= \frac{4}{1 + \omega^2} \\ S_{xx}(s) &= \frac{4}{(1 + s)(1 - s)} \end{aligned}$$

We know that the complex PSDs of the input process $w(t)$ and the output process $x(t)$ are related as follows:

$$S_{xx}(s) = H(s)H(-s)S_{ww}(s)$$

Since the complex PSD of the input process $w(t)$ is $S_{ww}(s) = 1$, we can write

$$
\begin{aligned}
S_{xx}(s) &= H(s)H(-s) \\
\frac{4}{(1+s)(1-s)} &= \frac{b^2}{(a+s)(a-s)}
\end{aligned}
$$

From the above we recognize that $b = \pm 2$ and $a = 1$.

(2c) (6 points) Is it possible to have a WSS process $z(t)$ whose autocorrelation function $R_{zz}(\tau) = R_{xx}(\tau) * R_{vv}(\tau)$, i.e., $R_{zz}(\tau)$ is the result of convolving the autocorrelation functions $R_{xx}(\tau)$ and $R_{vv}(\tau)$ given in Eq. (2.2)? If no, why not? If yes, show how to generate $z(t)$ starting from a WSS process $w(t)$ with autocorrelation function $R_{ww}(\tau) = \delta(\tau)$ (you can perform any necessary *causal* LTI filtering operations).

Solution: We would like to form a process $z(t)$ whose autocorrelation function and PSD are

$$\begin{aligned} R_{zz}(\tau) &= R_{xx}(\tau) * R_{vv}(\tau) \\ S_{zz}(j\omega) &= S_{xx}(j\omega) S_{vv}(j\omega) \end{aligned}$$

Suppose we filter the process $w(t)$ through a causal, LTI filter with frequency response $H(j\omega)$. Then we will obtain the following relation between the S_{ww} and S_{zz}

$$\begin{aligned} S_{zz}(j\omega) &= |H(j\omega)|^2 S_{ww}(j\omega) \\ S_{zz}(s) &= H(s)H(-s)S_{ww}(s) \end{aligned}$$

Since the complex PSD of the input process $w(t)$ is $S_{ww}(s) = 1$, we can write

$$\begin{aligned} S_{zz}(s) &= H(s)H(-s) \\ S_{xx}(s)S_{vv}(s) &= H(s)H(-s) \\ \frac{2}{(1+s)(1-s)} \frac{6}{(3+s)(3-s)} &= H(s)H(-s) \end{aligned}$$

Associating with $H(s)$ the poles in the left half plane we obtain

$$H(s) = \frac{2\sqrt{3}}{(1+s)(3+s)}$$

Problem 3 (15 points)

A DT wide-sense stationary random process $s[n]$ has zero mean and the autocovariance function $C_{ss}[m]$ shown in Figure 3.1. $C_{ss}[m] = 0$ for $|m| \geq 5$.

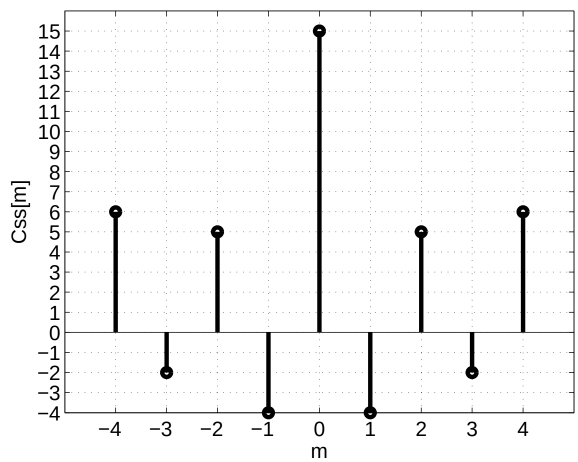

Figure 3.1

The process $s[n]$ is transmitted through a communication channel that introduces noise and a one-step delay. The received process $r[n]$ is related to $s[n]$ as indicated by Eq. (3.1). The noise process $w[n]$ has zero mean, autocovariance function $C_{ww}[m] = 3\delta[m]$. Furthermore, the noise process $w[\cdot]$ is uncorrelated with the process $s[\cdot]$.

$$r[n] = s[n-1] + w[n] \tag{3.1}$$

We wish to design an estimator $\widehat{s}[n]$ for $s[n]$. Throughout this problem the objective is to choose the parameters in the estimator to minimize the mean squared error $\mathcal{E}$ defined as

$$\mathcal{E} \triangleq E\{(s[n] - \widehat{s}[n])^2\}$$

(3a) (2 points) Express the covariance functions $C_{rs}[m]$ and $C_{rr}[m]$ in terms of the given autocovariance functions $C_{ss}[m]$ and $C_{ww}[m]$?

Solution: Since both $s[n]$ and $w[n]$ are zero mean, auto and cross covariance functions will equal auto and cross correlation functions.

$$
\begin{aligned}
C_{rs}[m] &= R_{rs}[m] \\
&= E[r[n+m]s[n]] \\
&= E[(s[n+m-1]+w[n+m])s[n]] \\
&= R_{ss}[m-1] \\
&= C_{ss}[m-1]
\end{aligned}
$$

$$
\begin{aligned}
C_{rr}[m] &= R_{rr}[m] \\
&= E[r(n+m)r(n)] \\
&= E[(s[n+m-1]+w[n+m])(s[n-1]+w[n])] \\
&= R_{ss}[m]+R_{ww}[m] \\
&= C_{ss}[m]+C_{ww}[m]
\end{aligned}
$$

(3b) (3 points) If the estimator is restricted to be of the form

$$\widehat{s}[n] = a_0 r[n] + a_1 r[n-1]$$

will $\mathcal{E}$ be minimized with the parameter values $a_0 = 0$ and $a_1 = 5$?

Circle: Yes/No

Clearly show how you arrived at your conclusion.

Solution: No. If $\widehat{s}[n]$ minimizes $\mathcal{E}$, then the estimation error must be orthogonal to both the data samples $r[n]$ and $r[n-1]$. We will show that orthogonality is not satisfied for the choice $a_0 = 0$ and $a_1 = 5$.

$$\begin{aligned} E[(s[n] - \widehat{s}[n])r[n]] \\ E[(s[n] - a_0 r[n] - a_1 r[n-1])r[n]] \\ E[(s[n] - a_1 r[n-1])r[n]] \\ R_{sr}[0] - a_1 R_{rr}[1] \\ -4 - 5(-4) \neq 0 \end{aligned}$$

Alternatively, note that if $a_0 = 0$ in the minimum, then a_1 must be the coefficient that minimizes the MSE when we use an estimator of the form $\widehat{s}[n] = a_1 r[n-1]$, but then we would need

$$a_1 = \frac{C_{sr}[1]}{C_{rr}[0]} = \frac{C_{ss}[-2]}{C_{rr}[0]} = \frac{5}{18}$$

(3c) (5 points) If the estimator is restricted to be of the form

$$\widehat{s}[n] = a_2 r[n]$$

determine the value of a_2 that minimizes $\mathcal{E}$.

Solution: The value of a_2 that minimizes $\mathcal{E}$ is given by

$$\begin{aligned} a_2 &= \frac{C_{sr}[0]}{C_{rr}[0]} \\ &= \frac{R_{sr}[0]}{R_{ss}[0] + R_{ww}[0]} = \frac{-2}{9} \end{aligned}$$

(3d) (5 points) If the estimator is of the form

$$\widehat{s}[n] = \frac{1}{2} r[n - n_o]$$

with $n_o \geq 0$, determine the value of n_o that minimizes $\mathcal{E}$.

Solution: Express the estimation error $\mathcal{E}$ in terms of n_o

$$\begin{aligned}
\mathcal{E} &= E[(s[n] - \frac{1}{2} r[n - n_o])^2] \\
&= E[s^2[n] - s[n]r[n - n_o] + \frac{1}{4} r[n - n_o]^2] \\
&= R_{ss}[0] - R_{sr}[n_o] + \frac{1}{4} R_{rr}[0] \\
&= R_{ss}[0] - R_{rs}[-n_o] + \frac{1}{4} R_{rr}[0] \\
&= R_{ss}[0] - R_{ss}[n_0 + 1] + \frac{1}{4} R_{rr}[0]
\end{aligned}$$

To minimize $\mathcal{E}$, we need to choose n_o so that $R_{ss}[n_0 + 1]$ is as large as possible and $n_o \geq 0$. Looking at Figure 3.1, we see that $n_0 = 3$ minimizes $\mathcal{E}$ while satisfying the constraint $n_o \geq 0$.

Note that the choice of $n_o = 3$ minimizes $\mathcal{E}$, but does not result in an estimation error that is orthogonal to $r[n-3]$. The reason we lost orthogonality is because $\widehat{s}[n] = \frac{1}{2} r[n-3]$ is not the LMMSE estimator of $s[n]$ given $r[n-3]$. The LMMSE estimator of $s[n]$ given $r[n-3]$ is $\widehat{s}[n] = \frac{C_{sr}[3]}{C_{rr}[0]} r[n-3] = \frac{1}{3} r[n-3]$.

Problem 4 (14 points)

Note: Nothing on this problem assumes or requires that you attended today's lecture.

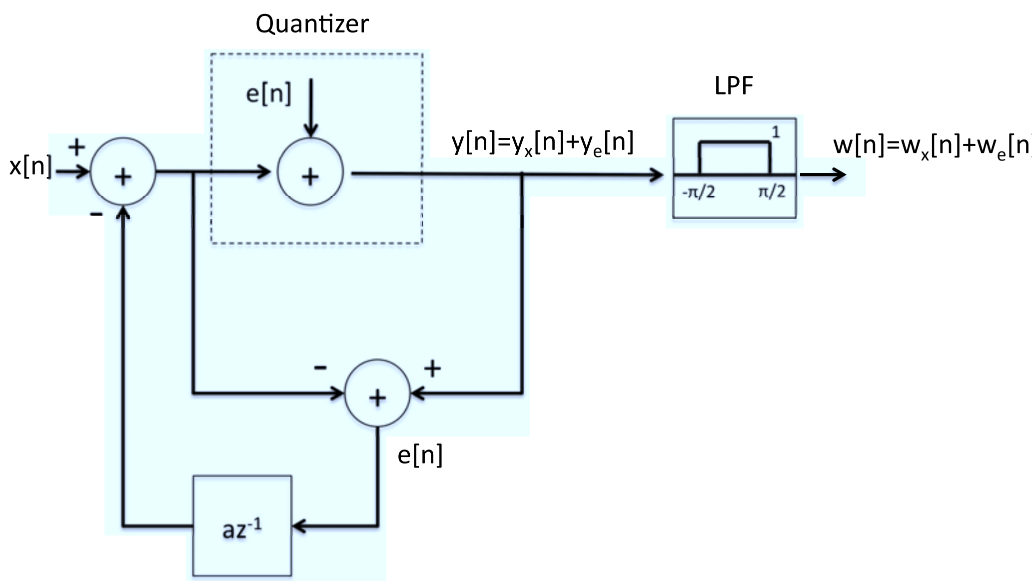

Figure 4.1

D/A converters utilizing oversampled noise shaping often use the structure in Figure 4.1 to quantize the output sequence to a 1-bit bitstream for which D/A conversion is then very simple. Its effectiveness assumes that $x[n]$ is highly oversampled and that most of the error introduced by the quantizer falls outside the signal band.

In Figure 4.1 the quantizer error is modeled as an additive, zero mean i.i.d sequence $e[n]$ that is uniformly distributed in amplitude between $\pm\frac{\triangle}{2}$ at each time instant n. Furthermore, $e[n]$ and $x[n]$ are assumed to be independent. The input $x[n]$ has mean $\mu_x = 0$ and variance $\sigma_x^2 = 1$. Moreover, the input $x[n]$ is bandlimited, i.e., the power spectral density $S_{xx}(e^{j\Omega})$ is zero for $\frac{\pi}{2} < |\Omega| < \pi$.

The output $y[n]$ is composed of two components: $y_x[n]$ due solely to $x[n]$, and $y_e[n]$ due solely to $e[n]$. Similarly, $w[n] = w_x[n] + w_e[n]$. Note that the transfer function from $x[n]$ to

$y_x[n]$ is unity, and the transfer function from $e[n]$ to $y_e[n]$ is $(1 - az^{-1})$.

(4a) (7 points) Determine the power spectral density of $y_e[n]$ as a function of Ω for $|\Omega| < \pi$.

Solution: We know that the PSD of $e[n]$ and the PDS of $y_e[n]$ are related as follows

$$S_{y_e y_e}(e^{j\Omega}) = H_{y_e}(e^{j\Omega}) H_{y_e}(e^{-j\Omega}) S_{ee}(e^{j\Omega})$$

Substituting $H_{y_e}(e^{j\Omega}) = (1 - ae^{-j\Omega})$ we get

$$\begin{aligned} S_{y_e y_e}(e^{j\Omega}) &= (1 - ae^{-j\Omega})(1 - ae^{j\Omega}) S_{ee}(e^{j\Omega}) \\ S_{y_e y_e}(e^{j\Omega}) &= (1 + a^2 - 2a\cos(\Omega)) \frac{\triangle^2}{12} \end{aligned}$$

(4b) (7 points) Determine $E\{w_x^2[n]\}$, $E\{w_e^2[n]\}$, and the value of the gain constant "a" that maximizes the signal-to-noise ratio (SNR) of $w[n]$ defined as

$$SNR = \frac{E\{w_x^2[n]\}}{E\{w_e^2[n]\}}$$

Solution:

$$\begin{aligned}
E\{w_x^2[n]\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{w_x w_x}(e^{j\Omega})\, d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LPF}(e^{j\Omega})\right|^2 S_{y_x y_x}(e^{j\Omega})\, d\Omega \\
&= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} S_{y_x y_x}(e^{j\Omega})\, d\Omega \\
&= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} S_{xx}(e^{j\Omega})\, d\Omega \\
&= \sigma_x^2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
E\{w_e^2[n]\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{w_e w_e}(e^{j\Omega})\, d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LPF}(e^{j\Omega})\right|^2 S_{y_e y_e}(e^{j\Omega})\, d\Omega \\
&= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} S_{y_e y_e}(e^{j\Omega})\, d\Omega \\
&= \frac{\triangle^2}{24\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1 + a^2 - 2a\cos(\Omega)\, d\Omega \\
&= \frac{\triangle^2}{24\pi}\left(\pi(1+a^2) - 4a\right)
\end{aligned}$$

The SNR is given by

$$SNR = \frac{1}{\frac{\triangle^2}{24\pi}\left(\pi(1+a^2) - 4a\right)}$$

To maximize the SNR we select the gain constant a that minimizes the expression $\pi(1+a^2)-4a$. Differentiating with respect to a and setting the derivative equal to 0 results in $a=\frac{2}{\pi}$. We know the value of $a=\frac{2}{\pi}$ minimizes the denominator because the second derivative of $\pi(1+a^2)-4a$ is greater than zero for this choice of a.