LECTURE 23

Last time:

- Finite-state channels

- Lower capacity and upper capacities
- Indecomposable channels

- Markov channels

Lecture outline

- Spreading over fading channels

- Channel model

- Upper bound to capacity

- Interpretation

Spreading over fading channels

Channel decorrelates in time T

Channel decorrelates in frequency W

Recall Markov channels: difficulty arises when we do not know the channel

Gilbert-Eliot channel: hypothesis testing for what channel state is

Consider several channels in parallel in frequency (recall that if channels are known, we can water-fill)

Channel model

Block fading in bandwidth and in time

Over each coherence bandwidth of size W, the channel experiences Rayleigh flat fading

All the channels over distinct coherence bandwidths are independent, yielding a block-fading model in frequency

We transmit over μ coherence bandwidths

The energy of the propagation coefficient $F[i]_j$ over coherence bandwidth i at sampled time j is σ_F

For input $X[i]_j$ at sample time j (we sample at the Nyquist rate W), the corresponding output is $Y[i]_j = F[i]_j X[i]_j + N[i]_j$, where the $N[i]_j$s are samples of WGN bandlimited to a bandwidth of W, with energy normalized to 1

Channel model

The time variations are block-fading in nature

The propagation coefficient of the channel remains constant for T symbols (the coherence interval), then changes to a value independent of previous values

Thus, $\underline{F}[i]_{jTW+1}^{(j+1)TW}$ is a constant vector and the $\underline{F}[i]_{jTW+1}^{(j+1)TW}$ are mutually independent for $j = 1, 2, \ldots$.

Signal constraints:

- For the signals over each coherence bandwidth, the second moment is upper bounded by $E[X^2] \leq \frac{\mathcal{E}}{\mu}$

- The amplitude is upper bounded by $\frac{\gamma}{\mathcal{E}}\sqrt{E[X^2]}$

Upper bound to capacity

Capacity is $C\left(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma\right)$

$$C\left(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma\right) = \lim_{k \to \infty} \max_{p_X}$$

$$\left(\frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{\mu} \frac{1}{T} I\left(\underline{X}[j]_{iTW+1}^{(i+1)TW} ; \underline{Y}[j]_{iTW+1}^{(i+1)TW} \right) \right)$$

$$(1)$$

where the fourth central moment of $X[j]_i$ is upper bounded by $\frac{\gamma}{\mu^2}$ and its average energy constraint is $\frac{\mathcal{E}}{\mu}$.

Since we have no sender channel side information and all the bandwidth slices are independent, we may use the fact that mutual information is concave in the input distribution to determine that selecting all the inputs to be IID maximizes the RHS of (1).

Upper bound to capacity

We first rewrite the mutual information term:

$$\frac{1}{T} I \left(\underline{X}[j]_{iTW+1}^{(i+1)TW} ; \underline{Y}[j]_{iTW+1}^{(i+1)TW} \right)$$

$$= \frac{1}{T} h \left(\underline{Y}[j]_{iTW+1}^{(i+1)TW} \right)$$

$$- \frac{1}{T} h \left(\underline{Y}[j]_{iTW+1}^{(i+1)TW} | \underline{X}[j]_{iTW+1}^{(i+1)TW} \right) \quad (2)$$

We may upper bound the first term of (2):

$$\frac{1}{T} h \left(\underline{Y}[j]_{iTW+1}^{(i+1)TW} \right)$$

$$\leq \frac{1}{2T} \ln \left((2\pi e)^{TW} \left| \bigwedge_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}} \right| \right)$$

Gaussian distribution bound

$$\leq \frac{1}{2T} \ln \left((2\pi e)^{TW} \prod_{i=1}^{TW} \left(\sigma_F^2 \sigma_{X[i]}^2 + 1 \right) \right)$$

from Hadamard's inequality

$$= \frac{W}{2} \ln (2\pi e) + \frac{1}{2T} \sum_{i=1}^{TW} \ln \left(\sigma_F^2 \sigma_{X[i]}^2 + 1 \right)$$

$$\leq \frac{W}{2} \ln (2\pi e) + \frac{W}{2} \ln \left(\sigma_F^2 \frac{\mathcal{E}}{\mu} + 1 \right) \quad (3)$$

from concavity of ln and our average energy constraint.

Upper bound to capacity

We now proceed to minimize the second term of (1)

Conditioned on $\underline{X}[j]_{iTW+1}^{(i+1)TW}$, $\underline{Y}[j]_{iTW+1}^{(i+1)TW}$ is Gaussian, since $\underline{F}[j]_{iTW+1}^{(i+1)TW}$ is Gaussian and $\underline{N}_{iTW+1}^{(i+1)TW}$ Gaussian and independent of F

$$\frac{1}{T} h \left(\underline{Y}[j]_{iTW+1}^{(i+1)TW} | \underline{X}[j]_{iTW+1}^{(i+1)TW} \right)$$
$$= \frac{1}{2T} E_{\underline{X}} \left[\ln \left((2\pi e)^T \left| \Lambda_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}} \right| \right) \right]$$
(4)

$\Lambda_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}}$ has k^{th} diagonal term $\sigma_F^2 x[k]^2 + 1$ and off-diagonal (k,j) term equal to $x(k)x(j)\sigma_F^2$, conditioned on

$$\underline{X}[j]_{iTW+1}^{(i+1)TW} = \underline{x} = [x(1), \ldots, x(TW)]$$

The eigenvalues λ_j of $\Lambda_{\underline{Y}}$ are 1 for $j = 1 \ldots TW - 1$ and $||\underline{x}||^2 \sigma_F^2 + 1$ for $j = TW$.

Upper bound to capacity

Hence, we may rewrite (3) as

$$\frac{1}{T} h \left(\underline{Y}_{iTW+1}^{(i+1)TW} \middle| \underline{X}_{iTW+1}^{(i+1)TW} \right)$$

$$= \frac{1}{2T} E_{\underline{X}} \left[\ln \left(\left\| \underline{x}_{iTW+1}^{(i+1)TW} \right\|^2 \sigma_F^2 + 1 \right) \right]$$

$$+ \frac{W}{2} \ln(2\pi e) \tag{5}$$

We seek to minimize the RHS of (5) subject to the second moment constraint holding with equality and the subject to the peak amplitude constraint

The distribution for X which minimizes the RHS of (5) subject to our constraints can be found using the concavity of the ln function

The distribution is such that the only values which $|X|$ can take are 0 and $\frac{\gamma}{\sqrt{\mu \mathcal{E}}}$ with probabilities $1 - \frac{\mathcal{E}^2}{\gamma^2}$ and $\frac{\mathcal{E}^2}{\gamma^2}$, respectively.

Upper bound to capacity

Thus, we may lower bound (5) by

$$\frac{1}{T} h \left(\underline{Y}_{iTW+1}^{(i+1)TW} | \underline{X}_{iTW+1}^{(i+1)TW} \right)$$

$$\geq \quad \frac{\mathcal{E}^2}{2T\gamma^2} \ln \left(TW \frac{\gamma^2}{\mu \mathcal{E}} \sigma_F^2 + 1 \right) + \frac{W}{2} \ln(2\pi e)$$

$$(6)$$

Combining (6), (3) and (1) yields

$$C \left(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma \right) \leq$$

$$\frac{\mu W}{2} \ln \left(\sigma_F^2 \frac{\mathcal{E}}{\mu} + 1 \right)$$

$$- \frac{\mu \mathcal{E}^2}{2T\gamma^2} \ln \left(TW \frac{\gamma^2}{\mu \mathcal{E}} \sigma_F^2 + 1 \right) \qquad (7)$$

Upper bound to capacity

What is the limit for μ infinite?

$\ln(1 + x) \simeq x$ for small x

First term goes to $\dfrac{\sigma_F^2 W \mathcal{E}}{2}$

Second term also goes to $\dfrac{\sigma_F^2 W \mathcal{E}}{2}$

Graphical interpretation

Interpretation

Over any channel (slice of of bandwidth of size W), we do not have enough energy to measure the channel satisfactorily

Necessary assumptions:

- energy scales per bandwidth slice over the whole bandwidth

- peak energy per bandwidth slice over the whole bandwidth

Interpretation

We may relax the assumption of the peak bandwidth

Assume second moment (variance) scales as $\frac{1}{\mu}$ and fourth moment (kurtosis) scales as $\frac{1}{\mu^2}$

The mutual information goes to 0 as $\mu \to \infty$

We may also relax the assumption regarding the channel block-fading in time and frequency as long as we have decorrelation

6.441 Information Theory
Spring 2010