**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**

6.436J/15.085J — Fall 2008

Problem Set 9 — due 11/19/2008

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**Exercise 1.** Let $\{X_n\}$ be a sequence of random variables defined on the same probability space.

(a) Suppose that $\lim_{n\to\infty} \mathbb{E}[|X_n|] = 0$. Show that $X_n$ converges to zero, in probability.

(b) Suppose that $X_n$ converges to zero, in probability, and that for some constant $c$, we have $|X_n| \le c$, for all $n$, with probability 1. Show that

$$\lim_{n\to\infty} \mathbb{E}[|X_n|] = 0.$$

(c) Suppose that each $X_n$ can only take the values 0 and 1 and, that $\mathbb{P}(X_n = 1) = 1/n$.

(i) Given an example in which we **have** almost sure convergence of $X_n$ to 0.

(ii) Given an example in which we **do not have** almost sure convergence of $X_n$ to 0.

**Exercise 2.** Let $X_1, X_2, \ldots$ be a sequence of independent random variables that are uniformly distributed between 0 and 1. For every $n$, we let $Y_n$ be the median of the values of $X_1, X_2, \ldots, X_{2n+1}$. [That is, we order $X_1, \ldots, X_{2n+1}$ in increasing order and let $Y_n$ be the $(n+1)$st element in this ordered sequence.] Show that that the sequence $Y_n$ converges to 1/2, in probability.
**Extra credit:** Prove that $Y_n \overset{\text{a.s.}}{\to} Y$.

**Exercise 3.** Let $X_1, X_2, \ldots$ be continuous random variables with probability density functions (PDFs) $f_{X_1}, f_{X_2}, \ldots$.

(a) Suppose that $\lim_{k\to\infty} f_{X_k}(x) = g(x)$, for all $x \in \Re$. Invoke a certain result on integration to show that $\int_{-\infty}^{\infty} g(x)\,dx \le 1$, and give an example to show that $g$ need not be a PDF.

(b) Suppose that:
(i) $\lim_{k\to\infty} f_{X_k}(x) = f_X(x)$, for all $x \in \Re$, where $f_X$ is the PDF of some random variable $X$, and
(ii) we have $f_{X_k}(x) \le h(x)$, for all $x \in \Re$, where $h$ is a function that satisfies $\int_{-\infty}^{\infty} h(x)\,dx < \infty$.

Show that $X_k$ converges to $X$, in distribution.

*Note:* The result of part (b) is actually true without the boundendness assumption, but the proof is harder.

**Exercise 4.** The following fact is known, and can be used in this problem: if a sequence of normal random variables $X_k$ converges in distribution to a random variable $X$, then $X$ is normal.

Suppose that for every $k$, the pair $(X_k, Y)$ has a bivariate normal distribution. Furthermore, suppose that the sequence $X_k$ converges to $X$, almost surely. Show that $(X, Y)$ has a bivariate normal distribution. *Hint:* Use the "right" definition of the bivariate normal.

**Exercise 5.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. normal random variables, with zero mean and unit variance. The corresponding characteristic function is $\mathbb{E}[e^{itX_1}] = e^{-t^2/2}$. We would like to define a new random variable

$$\sum_{k=1}^{\infty} \frac{X_k}{2^k}.$$

More precisely, we need to consider the finite sum

$$Y_n = \sum_{k=1}^{n} \frac{X_k}{2^k},$$

and investigate whether it converges to a limit in some sense.

(a) Use transforms to prove that $Y_n$ converges in distribution, and to identify the nature of the limit distribution. (Please state the facts that you are using.)

(b) Prove that $Y_n$ converges to a random variable $Y$ which is finite, with probability one. *Hint:* Consider the sum of $|X_k|/2^k$.

**Exercise 6.** (a) Consider two sequences of random variables, $\{X_i\}$ and $\{Y_i\}$. Suppose that $X_i$ converges to $a \in \mathbb{R}$, in probability, and $Y_i$ converges to $b \in \mathbb{R}$, in probability. Show that $X_i + Y_i$ converges to $a + b$, in probability.

(b) Suppose that $\{X_i\}$ is a sequence of independent identically distributed random variables that converges in probability to a random variable $X$. Show that $X$ must be a constant almost surely.