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6.013/ESD.013J Electromagnetics and Applications, Fall 2005

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**Lecture 4: The Scalar Electric Potential and the Coulomb Superposition Integral**

I. Quasistatics

Electroquasistatics (EQS)

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\mu_0 \vec{H}) \approx 0$$

$$\nabla \cdot (\epsilon_0 \vec{E}) = \rho$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}(\epsilon_0 \vec{E})$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Magnetquasistatics (MQS)

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\mu_0 \vec{H})$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}(\epsilon_0 \vec{E})$$

$$\nabla \cdot (\mu_0 \vec{H}) = 0$$

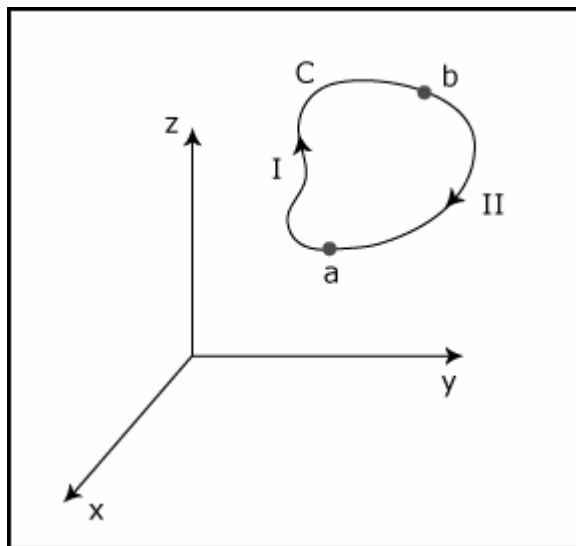
$$\nabla \cdot \vec{J} = 0$$

$$\nabla \cdot (\epsilon_0 \vec{E}) = \rho$$

II. Irrotational EQS Electric Field

1. Conservative Electric Field

$$\oint_c \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_s \mu_0 \vec{H} \cdot d\vec{a} \approx 0$$



$$\oint_C \vec{E} \cdot d\vec{s} = \int_a^b \vec{E} \cdot d\vec{s} + \int_b^a \vec{E} \cdot d\vec{s} = 0 \Rightarrow \underbrace{\int_a^b \vec{E} \cdot d\vec{s}}_{\text{Electromotive Force (EMF)}} = \int_a^b \vec{E} \cdot d\vec{s}$$

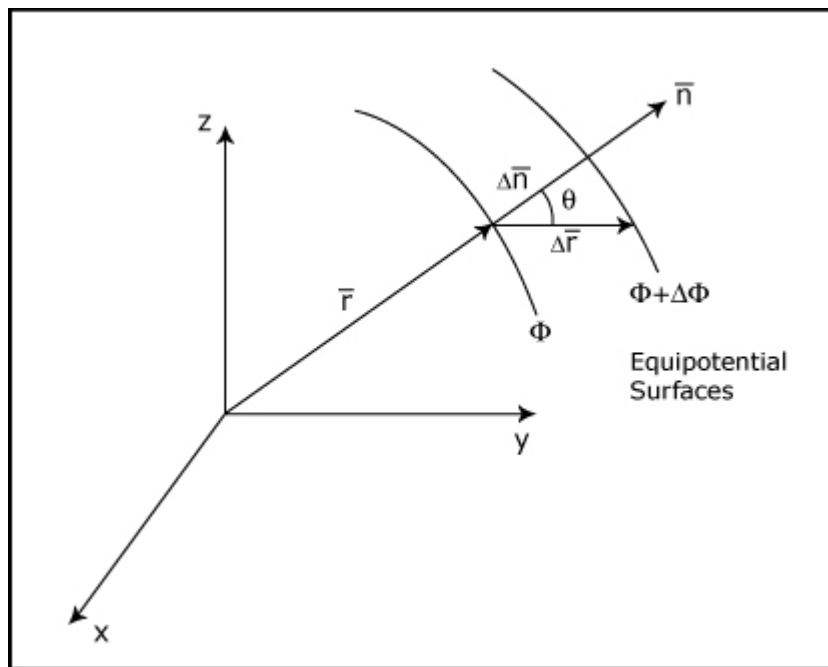
EMF between 2 points (a, b) independent of path  
 $\vec{E}$  field is conservative

$$\Phi(\vec{r}) - \Phi(\vec{r}_{\text{ref}}) = \int_{\vec{r}_{\text{ref}}}^{\vec{r}} \vec{E} \cdot d\vec{s}$$

↗ Scalar  
 electric potential

$$\int_a^b \vec{E} \cdot d\vec{s} = \int_a^{\vec{r}_{\text{ref}}} \vec{E} \cdot d\vec{s} + \int_{\vec{r}_{\text{ref}}}^b \vec{E} \cdot d\vec{s} = \Phi(a) - \Phi(\vec{r}_{\text{ref}}) + \Phi(\vec{r}_{\text{ref}}) - \Phi(b) = \Phi(a) - \Phi(b)$$

## 2. The Electric Scalar Potential



$$\vec{r} = x \vec{i}_x + y \vec{i}_y + z \vec{i}_z$$

$$\Delta \vec{r} = \Delta x \vec{i}_x + \Delta y \vec{i}_y + \Delta z \vec{i}_z$$

$$\Delta n = \Delta r \cos \theta$$

$$\begin{aligned}
\Delta\Phi &= \Phi(x + \Delta x, y + \Delta y, z + \Delta z) - \Phi(x, y, z) \\
&= \Phi(x, y, z) + \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z - \Phi(x, y, z) \\
&= \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z \\
&= \underbrace{\left[ \frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y + \frac{\partial\Phi}{\partial z}\bar{i}_z \right]}_{\text{grad } \Phi = \nabla\Phi} \cdot \Delta\bar{r}
\end{aligned}$$

$$\nabla = \bar{i}_x \frac{\partial}{\partial x} + \bar{i}_y \frac{\partial}{\partial y} + \bar{i}_z \frac{\partial}{\partial z}$$

$$\text{grad } \Phi = \nabla\Phi = \bar{i}_x \frac{\partial\Phi}{\partial x} + \bar{i}_y \frac{\partial\Phi}{\partial y} + \bar{i}_z \frac{\partial\Phi}{\partial z}$$

$$\int_{\bar{r}}^{\bar{r}+\Delta\bar{r}} \bar{E} \cdot d\bar{s} = \Phi(\bar{r}) - \Phi(\bar{r} + \Delta\bar{r}) = -\Delta\Phi = -\nabla\Phi \cdot \Delta\bar{r} = \bar{E} \cdot \Delta\bar{r}$$

$$\boxed{\bar{E} = -\nabla\Phi}$$

$$\Delta\Phi = \frac{\Delta\Phi}{\Delta n} \Delta r \cos\theta = \frac{\Delta\Phi}{\Delta n} \bar{n} \cdot \Delta\bar{r} = \nabla\Phi \cdot \Delta\bar{r}$$

$$\nabla\Phi = \frac{\Delta\Phi}{\Delta n} \bar{n} = \frac{\partial\Phi}{\partial n} \bar{n}$$

The gradient is in the direction perpendicular to the equipotential surfaces.

### III. Vector Identity

$$\nabla \times \bar{E} = 0$$

$$\bar{E} = -\nabla\Phi$$

$$\nabla \times (\nabla\Phi) = 0$$

#### IV. Sample Problem

$$\Phi(x, y) = \frac{V_0 xy}{a^2} \quad (\text{Equipotential lines hyperbolas: } xy = \text{constant})$$

$$\begin{aligned} \bar{E} = -\nabla\Phi &= -\left[ \frac{\partial\Phi}{\partial x} \bar{i}_x + \frac{\partial\Phi}{\partial y} \bar{i}_y \right] \\ &= \frac{-V_0}{a^2} \left( y \bar{i}_x + x \bar{i}_y \right) \end{aligned}$$

Electric Field Lines [lines tangent to electric field]

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \Rightarrow ydy = xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$y^2 - x^2 = y_0^2 - x_0^2$  [lines pass through point  $(x_0, y_0)$ ]  
(hyperbolas orthogonal to  $xy$ )

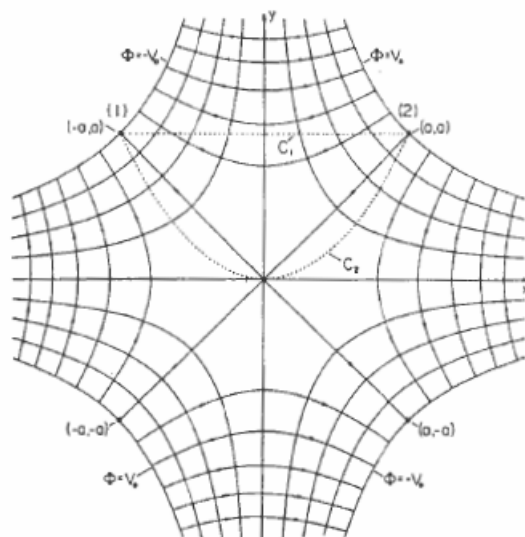


Figure 4.1.3 Cross-sectional view of surfaces of constant potential for two-dimensional potential given by (18).

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

## V. Poisson's Equation

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla\Phi) = \rho/\epsilon_0 \Rightarrow \nabla^2\Phi = -\rho/\epsilon_0$$

$$\begin{aligned}\nabla^2\Phi &= \nabla \cdot (\nabla\Phi) = \left[ \vec{i}_x \frac{\partial}{\partial x} + \vec{i}_y \frac{\partial}{\partial y} + \vec{i}_z \frac{\partial}{\partial z} \right] \cdot \left[ \frac{\partial\Phi}{\partial x} \vec{i}_x + \frac{\partial\Phi}{\partial y} \vec{i}_y + \frac{\partial\Phi}{\partial z} \vec{i}_z \right] \\ &= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

## VI. Coulomb Superposition Integral

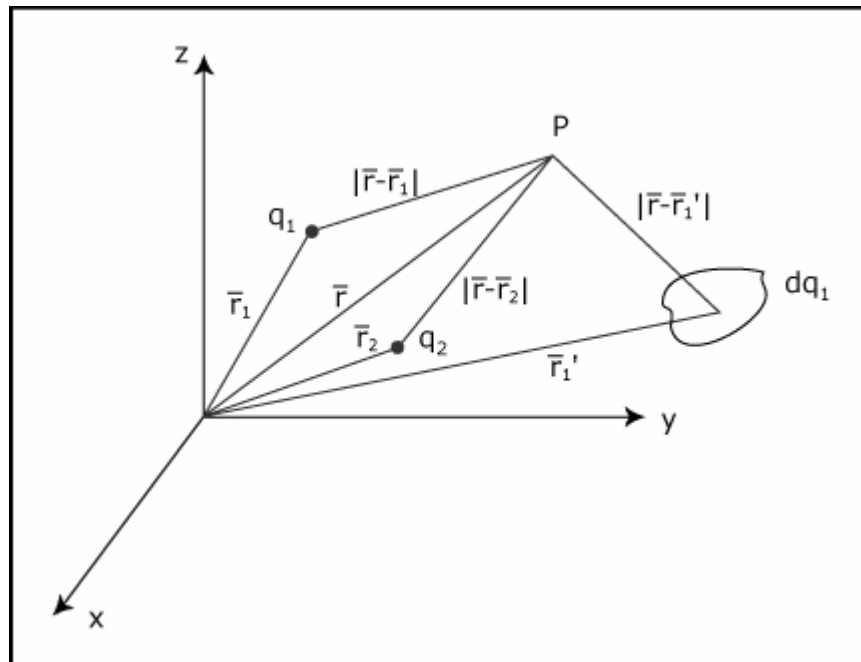
### 1. Point Charge

$$E_r = -\frac{\partial\Phi}{\partial r} = \frac{q}{4\pi\epsilon_0 r^2} \Rightarrow \Phi = \frac{q}{4\pi\epsilon_0 r} + C$$

Take reference  $\Phi(r \rightarrow \infty) = 0 \Rightarrow C = 0$

$$\Phi = \frac{q}{4\pi\epsilon_0 r}$$

### 2. Superposition of Charges



$$d\Phi_T(\mathbf{P}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q_1}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_1|} + \frac{q_2}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_2|} + \dots + \frac{dq_1}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_1'|} + \frac{dq_2}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_2'|} + \dots \right]$$

$$\Phi_T(\mathbf{P}) = \frac{1}{4\pi\epsilon_0} \left[ \sum_{n=1}^N \frac{q_n}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_n|} + \int_{\text{all line, surface, and volume charges}} \frac{dq}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[ \sum_{n=1}^N \frac{q_n}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_n|} + \int_L \frac{\lambda(\bar{\mathbf{r}}') d\mathbf{l}'}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} + \int_S \frac{\sigma_s(\bar{\mathbf{r}}') da'}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} + \int_V \frac{\rho(\bar{\mathbf{r}}') dV'}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \right]$$

Short-hand notation

$$\Phi(\bar{\mathbf{r}}) = \int_V \frac{\rho(\bar{\mathbf{r}}') dV'}{4\pi\epsilon_0 |\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$