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6.641 Electromagnetic Fields, Forces, and Motion, Spring 2005

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Lecture 3: Electroquasistatic and Magnetoquasistatic Fields and Boundary Conditions

I. Conditions for Electroquasistatic Fields

A. Order of Magnitude Estimate [Characteristic Length L , Characteristic time τ]

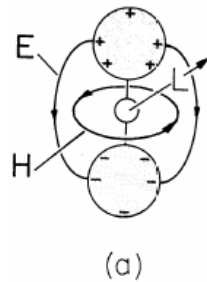


Figure 3.3.1 Prototype systems involving one typical length. (a) EQS system in which source of EMF drives a pair of perfectly conducting spheres having radius and spacing on the order of L .

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\nabla \cdot \bar{E} = \rho/\epsilon \Rightarrow \frac{E}{L} = \rho/\epsilon \Rightarrow E = \frac{\rho L}{\epsilon}$$

$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{H}{L} = \frac{\epsilon E}{\tau} \Rightarrow H = \frac{\epsilon E L}{\tau} = \frac{L^2 \rho}{\tau}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{E_{\text{error}}}{L} = \frac{\mu H}{\tau} = \frac{\mu \rho L^2}{\tau^2} \Rightarrow E_{\text{error}} = \frac{\mu \rho L^3}{\tau}$$

$$\frac{E_{\text{error}}}{E} = \frac{\mu \rho L^3}{\tau \rho L \epsilon} = \frac{\mu \epsilon L^2}{\tau^2} = \frac{L^2}{(c\tau)^2} ; \quad c = \frac{1}{\sqrt{\epsilon \mu}}$$

$$\frac{E_{\text{error}}}{E} \ll 1 \Rightarrow \frac{L}{c\tau} \ll 1$$

B. Estimate of Error introduced by EQS approximation

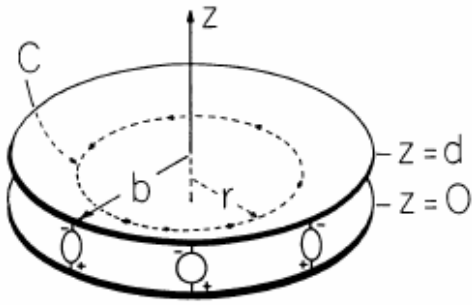


Figure 3.3.2 Plane parallel electrodes having no resistance, driven at their outer edges by a distribution of sources of EMF.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\bar{E} = \frac{V}{d} \bar{i}_z = E_0 \bar{i}_z$$

$$\sigma_{su} = \begin{cases} -\epsilon E_0 & z = d \\ +\epsilon E_0 & z = 0 \end{cases}$$

$$K_r 2\pi b + \pi b^2 \frac{d\sigma_{su}}{dt} = 0 \Rightarrow K_r = -\frac{b}{2} \frac{d\sigma_{su}}{dt} = -\frac{b}{2} \epsilon \frac{dE_0}{dt}$$

$$\oint_C \bar{H} \cdot d\bar{s} = \int_S \frac{\partial}{\partial t} (\epsilon \bar{E}) \cdot d\bar{a} \Rightarrow H_\phi 2\pi r = \pi r^2 \epsilon \frac{dE_0}{dt} \Rightarrow H_\phi = \frac{r}{2} \epsilon \frac{dE_0}{dt}$$

$$\oint_C \bar{E} \cdot d\bar{s} = -\int_S \mu \frac{\partial \bar{H}}{\partial t} \cdot d\bar{a}$$

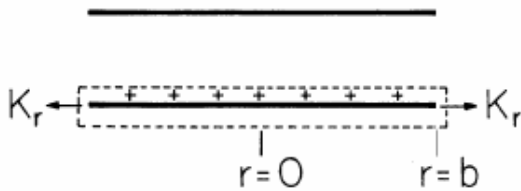


Figure 3.3.3 Parallel plates of Figure 3.3.2, showing volume containing lower plate and radial surface current density at its periphery.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

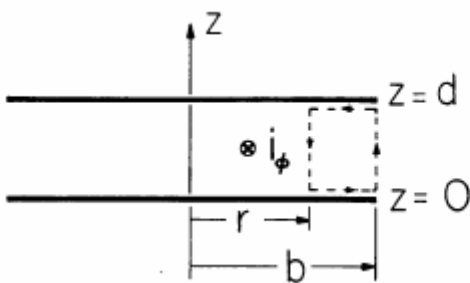


Figure 3.3.4 Cross-section of system shown in Figure 3.3.2 showing surface and contour used in evaluating correction E field.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$[E_z(b) - E_z(r)]d = + \frac{\mu\epsilon}{2} \int_r^b r' dr' d \frac{d^2 E_0}{dt^2}$$

$$= \frac{\mu\epsilon d}{4} (b^2 - r^2) \frac{d^2 E_0}{dt^2}$$

$$E_z(r) = E_0 + \frac{\epsilon\mu}{4} \frac{d^2 E_0}{dt^2} (r^2 - b^2)$$

If $E_0(t) = A \cos \omega t$

$$\frac{|E_{\text{error}}|}{E_0} = \frac{\epsilon\mu}{4E_0} \frac{d^2 E_0}{dt^2} (b^2 - r^2) = \frac{1}{4} \omega^2 \epsilon\mu (b^2 - r^2)$$

$$\frac{|E_{\text{error}}|}{E_0} \ll 1 \Rightarrow \frac{\omega^2 \epsilon\mu b^2}{4} \ll 1$$

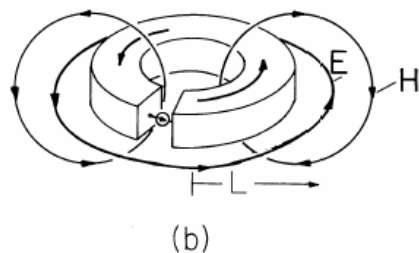
$$f\lambda = c = \frac{1}{\sqrt{\epsilon\mu}}$$

$$\frac{\omega}{2\pi} \lambda = c \Rightarrow \omega = \frac{2\pi c}{\lambda} \Rightarrow \frac{\omega^2 \epsilon\mu b^2}{4} = \frac{\pi^2}{\lambda^2} b^2 \ll 1 \Rightarrow b \ll \frac{\lambda}{\pi}$$

$$f=1 \text{ MHz in free space} \Rightarrow \lambda = \frac{3 \times 10^8}{10^6} = 300 \text{ m}$$

If $b \ll 100 \text{ m}$ EQS approximation is valid.

II. Conditions for Magnetoquasistatic Fields



(b) MQS system consisting of perfectly conducting loop driven by current source. The radius of the loop and diameter of its cross-section are on the order of L .

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\nabla \times \bar{H} = \bar{J} \Rightarrow \frac{H}{L} = J \Rightarrow H = JL$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{E}{L} = \frac{\mu H}{\tau} \Rightarrow E = \frac{\mu HL}{\tau} = \frac{\mu JL^2}{\tau}$$

$$\nabla \times \bar{H}_{\text{error}} = \varepsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{\bar{H}_{\text{error}}}{L} = \frac{\varepsilon E}{\tau} \Rightarrow H_{\text{error}} = \frac{\varepsilon EL}{\tau^2} = \frac{\varepsilon \mu JL^3}{\tau^2}$$

$$\frac{H_{\text{error}}}{H} = \frac{\varepsilon \mu JL^3}{\tau^2 JL} = \frac{\varepsilon \mu L^2}{\tau^2} = \frac{L^2}{(c\tau)^2} \ll 1 \Rightarrow L \ll c\tau$$

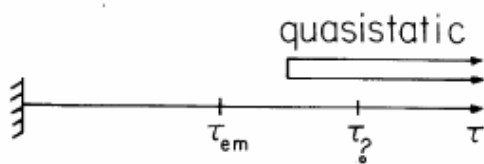


Figure 3.4.1 Range of characteristic times over which quasistatic approximation is valid. The transit time of an electromagnetic wave is τ_{em} while $\tau_?$ is a time characterizing the dynamics of the quasistatic system.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\tau_{em} = \frac{L}{c} = L\sqrt{\varepsilon\mu}$$

III. Boundary Conditions

1. Gauss' Continuity Condition

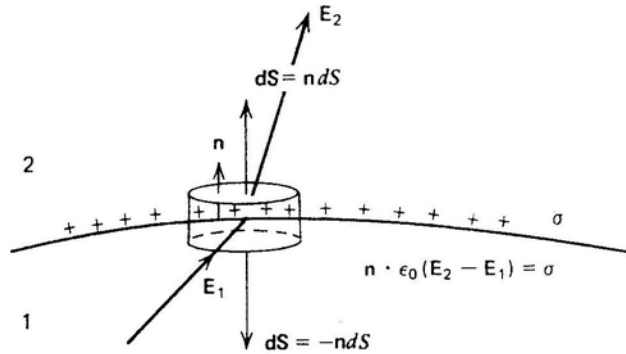


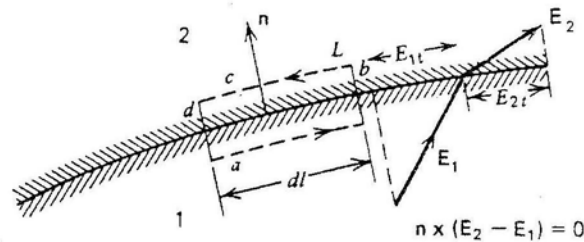
Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\epsilon_0 \mathbf{E}$ is discontinuous in the surface charge density.

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$$\oint_S \epsilon_0 \bar{\mathbf{E}} \cdot \overline{d\mathbf{a}} = \int_S \sigma_s dS \Rightarrow \epsilon_0 (E_{2n} - E_{1n}) dS = \sigma_s dS$$

$$\epsilon_0 (E_{2n} - E_{1n}) = \sigma_s \Rightarrow \bar{\mathbf{n}} \cdot [\epsilon_0 (\bar{\mathbf{E}}_2 - \bar{\mathbf{E}}_1)] = \sigma_s$$

2. Continuity of Tangential $\bar{\mathbf{E}}$



(a)

Figure 3-12 (a) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary.

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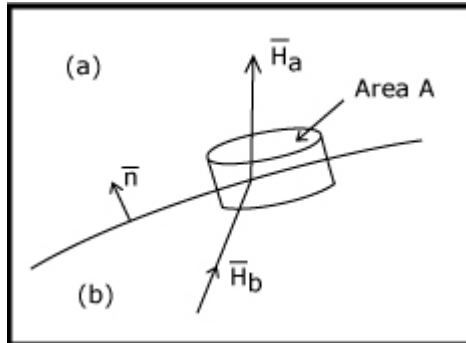
$$\oint_C \bar{\mathbf{E}} \cdot \overline{d\mathbf{s}} = (E_{1t} - E_{2t}) dl = 0 \Rightarrow E_{1t} - E_{2t} = 0$$

$$\bar{\mathbf{n}} \times (\bar{\mathbf{E}}_1 - \bar{\mathbf{E}}_2) = 0$$

Equivalent to $\Phi_1 = \Phi_2$ along boundary

3. Normal \bar{H}

$$\oint_S \mu_0 \bar{H} \cdot d\bar{a} = 0$$



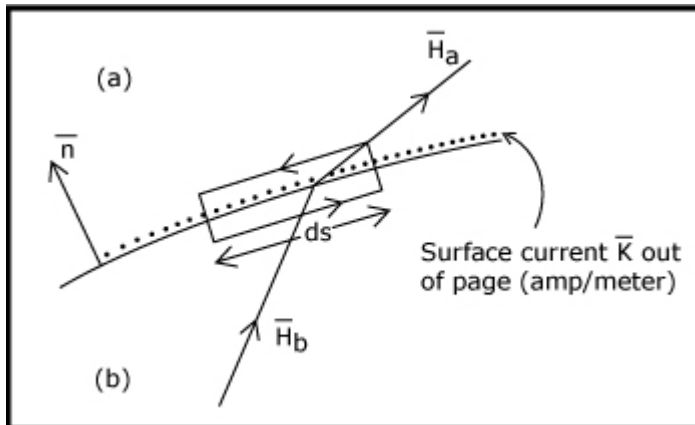
$$\mu_0 (H_{an} - H_{bn}) A = 0$$

$$H_{an} = H_{bn}$$

$$\bar{n} \cdot [\bar{H}_a - \bar{H}_b] = 0$$

4. Tangential \bar{H}

$$\oint_C \bar{H} \cdot d\bar{s} = \int_S \bar{J} \cdot d\bar{a} + \frac{d}{dt} \int_S \epsilon_0 \bar{E} \cdot d\bar{a}$$

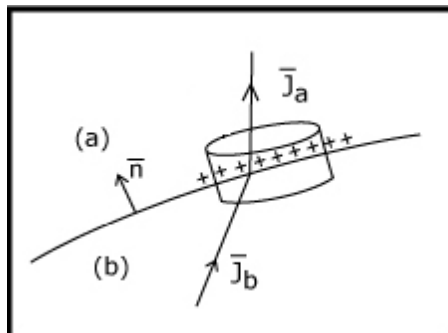


$$H_{bt} ds - H_{at} ds = K ds$$

$$H_{bt} - H_{at} = K$$

$$\bar{n} \times [\bar{H}_a - \bar{H}_b] = \bar{K}$$

5. Conservation of Charge Boundary Condition

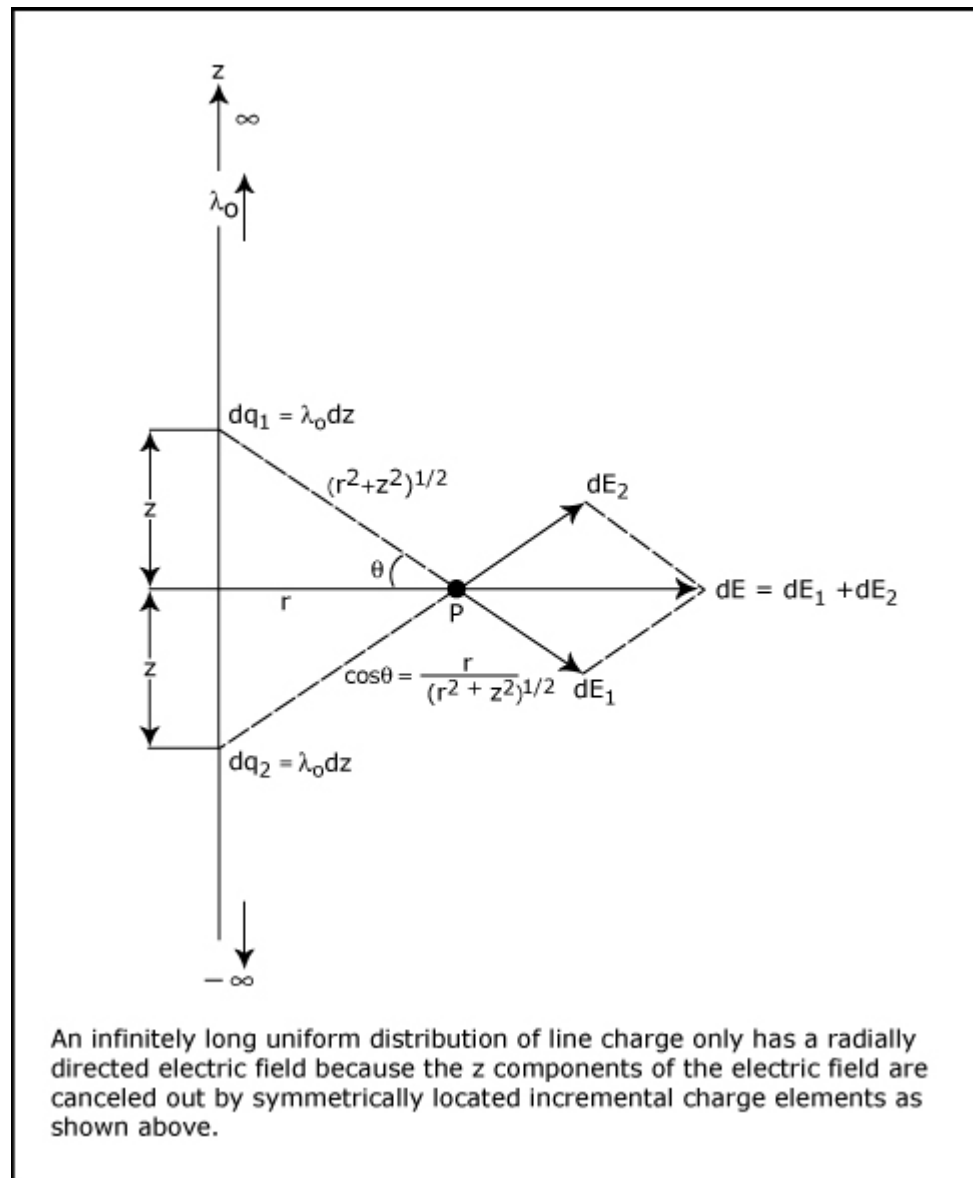


$$\oint_S \bar{J} \cdot d\bar{a} + \frac{d}{dt} \int_V \rho dV = 0$$

$$\bar{n} \cdot [\bar{J}_a - \bar{J}_b] + \frac{\partial}{\partial t} \sigma_s = 0$$

6. Electric Field from a Sheet of Surface Charge

a. Electric Field from a Line Charge



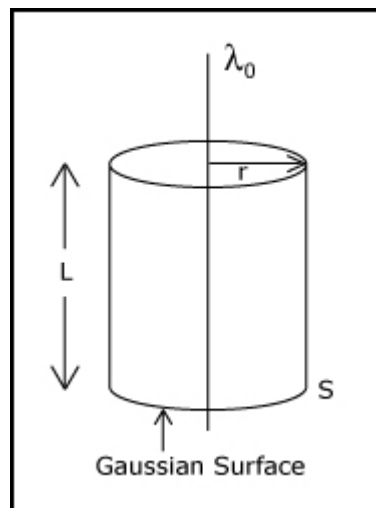
$$dE_r = \frac{dq}{4\pi\epsilon_0(r^2 + z^2)} \cos\theta = \frac{\lambda_0 r dz}{4\pi\epsilon_0(r^2 + z^2)^{3/2}}$$

$$E_r = \int_{z=-\infty}^{+\infty} dE_r = \frac{\lambda_0 r}{4\pi\epsilon_0} \int_{z=-\infty}^{+\infty} \frac{dz}{(r^2 + z^2)^{3/2}}$$

$$= \frac{\lambda_0 r}{4\pi\epsilon_0} \frac{z}{r^2(z^2 + r^2)^{1/2}} \Bigg|_{z=-\infty}^{+\infty}$$

$$= \frac{\lambda_0}{2\pi\epsilon_0 r}$$

Another way: Gauss' Law



$$\int_S \epsilon_0 \vec{E} \cdot d\vec{a} = \epsilon_0 E_r 2\pi r L = \lambda_0 L$$

$$E_r = \frac{\lambda_0}{2\pi\epsilon_0 r}$$

b. Electric Field from a Sheet Charge

(a) Diagram showing a uniformly charged sheet of infinite extent in the yz-plane. A differential line charge element $d\lambda = \sigma_0 dx$ is shown. The electric field contribution dE at a point P is decomposed into dE_1 and dE_2 . The distance from the element to P is $(x^2 + y^2)^{1/2}$. The resulting electric field E_y is shown as a constant $\frac{\sigma_0}{2\epsilon_0}$ for $y > 0$ and $-\frac{\sigma_0}{2\epsilon_0}$ for $y < 0$.

(b) Diagram showing two parallel sheets of surface charge with densities σ_0 and $-\sigma_0$. The electric field E_y is $\frac{\sigma_0}{2\epsilon_0}$ in region I ($y < -a$), $\frac{\sigma_0}{\epsilon_0}$ in region II ($-a < y < a$), and $\frac{\sigma_0}{2\epsilon_0}$ in region III ($y > a$).

(c) Diagram showing a volume charge distribution ρ_0 between $y = -a$ and $y = a$. A differential surface charge element $d\sigma = \rho_0 dy'$ is shown. The electric field contribution $dE = \frac{\rho_0 dy'}{2\epsilon_0} i_y$ is shown. The resulting electric field E_y is $-\frac{\rho_0 a}{\epsilon_0}$ for $y < -a$, $\frac{\rho_0 y}{\epsilon_0}$ for $-a < y < a$, and $\frac{\rho_0 a}{\epsilon_0}$ for $y > a$.

(a) The electric field from a uniformly surface charged sheet of infinite extent is found by summing the contributions from each incremental line charge element. Symmetrically placed line charge elements have x field components that cancel, but y field components that add. (b) Two parallel but oppositely sheets of surface charge have fields that add in the region between the sheets but cancel outside. (c) The electric field from a volume charge distribution is obtained by summing the contributions from each incremental surface charge element.

$$dE_y = \frac{d\lambda}{2\pi\epsilon_0 (x^2 + y^2)^{3/2}} \cos\theta = \frac{\sigma_0 y dx}{2\pi\epsilon_0 (x^2 + y^2)^{3/2}}$$

$$E_y = \int_{x=-\infty}^{+\infty} dE_y = \frac{\sigma_0 y}{2\pi\epsilon_0} \int_{x=-\infty}^{+\infty} \frac{dx}{x^2 + y^2}$$

$$= \frac{\sigma_0 y}{2\pi\epsilon_0} \frac{1}{y} \tan^{-1} \frac{x}{y} \Big|_{-\infty}^{+\infty}$$

$$= \begin{cases} \frac{\sigma_0}{2\epsilon_0} & y > 0 \\ -\frac{\sigma_0}{2\epsilon_0} & y < 0 \end{cases}$$

Checking Boundary condition at $y=0$

$$E_y(y = 0_+) - E_y(y = 0_-) = \frac{\sigma_0}{\epsilon_0}$$

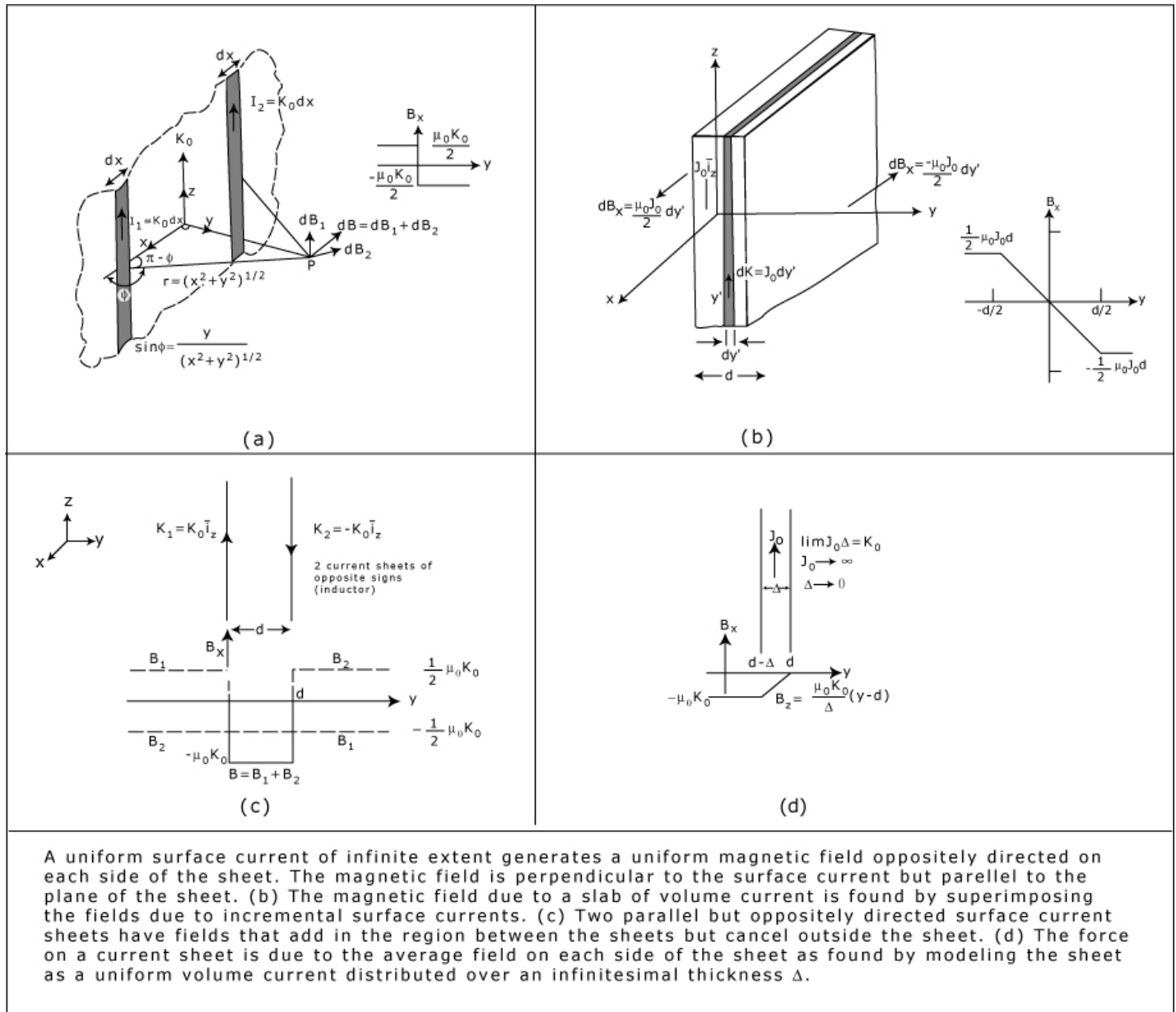
$$\frac{\sigma_0}{2\epsilon_0} - \left(-\frac{\sigma_0}{2\epsilon_0}\right) = \frac{\sigma_0}{\epsilon_0}$$

c. Two sheets of Surface Charge (Capacitor)

$$\bar{E}_1 = \begin{cases} \frac{\sigma_0}{2\epsilon_0} \bar{i}_y & y > -a \\ -\frac{\sigma_0}{2\epsilon_0} \bar{i}_y & y < -a \end{cases}, \bar{E}_2 = \begin{cases} -\frac{\sigma_0}{2\epsilon_0} \bar{i}_y & y > a \\ \frac{\sigma_0}{2\epsilon_0} \bar{i}_y & y < a \end{cases}$$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \begin{cases} \frac{\sigma_0}{\epsilon_0} \bar{i}_y & |y| < a \\ 0 & |y| > a \end{cases}$$

7. Magnetic Field from a Sheet of Surface Current



From a line current I

$$H_\phi = \frac{I}{2\pi r}$$

$$\bar{i}_\phi = -\sin \phi \bar{i}_x + \cos \phi \bar{i}_y$$

Thus from 2 symmetrically located line currents

$$dH_x = \frac{dI}{2\pi(x^2 + y^2)^{1/2}} (-\sin \phi)$$

$$= -\frac{K_0 dx}{2\pi} \frac{y}{x^2 + y^2}$$

$$H_x = -\frac{K_0}{2\pi} y \int_{x=-\infty}^{+\infty} \frac{dx}{x^2 + y^2}$$

$$= -\frac{K_0 y}{2\pi} \frac{1}{y} \tan^{-1} \frac{x}{y} \Big|_{x=-\infty}^{+\infty}$$

$$= \begin{cases} -\frac{K_0}{2} & y > 0 \\ +\frac{K_0}{2} & y < 0 \end{cases}$$

Check boundary condition at $y=0$:

$$H_x(y = 0_+) - H_x(y = 0_-) = -K_0$$

$$-\frac{K_0}{2} - \left(\frac{K_0}{2}\right) = -K_0$$