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6.641, Electromagnetic Fields, Forces, and Motion
 Prof. Markus Zahn
Quiz 2: Solution

1) (a) at $0 < x < d$ $\bar{H} = H_y \bar{i}_y$

at $x < 0$, $\mu \rightarrow \infty$ $\bar{H} = 0$

at $d < x$, $\sigma \rightarrow \infty$ $\bar{H} = 0$

(b) at $x = 0$ $H_y(0^+) - H_y(0^-) = K_z(0)$

$H_y(0^+) = K_0 \cos \omega t$

at $x = d$ $B_x(d^+) = B_x(d^-) = 0$

$-H_y(d^-) = K_z(d)$

(c) at $0 < x < d$ $\bar{J} = J_z \bar{i}_z$, $\bar{E} = E_z \bar{i}_z$

at $x = d$ $E_z(d^+) = E_z(d^-) = 0$

(d) $\nabla^2 \bar{H} - \omega_p^2 \epsilon_0 \mu_0 \bar{H} = 0$; $\nabla \times \bar{H} = \bar{J} \Rightarrow J_z = \frac{\partial H_y}{\partial x}$; $\frac{\partial \bar{J}}{\partial t} = \omega_p^2 \epsilon_0 \bar{E} \Rightarrow E_z = \frac{1}{\omega_p^2 \epsilon_0} \frac{\partial J_z}{\partial t}$

$$\frac{d^2 H_y}{dx^2} - \frac{\omega_p^2}{c^2} H_y = 0, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$H_y = A_1 e^{\omega_p x/c} + A_2 e^{-\omega_p x/c}$$

$$J_z = B_1 e^{\omega_p x/c} + B_2 e^{-\omega_p x/c} = \frac{\omega_p}{c} (A_1 e^{\omega_p x/c} - A_2 e^{-\omega_p x/c})$$

$$E_z = C_1 e^{\omega_p x/c} + C_2 e^{-\omega_p x/c} = \frac{1}{\omega_p c \epsilon_0} \left(\frac{\partial A_1}{\partial t} e^{\omega_p x/c} - \frac{\partial A_2}{\partial t} e^{-\omega_p x/c} \right)$$

$$(e) \quad \left. \begin{array}{l} H_y(x=0) = K_0 \cos \omega t \\ E_z(x=d) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} A_1 = \frac{e^{-2\omega_p d/c}}{1 + e^{-2\omega_p d/c}} K_0 \cos \omega t \\ A_2 = \frac{1}{1 + e^{-2\omega_p d/c}} K_0 \cos \omega t \end{array}$$

$$H_y = \frac{K_0 \cos \omega t}{1 + e^{-2\omega_p d/c}} \left(e^{-2\omega_p d/c} e^{\omega_p x/c} + e^{-\omega_p x/c} \right) = \frac{K_0 \cos \omega t \cosh(\omega_p (x-d)/c)}{\cosh(\omega_p (d/c))}$$

$$J_z = \frac{\omega_p}{c} \frac{K_0 \cos \omega t}{1 + e^{-2\omega_p d/c}} \left(e^{-2\omega_p d/c} e^{\omega_p x/c} - e^{-\omega_p x/c} \right) = \frac{\omega_p}{c} \frac{K_0 \cos(\omega_p d/c) \sinh(\omega_p (x-d)/c)}{\cosh(\omega_p d/c)}$$

$$E_z = \frac{-\omega K_0 \sin \omega t}{\omega_p c \epsilon_0 (1 + e^{-2\omega_p d/c})} \left(e^{-2\omega_p d/c} e^{\omega_p x/c} - e^{-\omega_p x/c} \right) = -\frac{K_0 \omega}{\omega_p \epsilon_0 c} \frac{\sin(\omega_p d/c) \sinh(\omega_p (x-d)/c)}{\cosh(\omega_p (d/c))}$$

$$(f) \quad \bar{K}(x=d) = \bar{i}_z \frac{-K_0 \cos \omega t}{1 + e^{-2\omega_p d/c}} 2e^{-\omega_p d/c} = \bar{i}_z \frac{-K_0 \cos \omega t}{\cosh(\omega_p d/c)}$$

$$(g) \quad \bar{f} = \frac{\bar{K} \times \bar{B}}{2} \Big|_{x=d} = \frac{1}{2} K_z \bar{i}_z \times \mu_0 H_y \bar{i}_y \Big|_{x=d} = \frac{\bar{i}_x}{2} \mu_0 K_z^2 \Big|_{x=d}$$

$$= \bar{i}_x \frac{2\mu_0 K_0^2 \cos^2 \omega t}{(1 + e^{-2\omega_p d/c})^2} e^{-2\omega_p d/c}$$

$$= \bar{i}_x \frac{1}{2} \frac{\mu_0 K_0^2 \cos^2 \omega t}{\cosh^2(\omega_p d/c)}$$

$$2) \quad (a) \quad \nabla \cdot \bar{B} = 0, \quad \bar{B} = \mu_0 \bar{H}, \quad \bar{H} = -\nabla \chi \Rightarrow \nabla^2 \chi = 0, \quad r < R$$

$$\bar{B} = \mu_0 (\bar{H} + \bar{M}), \quad \nabla \cdot \bar{H} = -\nabla \cdot \bar{M} = 0 \Rightarrow \nabla^2 \chi = 0, \quad r > R$$

(b) at $r = R$ $H_\theta(R^-) = H_\theta(R^+)$ (no surface current)

$$B_r(R^-) = B_r(R^+)$$

$$(c) \quad \chi = \begin{cases} Ar \cos \theta, & r < R \\ \frac{B}{r^2} \cos \theta, & r > R \end{cases} \quad \bar{H} = -\nabla\chi = -\bar{i}_r \frac{\partial\chi}{\partial r} - \bar{i}_\theta \frac{1}{r} \frac{\partial\chi}{\partial\theta}$$

$$\bar{H} = \begin{cases} -\bar{i}_r A \cos \theta + \bar{i}_\theta A \sin \theta, & r < R \\ 2\bar{i}_r \frac{B}{r^3} \cos \theta + \bar{i}_\theta \frac{B}{r^3} \sin \theta, & r > R \end{cases}$$

use boundary conditions $A = \frac{B}{R^3}$, $-A \cos \theta = \frac{2B}{R^3} \cos \theta + M_0 \cos \theta$

$$\Rightarrow A = -\frac{M_0}{3} \quad B = -\frac{R^3}{3} M_0$$

$$\chi = \begin{cases} -\frac{M_0}{3} r \cos \theta, & r < R \\ -\frac{M_0 R^3}{3r^2} \cos \theta, & r > R \end{cases} \quad \bar{H} = \begin{cases} \bar{i}_r \frac{M_0}{3} \cos \theta - \bar{i}_\theta \frac{M_0}{3} \sin \theta, & r < R \\ -\bar{i}_r \frac{2M_0 R^3}{3r^3} \cos \theta - \bar{i}_\theta \frac{M_0 R^3}{3r^3} \sin \theta, & r > R \end{cases}$$

$$(d) \quad \frac{|\bar{m}_e| \cos \theta}{4\pi r^2} = -\frac{|M_0 R^3|}{3r^2} \cos \theta \Rightarrow \bar{m}_e = -M_0 \frac{4\pi R^3}{3} \bar{i}_z$$

3) (a) $H = \frac{Ni}{g}$ in air gap and the magnetic material gap, $B_{\mu_0} = \frac{\mu_0 Ni}{g}$, $B_\mu = \frac{\mu Ni}{g}$

$$\Phi = \mu_0 H w x + \mu H w (D - x) = \frac{Ni w}{g} (\mu_0 x + \mu (D - x))$$

$$\lambda = N\Phi = (\mu_0 x + \mu (D - x)) \frac{N^2 i w}{g}$$

(b) $dW' = \lambda di + f_m dx$

$$W' = \int \lambda di + \int f_m dx = (\mu_0 x + \mu (D - x)) \frac{N^2 w}{2g} I_0^2$$

$$f_m = \left. \frac{dW'}{dx} \right|_i = \frac{-(\mu - \mu_0) N^2 w}{2g} I_0^2$$

$$\begin{aligned} \text{(c)} \quad v &= \left. \frac{d\lambda}{dt} \right|_{i=I_0} = \frac{d}{dt} (\mu D + (\mu_0 - \mu) (x_0 + x' \cos \omega t)) \frac{N^2 I_0 w}{g} \\ &= (\mu - \mu_0) \omega \sin \omega t \frac{N^2 I_0 w}{g} x' \end{aligned}$$

$$\text{(d) In a cycle } \Delta W' = 0 \quad \oint \lambda di + \oint f_m dx = 0$$

$$\begin{aligned} \text{total work } -\oint f_m dx &= +\oint \lambda di \\ &= -\left[\int_{x_1}^{x_2} f_m(I_1) dx + \int_{x_2}^{x_1} f_m(I_2) dx \right] \\ &= -\frac{\mu - \mu_0}{2g} N^2 w (x_2 - x_1) (I_2^2 - I_1^2) \end{aligned}$$