6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Problem Set 2 - Solutions	
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Problem 2.1

A

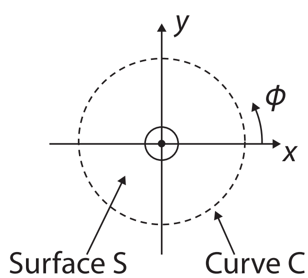

Figure 1: Surface S and contour C for using Ampere's law (Image by MIT OpenCourseWare.)

Step 1: Find field of z-directed line current, $\overrightarrow{I}$, at $x = y = 0$.

$$\overrightarrow{I} = I\hat{i}_Z$$

By symmetry: $\overrightarrow{H} = H_\phi \hat{i}_\phi$ in cylindrical coordinates. By Ampere:

$$\oint_C \overrightarrow{H} \cdot d\overrightarrow{l} = \underbrace{\int_S \overrightarrow{J} \cdot d\overrightarrow{a}}_{\text{current going through } S}$$

$$(2\pi r)H_\phi = I$$

$$H_\phi = \frac{I}{2\pi r}\hat{i}_\phi \tag{1}$$

$$\hat{i}_\phi = \frac{-y}{\sqrt{x^2+y^2}}\hat{i}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{i}_y$$

$$\overrightarrow{H} = \frac{I}{2\pi(x^2+y^2)}(-y\hat{i}_x + x\hat{i}_y)$$

Step 2: Find solution by adding two translated $\overrightarrow{H}$ fields:

$$\overrightarrow{H}_{\text{total}} = \overrightarrow{H}_{I_1} + \overrightarrow{H}_{I_2}$$

$$= \frac{I_1}{2\pi\left[x^2 + (y-\frac{d}{2})^2\right]}\left[-(y-\frac{d}{2})\hat{i}_x + x\hat{i}_y\right] + \frac{I_2}{2\pi\left[x^2 + (y-\frac{d}{2})^2\right]}\left[-(y+\frac{d}{2})\hat{i}_x + x\hat{i}_y\right]$$

B

Step 3: We want field in $y = 0$ plane so $y \to 0$.

i

$$I_1 = I, I_2 = 0$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{2\pi(x^2 + \frac{d^2}{4})} \left[\frac{d}{2}\hat{i}_x + x\hat{i}_y\right]$$

ii

$$I_1 = I, I_2 = I$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{\pi(x^2 + \frac{d^2}{4})} \left[x\hat{i}_y\right]$$

iii

$$I_1 = I, I_2 = -I$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{2\pi(x^2 + \frac{d^2}{4})} [d\hat{i}_x]$$

C

$$\overrightarrow{F} = q\overrightarrow{v} \times (\mu_0 \overrightarrow{H})$$

so

$$d\overrightarrow{F} = dq \left[\overrightarrow{v} \times \left(\mu_0 \overrightarrow{H}\right)\right]$$

In our problem the line current is a moving line charge, so $dq = \lambda dl$

$$d\overrightarrow{F} = \lambda\overrightarrow{v} \times \left[\mu_0 \overrightarrow{H}\right] dl = \bar{I} \times (\mu_0 \overrightarrow{H}) dl$$

$$\overrightarrow{F} = \int_0^L \overrightarrow{I} \times (\mu_0 \overrightarrow{H}) dl = (\overrightarrow{I} \times \mu_0 \overrightarrow{H}) L$$

So

$$\frac{\text{Force}}{\text{Length}} = \overrightarrow{I} \times (\mu_0 \overrightarrow{H})$$

We don't need to know $\overrightarrow{H}$ since I_1 cannot exert a net force on itself. So we need a field of I_2

$$\overrightarrow{H} = \frac{I_2 \left[-\left(y + \frac{d}{2}\right)\hat{i}_x + x\hat{i}_y\right]}{2\pi\left(x^2 + \left(y + \frac{d}{2}\right)^2\right)}$$

I_1 is at $x = 0, y = \frac{d}{2}$, so $x \to 0, y \to \frac{d}{2}$ above.

$$\overrightarrow{H} = \frac{-I_2 d}{2\pi d^2}\hat{i}_x = \frac{-I_2}{2\pi d}\hat{i}_x$$

$$\frac{\text{Force}}{\text{Length}} = (\overrightarrow{I_1}) \times (\mu_0 \overrightarrow{H}) = (I_1 \hat{i}_z) \times \left(\frac{-\mu_0 I_2}{2\pi d}\right)\hat{i}_x$$

$$= \frac{-\mu_0 I_1 I_2}{2\pi d}\hat{i}_y$$

$$(i) \qquad I_1 = I, \quad I_2 = 0$$

$$\frac{\text{Force}}{\text{Length}} = 0$$

$$(ii) \qquad I_1 = I, \quad I_2 = I$$

$$\frac{\text{Force}}{\text{Length}} = \frac{-\mu_0 I^2}{2\pi d}\hat{i}_y$$

$$(iii) \qquad I_1 = I, \quad I_2 = -I$$

$$\frac{\text{Force}}{\text{Length}} = +\frac{\mu_0 I^2}{2\pi d}\hat{i}_y$$

Problem 2.2

A

The idea here is similar to applying the chain rule in a 1D problem

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \left[\frac{d}{df}\left(\frac{1}{f(x)}\right)\right]\left[\frac{df}{dx}\right] = \frac{-f^{'}(x)}{f^2(x)}$$

$f(x)$ corresponds to $|\overrightarrow{r} - \overrightarrow{r}^{'}|$. So, by diff. $f(x)$ we get part of the answer to the derivative of $\frac{1}{f(x)}$. But we can just do it directly too.

$$|\overrightarrow{r} - \overrightarrow{r}^{'}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\nabla\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}\right] = \hat{i}_x\frac{\partial}{\partial x}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}\right] + \hat{i}_y\frac{\partial}{\partial y}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}\right] + \hat{i}_z\frac{\partial}{\partial z}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}\right]$$

So we can apply the trick above by just considering x, y, and z components separately.

$$\frac{\partial}{\partial x}|\overrightarrow{r} - \overrightarrow{r}^{'}| = \frac{\partial}{\partial x}\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)$$

$$= \frac{x - x^{'}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{x - x^{'}}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}$$

Similarly, $\frac{\partial}{\partial y}|\overrightarrow{r} - \overrightarrow{r}^{'}| = \frac{y-y^{'}}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}$ and $\frac{\partial}{\partial z}|\overrightarrow{r} - \overrightarrow{r}^{'}| = \frac{z-z^{'}}{|\overrightarrow{r} - \overrightarrow{r}^{'}|}$.

$$\frac{\partial}{\partial x}\left(\frac{1}{|\overrightarrow{r} - \overrightarrow{r}^{'}}\right) = \frac{-\frac{\partial}{\partial x}|\overrightarrow{r} - \overrightarrow{r}^{'}|}{|\overrightarrow{r} - \overrightarrow{r}^{'}|^2}$$

and so on for y and z.

$$|\overrightarrow{r} - \overrightarrow{r}^{'}|^2 = (x - x^{'})^2 + (y - y^{'})^2 + (z - z^{'})^2$$

so:

$$\nabla\left[\frac{1}{|\overrightarrow{r}-\overrightarrow{r}'|}\right]=\frac{-(x-x')\hat{i}_x}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}+\text{...similar terms for } y \text{ and } z$$

Denominators $= |\overrightarrow{r}-\overrightarrow{r}'|^{\frac{3}{2}}$. Thus,

$$\nabla\left[\frac{1}{|\overrightarrow{r}-\overrightarrow{r}'|}\right]=\frac{-(\overrightarrow{r}-\overrightarrow{r}')}{|\overrightarrow{r}-\overrightarrow{r}'|^3}=\frac{-1}{|\overrightarrow{r}-\overrightarrow{r}'|^2}\frac{(\overrightarrow{r}-\overrightarrow{r}')}{|\overrightarrow{r}-\overrightarrow{r}'|}$$

$$=\frac{-\hat{i}_{r'r}}{|\overrightarrow{r}-\overrightarrow{r}'|^2}$$

B

Follows from (A) immediately by substitution. Remember ∇ is derived in terms of unprimed x, y, z. ∇ does not affect x', y', z'.

C

$$\Phi(\overrightarrow{r})=\int_{V'}\frac{\rho(\overrightarrow{r}')dV'}{4\pi\varepsilon_0|\overrightarrow{r}-\overrightarrow{r}'|}$$

$\rho(\overrightarrow{r}')$ = charge density in $\frac{C}{m^3}$. We have λ in units of $\frac{C}{m}$. In this sense, $\rho\to\infty$ at the ring. We can represent this in cylindrical coordinates by $\rho(\overrightarrow{r}')=\lambda_0\delta(z)\delta(r-a)$. Then we can evaluate the triple integral

$$\int\int\int\frac{\lambda_0\delta(z)\delta(r-a)rd\phi d\theta dr}{4\pi\varepsilon_0|\overrightarrow{r}-\overrightarrow{r}'|}$$

But, we can skip that unnecessary work by simply considering infinitesimal charges $(ad\phi)\lambda_0$ around the ring.

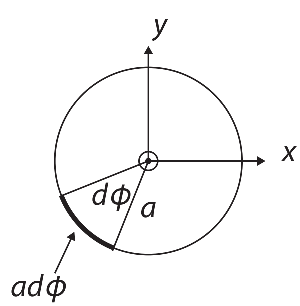

Figure 2: A ring of line charge with infinitesimal charge elements. (Image by MIT OpenCourseWare.)

We only care about z axis in this as well, so, by symmetry, there is no field in x and y directions.

$$\Phi(\overrightarrow{r})=\int_0^{2\pi}\frac{\lambda_0(ad\phi)}{4\pi\varepsilon_0\underbrace{(a^2+z^2)^{\frac{1}{2}}}_{\text{distance from the charge }\lambda_0ad\phi\text{ to the point }z\text{ on the }z\text{-axis}}}$$

$$\Phi(\overrightarrow{r})=\frac{\lambda_0a}{2\varepsilon_0(a^2+z^2)^{\frac{1}{2}}}$$

on z-axis.

$$\overrightarrow{E} = -\nabla\Phi(\overrightarrow{r}) = -\left(\hat{i}_x \cancelto{0}{\frac{\partial}{\partial x}}\Phi + \hat{i}_y \cancelto{0}{\frac{\partial}{\partial y}}\Phi + \hat{i}_z \frac{\partial}{\partial z}\Phi\right)$$

$$\overrightarrow{E} = -\hat{i}_z \frac{\partial}{\partial z}\left(\frac{\lambda_0 a}{2\varepsilon_0 (a^2+z^2)^{\frac{1}{2}}}\right)$$

$$\overrightarrow{E} = \hat{i}_z \frac{a\lambda_0 z}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}}$$

Using Eq. (5) with z component only (symmetry) and with $\rho(\overrightarrow{r}')dV' \to \lambda_0 a d\phi$

$$E_z(z) = \int_0^{2\pi} \frac{\lambda_0 a d\phi \cos\theta}{4\pi\varepsilon_0 (z^2+a^2)}, \cos\theta = \frac{z}{(a^2+z^2)^{\frac{1}{2}}}$$

$$= \int_0^{2\pi} \frac{\lambda_0 a z}{(a^2+z^2)^{\frac{3}{2}}} \frac{d\phi}{4\pi\varepsilon_0}$$

$$= \frac{\lambda_0 a z}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}}$$

Limit $|z| \to \infty$

$$\sqrt{a^2+z^2} \to |z|$$

$$\Phi(z) \approx \frac{\lambda_0 a}{2\varepsilon_0 (a^2+z^2)^{\frac{1}{2}}} \approx \frac{2\pi\lambda_0 a}{4\pi\varepsilon_0 |z|} \approx \frac{Q}{4\pi\varepsilon_0 |z|}$$

$Q = 2\pi\lambda_0 a$ (total charge on loop). $\Phi(z)$ looks like potential from point charge in far field.

$$E_z = \frac{\lambda_0 a z}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}} \approx \frac{\lambda_0 a z}{2\varepsilon_0 |z|^3} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 |z|^2} & z > 0 \\ \frac{-Q}{4\pi\varepsilon_0 |z|^2} & z < 0 \end{cases}$$

D

From (C), $\Phi = \frac{\lambda_0 r}{2\varepsilon_0 (r^2+z^2)^{\frac{1}{2}}}$ for a ring of radius r. But now we have σ_0, not λ_0. How do we express λ_0 in terms of σ_0? Take a ring of width dr in the disk (see figure). Total charge = $\underbrace{(r)(2\pi)}_{\text{circumference}} (dr)\sigma_0$. Line charge

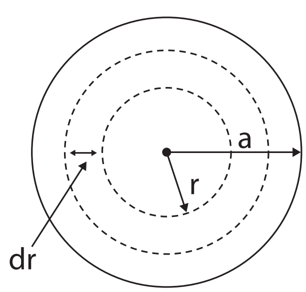

Figure 3: A ring of width dr in the disk (Image by MIT OpenCourseWare.)

density $= \lambda_0 = \frac{\text{total charge}}{\text{length}} = \sigma_0 dr$ So: $\lambda_0 = \sigma_0 dr$

$$\Phi = \frac{\sigma_0 r dr}{2\varepsilon_0 (r^2 + z^2)^{\frac{1}{2}}}$$

$$\Phi_{\text{total}} = \int_0^a \frac{\sigma_0 r dr}{2\varepsilon_0 (r^2 + z^2)^{\frac{1}{2}}} = \frac{\sigma_0}{2\varepsilon_0} \int_0^a \frac{r dr}{(r^2 + z^2)^{\frac{1}{2}}}$$

$$= \frac{\sigma_0}{2\varepsilon_0} \left[\sqrt{r^2 + z^2} \right] \Big|_{r=0}^{r=a} = \frac{\sigma_0}{2\varepsilon_0} \left[\sqrt{a^2 + z^2} - |z| \right]$$

$$\overrightarrow{E} = -\nabla \Phi_{\text{total}} = \frac{\sigma_0}{2\varepsilon_0} z \left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{a^2 + z^2}} \right] \overline{i_z}$$

E

As $z \to \infty$,

$$(a^2 + z^2)^{\frac{1}{2}} \to z + \frac{a^2}{2z}; \quad (a^2 + z^2)^{-\frac{1}{2}} \to \frac{1}{z} \left(1 - \frac{a^2}{2z^2} \right)$$

$$\Phi_{\text{total}} \to \frac{\pi a^2 \sigma_0}{4\epsilon_0 \pi z}$$

$$\bar{E} \to \frac{\pi a^2 \sigma_0}{4\pi\varepsilon_0 z^2} \overline{i_z}$$

just like a point charge of $\sigma_0 \pi a^2$.

F

As $a \to \infty$, z in the $\sqrt{a^2 + z^2}$ can be neglected, so

$$\Phi_{\text{total}} \to \frac{\sigma_0}{2\varepsilon_0} [a - |z|]$$

$$E_z \to \frac{\sigma_0 z}{2\varepsilon_0} \left[\frac{1}{|z|} - 0 \right] = \begin{cases} \frac{\sigma_0}{2\varepsilon_0} & z > 0 \\ \frac{-\sigma_0}{2\varepsilon_0} & z < 0 \end{cases}$$

just like sheet charge.

Problem 2.3

A

By the divergence theorem:

i

$$\int_V \nabla \cdot (\nabla \times \overrightarrow{A}) dV = \oint_S (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a}$$

where S encloses V. By Stokes' Theorem:

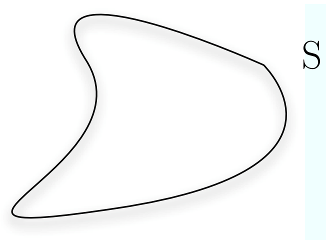

Figure 4: Closed surface S (Image by MIT OpenCourseWare.)

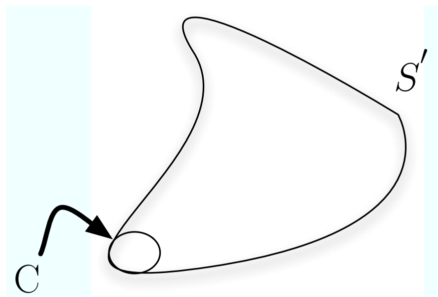

Figure 5: Open surface $S^{'}$ (Image by MIT OpenCourseWare.)

ii

$$\int_{S'} (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a} = \oint_C \overrightarrow{A} \cdot d\overrightarrow{l}$$

Suppose S is as in figure 4

and $S^{'}$ is as in figure 5

i.e. $S^{'}$ is the same as S, except for the curve C, which makes $S^{'}$ slightly unclosed. Now consider limit as $C \rightarrow 0$ (Figure 6)

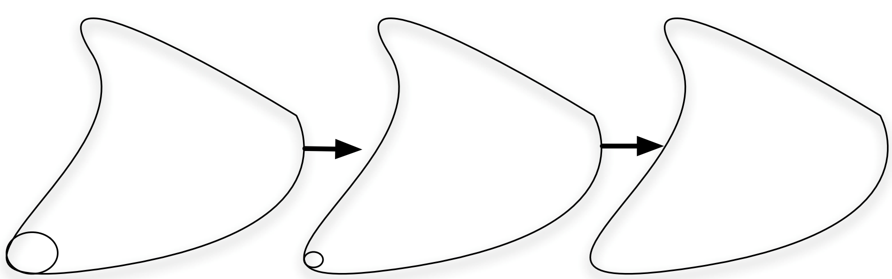

Figure 6: Limit as $C \rightarrow 0$ (Image by MIT OpenCourseWare.)

In limit $C \rightarrow 0, S^{'} \rightarrow S$. If C is 0, then $\oint_C \overrightarrow{A} \cdot d\overrightarrow{l} = 0$. By equation (ii), $\oint_S (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a} = 0$. By equation (i), $\int_V \nabla \cdot (\nabla \times \overrightarrow{A}) dV = 0$. Since V can be any volume, argument of integral must be identically 0.

$$\nabla \cdot (\nabla \times \overrightarrow{A}) = 0$$

B

$$\overrightarrow{A} = A_x \hat{i}_x + A_y \hat{i}_y + A_z \hat{i}_z$$

$$\nabla \cdot (\nabla \times \overrightarrow{A}) = \nabla \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{i}_z \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$= 0$ because of interchangeability of partial derivatives

In cylindrical coordinates

$$\bar{A} = A_r \bar{i}_r + A_\phi \bar{i}_\phi + A_z \bar{i}_z$$

$$\nabla \times \bar{A} = \bar{i}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \bar{i}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \bar{i}_z \frac{1}{r} \left[\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla \cdot (\nabla \times \bar{A}) = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial A_z}{\partial \phi} - r \frac{\partial A_\phi}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \frac{\partial}{\partial z} \left[\frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right]$$

$$= \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} - \frac{r}{r} \frac{\partial^2 A_\phi}{\partial r \partial z} - \frac{1}{r} \frac{\partial A_\phi}{\partial z} + \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z} - \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} + \frac{\partial^2 A_\phi}{\partial r \partial z} + \frac{1}{r} \frac{\partial A_\phi}{\partial z} - \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z}$$

$= 0$

Problem 2.4

A

$$Q = \int_{-\infty}^{\infty} \lambda(z) dz = \int_{-a}^{a} \frac{\lambda_0 z}{a} dz = \frac{\lambda_0 z^2}{2a} \Big|_{-a}^{+a} = 0$$

B

$$\Phi(\bar{r}) = \int_{V'} \frac{\rho(\bar{r}') dv'}{4\pi\varepsilon_0 |\bar{r} - \bar{r}'|}$$

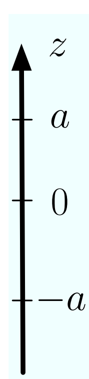

Figure 7: Axis showing extent of line charge $-a < z < a$ for Problem 2.4B (Image by MIT OpenCourseWare.)

$$\Phi(r = 0, z) = \int_{-a}^{a} \frac{\lambda_0 z' dz'}{4\pi\varepsilon_0 |z - z'| a}$$

$$\Phi(r=0,z) = \frac{\lambda_0}{4\pi\varepsilon_0 a}\int_{-a}^{a}\frac{z^{'}}{z-z^{'}}dz^{'} = \left[z\ln\left(\frac{z+a}{z-a}\right)-2a\right]\frac{\lambda_0}{4\pi\varepsilon_0 a} \text{ for } z>a$$

$$\bar{E} = -\nabla\Phi$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 a}\frac{\partial}{\partial z}\left[z\ln\left(\frac{z+a}{z-a}\right)-2a\right]\hat{i}_z$$

$$= -\frac{\lambda_0}{4\pi\varepsilon_0 a}\left[\ln\left(\frac{z+a}{z-a}\right)-\frac{2az}{(z+a)(z-a)}\right]\bar{i}_z$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 a}\left[\frac{2az}{(z+a)(z-a)}-\ln\left(\frac{z+a}{z-a}\right)\right]\bar{i}_z$$

C

$$z\to\infty : \ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3\ldots \quad x\ll 1$$

$$\Phi(z\to\infty) = \frac{\lambda_0}{4\pi\varepsilon_0 a}\left[z(\frac{a}{z}-\frac{1}{2}(\frac{a}{z})^2+\frac{1}{3}(\frac{a}{z})^3-(\frac{-a}{z}-\frac{1}{2}(\frac{a}{z})^2-\frac{1}{3}(\frac{a}{z})^3))-2a\right]$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 a}z\frac{2}{3}\frac{a^3}{z^3} = \frac{\frac{2}{3}a^2\lambda_0}{4\pi\varepsilon_0 z^2}$$

$$E_z(z\to\infty) = \frac{\lambda_0}{4\pi\varepsilon_0 a}\left[\frac{2az}{z^2}(1+(\frac{a}{z})^2))-(\frac{a}{z}-\frac{1}{2}(\frac{a}{z})^2+\frac{1}{3}(\frac{a}{z})^3+\frac{a}{z}+\frac{1}{2}(\frac{a}{z})^2+\frac{1}{3}(\frac{a}{z})^3)\right]$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 a}\frac{4}{3}(\frac{a}{z})^3 = \frac{\frac{4}{3}a^2\lambda_0}{4\pi\varepsilon_0 z^3}$$

It's a solution for dipole along z axis

D

$$p = \frac{2}{3}a^2\lambda_0$$

Check: Zahn, pp. 139-140

$$\bar{p} = \int_{\text{all } q}\bar{r}dq \Rightarrow p_z = \int_{-a}^{a} z\lambda(z)dz = \int_{-a}^{a}\frac{\lambda_0 z^2}{a}dz = \frac{\lambda_0 z^3}{3a}\Big|_{-a}^{a} = \frac{2}{3}\lambda_0 a^2$$