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6.641 Electromagnetic Fields, Forces, and Motion  
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**Lecture 11: Solutions to Laplace's Equation In Polar and Spherical Coordinates**

I. Solution's to Laplace's Equation in Polar Coordinates  $\left(\frac{\partial}{\partial z} = 0\right)$

A. Product Solution

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

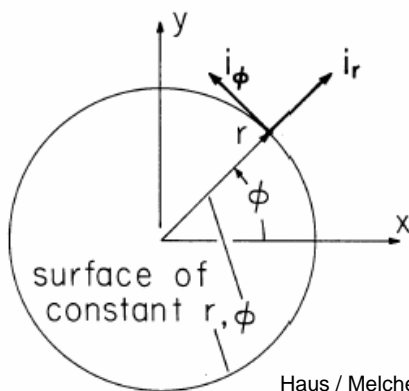
$$\Phi(r, \phi) = R(r) F(\phi)$$

$$\frac{F(\phi)}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{R(r)}{r^2} \frac{d^2 F}{d\phi^2} = 0 \quad \left| \begin{array}{l} \text{Multiply through} \\ \text{by } \frac{r^2}{R(r)F(\phi)} \end{array} \right.$$

$$\underbrace{\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right)}_{m^2} + \underbrace{\frac{1}{F} \frac{d^2 F}{d\phi^2}}_{-m^2} = 0$$

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = m^2 \Rightarrow r \frac{d}{dr} \left( r \frac{dR}{dr} \right) - m^2 R = 0$$

$$\frac{1}{F} \frac{d^2 F}{d\phi^2} = -m^2 \Rightarrow \frac{d^2 F}{d\phi^2} + m^2 F = 0$$



Haus / Melcher **Figure 5.7.1** Polar coordinate system.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

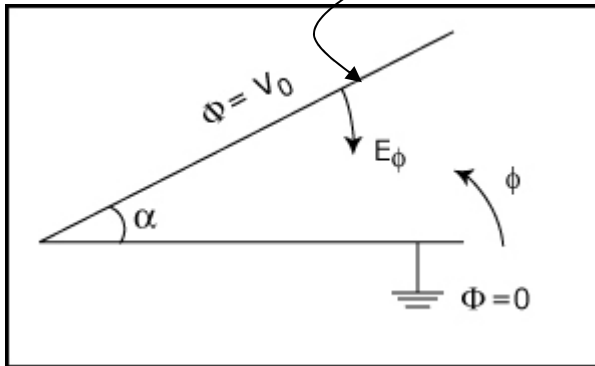
B.  $m=0$  Solutions (Zero Separation Constant Solutions)

$$r \frac{dR}{dr} = C \Rightarrow R = C \ln r + D$$

$$\frac{d^2F}{d\phi^2} = 0 \Rightarrow F = A\phi + B$$

$$\Phi(r, \phi) = R(r)F(\phi) = A_1 + A_2\phi + A_3 \ln r + A_4 \phi \ln r$$

↖ Potential of line charge



$$\left. \begin{array}{l} \Phi(\phi=0) = 0 \\ \Phi(\phi=\alpha) = V_0 \end{array} \right\} \Rightarrow \Phi(\phi) = \frac{V_0\phi}{\alpha} \Rightarrow \vec{E} = -\nabla\Phi = -\left[ \hat{i}_r \frac{\partial\Phi}{\partial r} + \hat{i}_\phi \frac{1}{r} \frac{\partial\Phi}{\partial\phi} + \hat{i}_z \frac{\partial\Phi}{\partial z} \right]$$

$$E_\phi = -\frac{1}{r} \frac{\partial\Phi}{\partial\phi} = -\frac{V_0}{\alpha r}$$

$$\sigma_s(r, \phi=0) = \epsilon_0 E_\phi(r, \phi=0) = -\frac{\epsilon_0 V_0}{\alpha r}$$

$$\sigma_s(r, \phi=\alpha) = -\epsilon_0 E_\phi(r, \phi=\alpha) = +\frac{\epsilon_0 V_0}{\alpha r}$$

C.  $m \neq 0$  Solutions (Non-Zero Separation Constant Solutions)

$$r \frac{d}{dr} \left( r \frac{dR}{dr} \right) - m^2 R = 0$$

Try  $R = Ar^n$

$$r \frac{d}{dr} [nAr^n] - m^2 Ar^n = 0$$

$$n^2 r^n - m^2 r^n = 0 \Rightarrow n = \pm m$$

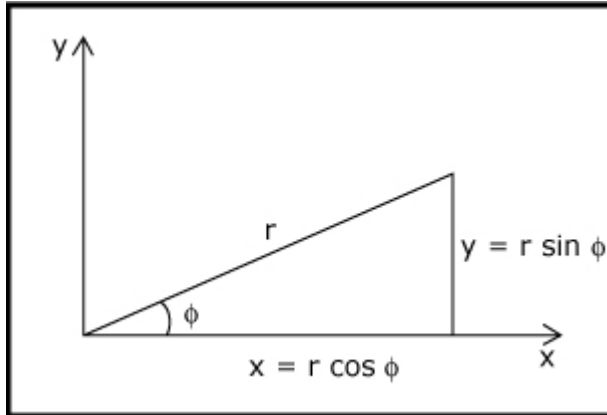
$$R(r) = A_3 r^m + A_4 r^{-m}$$

$$\frac{d^2F}{d\phi^2} + m^2 F = 0$$

$$F = A_1 \sin m\phi + A_2 \cos m\phi$$

$$\Phi(r, \phi) = R(r)F(\phi) = [A_1 \sin m\phi + A_2 \cos m\phi][A_3 r^m + A_4 r^{-m}]$$

$$= A \sin m\phi r^m + B \sin m\phi r^{-m} + C \cos m\phi r^m + D \cos m\phi r^{-m}$$

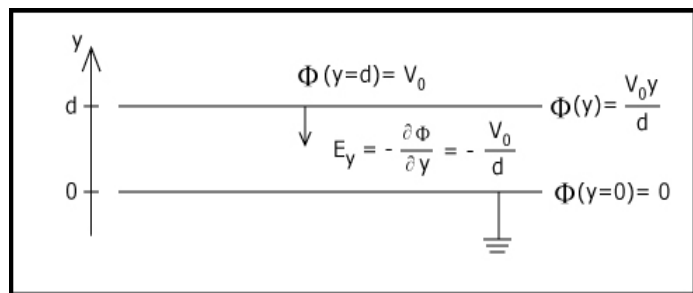


D. Selected Solutions

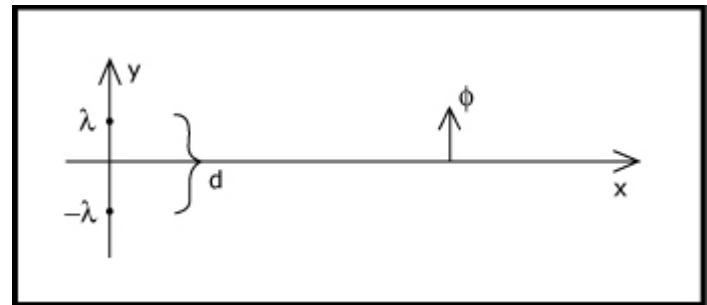
m=1

$$\Phi(r, \phi) = Ar \sin \phi = Ay$$

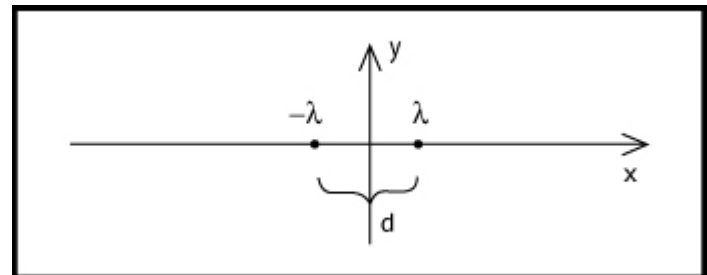
$$\Phi(r, \phi) = Cr \cos \phi = Cx$$



$$\Phi(r, \phi) = \frac{B \sin \phi}{r} \Rightarrow \text{Line dipole oriented in } y \text{ direction}$$



$$\Phi(r, \phi) = \frac{D \cos \phi}{r} \Rightarrow \text{Line dipole oriented in } x \text{ direction}$$

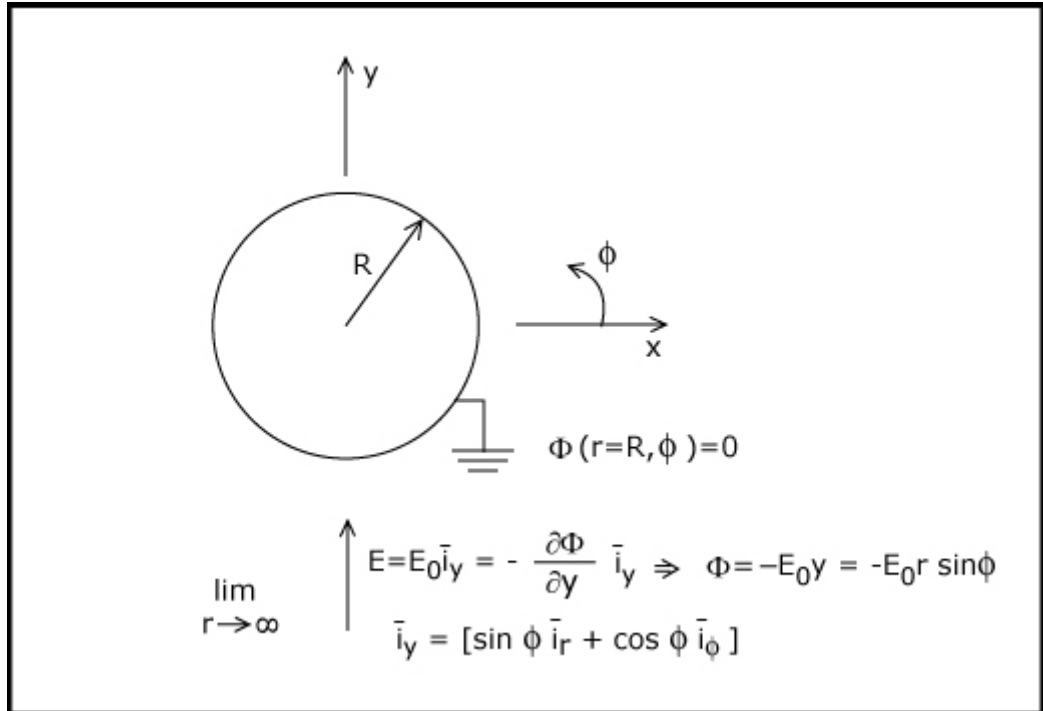


m=2

$$\Phi = Ar^2 \sin 2\phi = 2Ar^2 \sin \phi \cos \phi = 2Axy$$

Generally m an integer if  $\Phi(\phi = 0) = \Phi(\phi = 2\pi)$

E. Grounded Cylinder in a Uniform y Directed Electric Field



$$\Phi = \left[ -E_0 r + \frac{A}{r} \right] \sin \phi \quad r \geq R$$

$$\Phi(r = R, \phi) = 0 \Rightarrow -E_0 R + \frac{A}{R} = 0 \Rightarrow A = E_0 R^2$$

$$\Phi = -E_0 \left( r - \frac{R^2}{r} \right) \sin \phi \quad r \geq R$$

$$\bar{E} = -\nabla \Phi = - \left[ \frac{\partial \Phi}{\partial r} \bar{i}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \bar{i}_\phi + \frac{\partial \Phi}{\partial z} \bar{i}_z \right]$$

$$= E_0 \left\{ \left[ 1 + \frac{R^2}{r^2} \right] \sin \phi \bar{i}_r - \left[ 1 - \frac{R^2}{r^2} \right] \cos \phi \bar{i}_\phi \right\} \quad r > R$$

$$\sigma_s(r = R, \phi) = \epsilon_0 [E_r(r = R_+, \phi) - E_r(r = R_-, \phi)] = 2\epsilon_0 E_0 \sin \phi$$

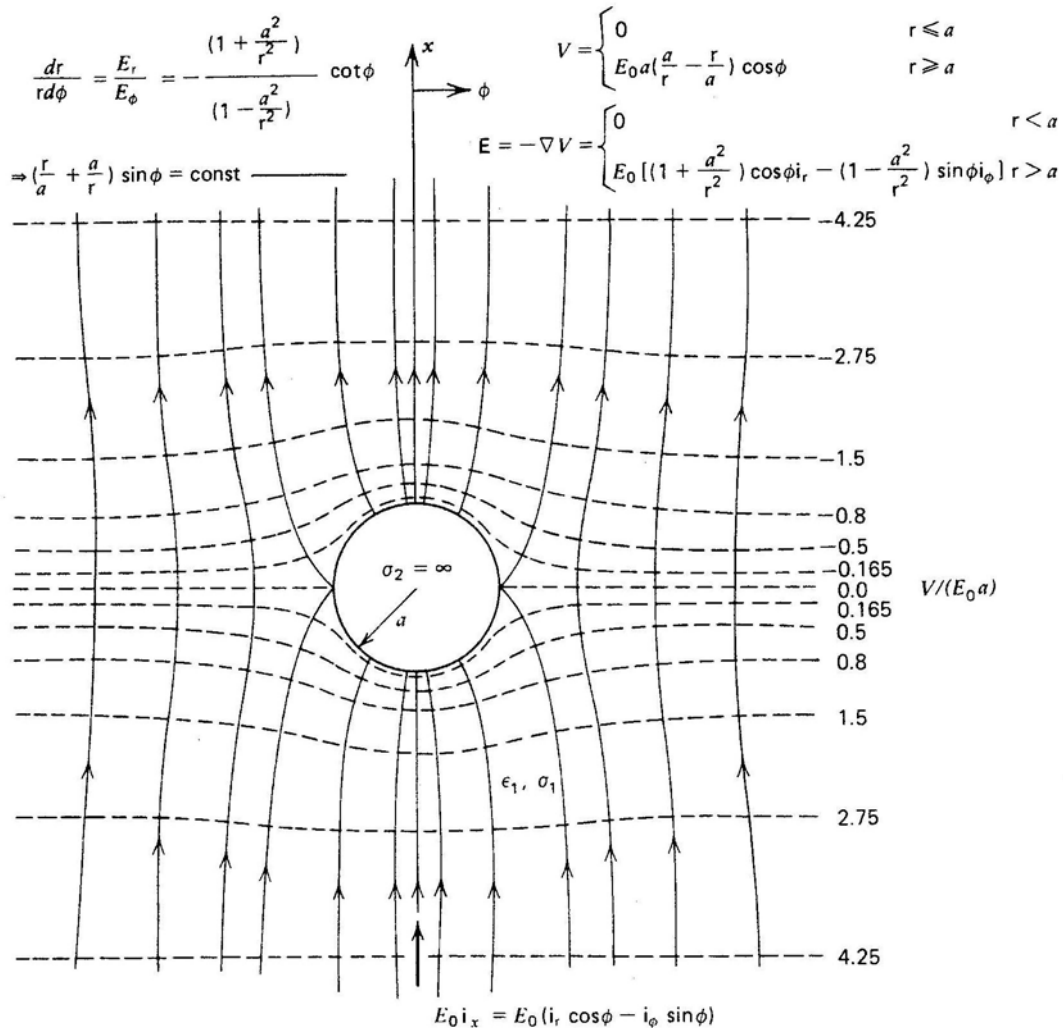


Figure 4-8 Steady-state field and equipotential lines about a (a) perfectly conducting or (b) perfectly insulating cylinder in a uniform electric field.

## II. Solutions to Laplace's Equation in Spherical Coordinates $\left(\frac{\partial}{\partial \phi} = 0\right)$

### A. Product Solution

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(r, \theta) = R(r) \Theta(\theta)$$

$$\frac{\Theta(\theta)}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0 \quad \left| \begin{array}{l} \text{Multiply through} \\ \text{by } \frac{r^2}{R(r)\Theta(\theta)} \end{array} \right.$$

$$\underbrace{\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right)}_{n(n+1)} + \underbrace{\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right)}_{-n(n+1)} = 0$$

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - n(n+1) R = 0$$

Try  $R = Ar^p$

$$Ap(p+1)r^p - n(n+1)Ar^p = 0 \Rightarrow p = n, -(n+1)$$

$$R(r) = Ar^n + Br^{-(n+1)}$$

$$\frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + n(n+1) \sin \theta \Theta = 0 \quad [\text{Legendre's Equation}]$$

In 6.641, only responsible for  $n=1$  solution

$$\Rightarrow \Theta(\theta) = \cos \theta$$

$\Phi = Ar \cos \theta = Az$  is potential of uniform  $z$  directed electric field

$\Phi = \frac{B \cos \theta}{r^2}$  is potential of point electric dipole

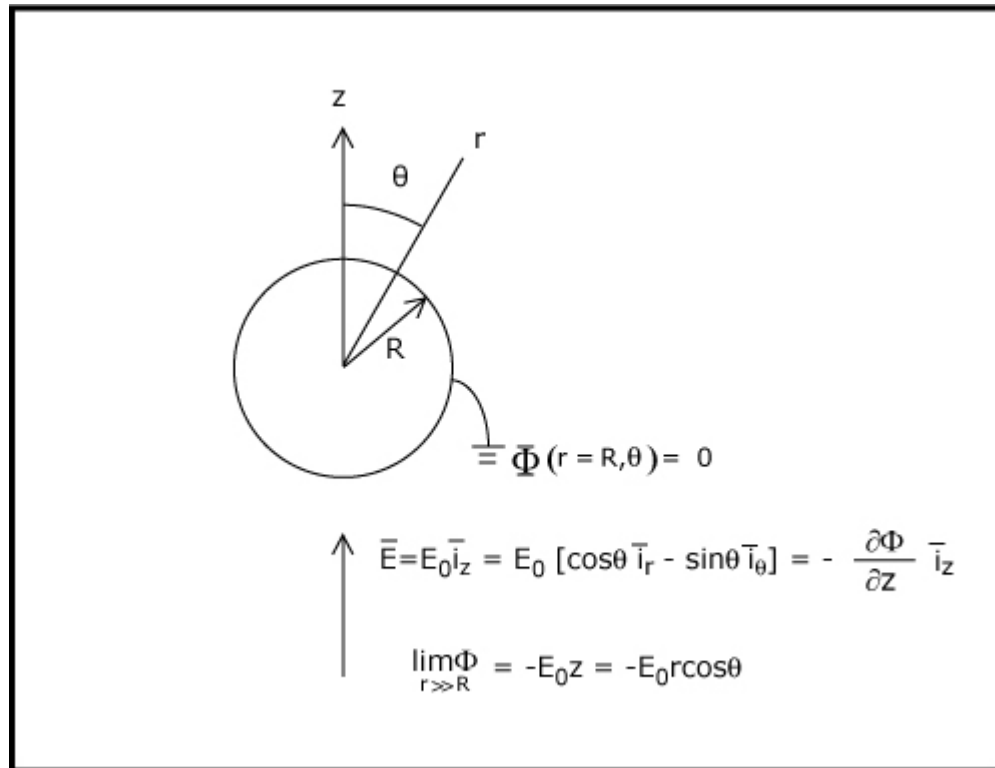
$n=1$

$$\Phi(r, \theta) = \left( Ar + \frac{B}{r^2} \right) \cos \theta$$

## B. Grounded Sphere in a Uniform $z$ Directed Electric Field

$$z = r \cos \theta$$

$$\nabla z = \bar{i}_z = \nabla(r \cos \theta) = \bar{i}_r \frac{\partial}{\partial r} (r \cos \theta) + \bar{i}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (r \cos \theta) = \bar{i}_r \cos \theta - \bar{i}_\theta \sin \theta$$



$$r \geq R$$

$$\Phi(r, \theta) = -E_0 r \cos \theta + \frac{A \cos \theta}{r^2}$$

$$\Phi(r = R, \theta) = 0 = \left( -E_0 R + \frac{A}{R^2} \right) \cos \theta \Rightarrow A = E_0 R^3$$

$$\Phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta \quad r \geq R$$

$$\bar{E} = -\nabla \Phi = - \left[ \frac{\partial \Phi}{\partial r} \bar{i}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \bar{i}_\theta \right]$$

$$= E_0 \left[ \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta \bar{i}_r - \left( 1 - \frac{R^3}{r^3} \right) \sin \theta \bar{i}_\theta \right]$$

$$\sigma_s(r = R, \theta) = \epsilon_0 E_r(r = R, \theta) = 3\epsilon_0 E_0 \cos \theta$$

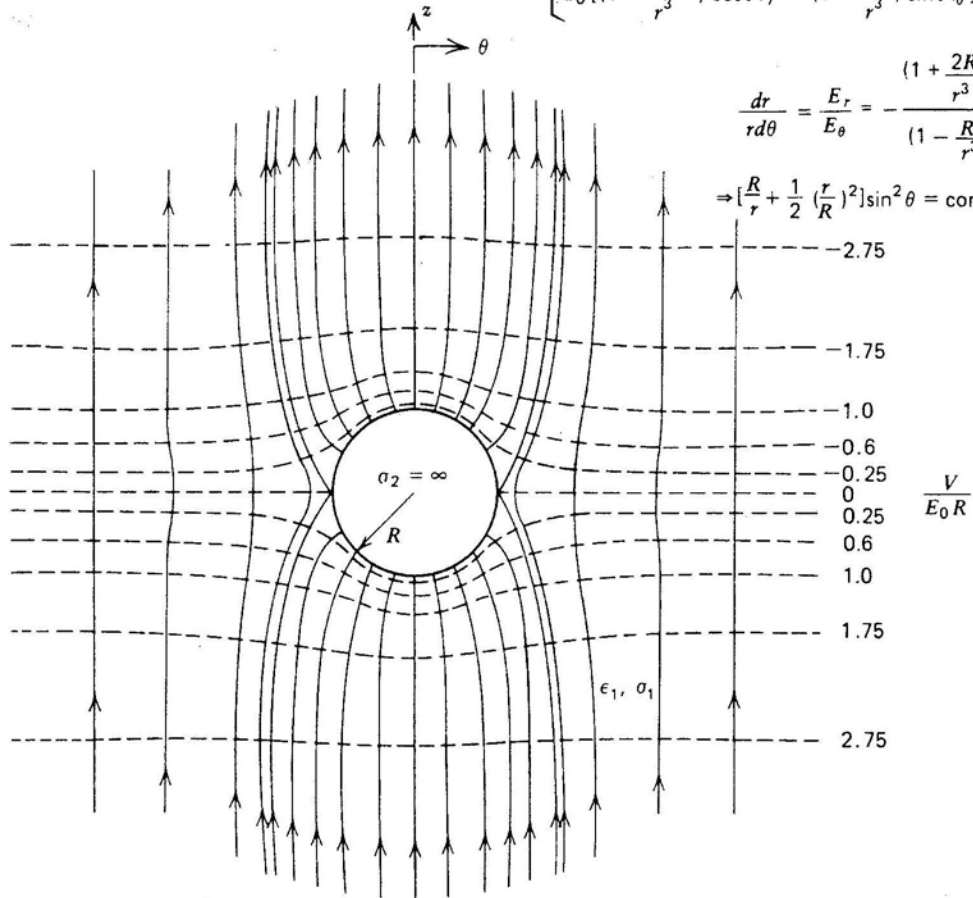


$$V = \begin{cases} 0 & r \leq R \\ -E_0 R \left( \frac{r}{R} - \frac{R^2}{r^2} \right) \cos \theta & r \geq R \end{cases}$$

$$\mathbf{E} = -\nabla V = \begin{cases} 0 & r < R \\ E_0 \left[ \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta \mathbf{i}_r - \left( 1 - \frac{R^3}{r^3} \right) \sin \theta \mathbf{i}_\theta \right] & r > R \end{cases}$$

$$\frac{dr}{rd\theta} = \frac{E_r}{E_\theta} = -\frac{\left( 1 + \frac{2R^3}{r^3} \right)}{\left( 1 - \frac{R^3}{r^3} \right)} \cot \theta$$

$$\Rightarrow \left[ \frac{R}{r} + \frac{1}{2} \left( \frac{r}{R} \right)^2 \right] \sin^2 \theta = \text{const}$$



$$E_0 \mathbf{i}_z = E_0 (i_r \cos \theta - i_\theta \sin \theta)$$