6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2009
Problem Set 6 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 6.1

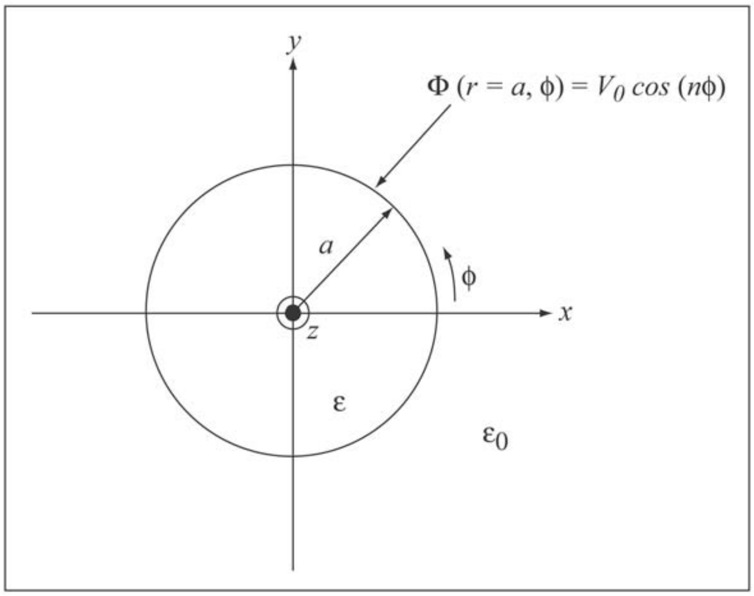

Figure 1: The potential on the surface of a dielectric cylinder in free space is imposed (Image by MIT OpenCourseWare)

A

As always, we start by finding boundary conditions.

I. $\Phi(r = a) = V_0 \cos n\phi$

II. Φ for $r < a$ is finite

III. $\Phi(r = \infty) = 0$

Boundary conditions 1 and 2 suggest for $r \leq a$

$$\Phi(r, \phi) = Ar^n \cos n\phi$$

$$\Phi(r = a, \phi) = Aa^n \cancel{\cos n\phi} = V_0 \cancel{\cos n\phi}$$

$$\Rightarrow A = \frac{V_0}{a^n} \quad r \leq a$$

Boundary condition III suggests for $r \geq a$

$$\Phi(r,\phi) = Br^{-n}\cos n\phi$$

$$\Phi(r=a,\phi) = Ba^{-n}\cancel{\cos n\phi} = V_0\cancel{\cos n\phi}$$

$$\Rightarrow B = V_0 a^n \quad r \geq a$$

So

$$\Phi(r,\phi) = \begin{cases} \frac{V_0}{a^n} r^n \cos n\phi & r \leq a \\ V_0 \frac{a^n}{r^n} \cos n\phi & r \geq a \end{cases}$$

$$\vec{E} = -\nabla\Phi = -\left(\hat{i}_r \frac{\partial \Phi}{\partial r} + \hat{i}_\phi \frac{1}{r}\frac{\partial \Phi}{\partial \phi} + \hat{i}_z \cancelto{0}{\frac{\partial \Phi}{\partial z}}\right)$$

$$\vec{E} = \begin{cases} \frac{V_0 n}{a}\left(\frac{r}{a}\right)^{n-1}\left[-\cos(n\phi)\hat{i}_r + \sin(n\phi)\hat{i}_\phi\right] & r \leq a \\ \frac{V_0 n}{a}\left(\frac{a}{r}\right)^{n+1}\left[\cos(n\phi)\hat{i}_r + \sin(n\phi)\hat{i}_\phi\right] & r \geq a \end{cases}$$

B

$$\sigma_f = \hat{i}_r \cdot (\varepsilon_0 \vec{E}(r=a^+) - \varepsilon \vec{E}(r=a^-))$$

$$= \frac{\varepsilon_0 V_0 n}{a}\Bigg(\underbrace{\frac{a}{a}}_{=1}\Bigg)\cos(n\phi) + \frac{\varepsilon V_0 n}{a}\Bigg(\underbrace{\frac{a}{a}}_{=1}\Bigg)\cos(n\phi)$$

The answer:

$$\sigma_f = \frac{V_0 n}{a}(\varepsilon + \varepsilon_0)\cos(n\phi)$$

From Table 6.8.1 of Haus & Melcher (an aside for cultural purposes):

$$\sigma_{\text{total}} = \hat{i}_r \cdot (\varepsilon_0 \vec{E}(r=a^+) - \varepsilon_0 \vec{E}(r=a^-)) \qquad \text{[polarization plus free surface charge densities]}$$

$$\sigma_{\text{total}} = \frac{2V_0 n\varepsilon_0}{a}\cos(n\phi)$$

$$\sigma_p = \sigma_{\text{total}} - \sigma_f = \frac{V_0 n}{a}(\varepsilon_0 - \varepsilon)\cos(n\phi) \qquad \text{[polarization surface charge density]}$$

C

$$\Phi(r=a,\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{2\pi n}{2\pi}\phi\right) + B_n \sin\left(\frac{2\pi n}{2\pi}\phi\right)\right)$$

Average Φ over $2\pi = 0$, so $A_0 = 0$. $\Phi(r=a,\phi)$ is purely even, so $B_n = 0$.

D

$$\Phi(r=a,\phi)=\sum_{m=1}^{\infty}A_n\cos(n\phi)=\begin{cases}\frac{V_0}{2} & -\frac{\pi}{2}<\phi<\frac{\pi}{2}\\ -\frac{V_0}{2} & \frac{\pi}{2}<\phi<\frac{3\pi}{2}\end{cases}$$

If we multiply both sides by $\cos(m\phi)$, integrate over one period (2π) in ϕ, and use the orthogonality relation $\int_0^{2\pi}\cos(n\phi)\cos(m\phi)\,d\phi=\begin{cases}0 & m\neq n\\ \pi & m=n\end{cases}$

$$A_m=\frac{2V_0}{m\pi}$$

$$\Phi(r=a,\phi)=\sum_{n=1,3,5,\ldots}\frac{2V_0}{n\pi}\cos(n\phi)$$

We omit work: nearly identical to example done in tutorial.

We have same boundary conditions as in (a), but boundary condition 1 is $\Phi(r=a,\phi)=\sum_{n=1,3,5\ldots}\frac{2V_0}{n\pi}\cos(n\phi)$

$$\Phi=\begin{cases}\sum_{n=1,3,5,\ldots}\frac{2V_0}{n\pi}\left(\frac{r}{a}\right)^n\cos(n\phi) & r\leq a\\ \sum_{n=1,3,5,\ldots}\frac{2V_0}{n\pi}\left(\frac{a}{r}\right)^n\cos(n\phi) & r\geq a\end{cases}$$

where V_0 of $(a)\to\frac{2V_0}{n\pi}$ in each term of the series.

Problem 6.2

A

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}$$

Azimuthal symmetry $\Rightarrow\frac{\partial\Phi}{\partial\phi}=0$.

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}=0$$

Guess $\Phi=R(r)Z(z)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\left(R(r)Z(z)\right)}{\partial r}\right)+\frac{\partial^2\left(R(r)Z(z)\right)}{\partial z^2}=0$$

$$\frac{Z(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)=-R(r)\frac{\partial^2Z(z)}{\partial z^2}$$

Divide by $Z(z)R(r)$:

$$\underbrace{\frac{1}{rR(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)}_{\text{only } r}=\underbrace{-\frac{1}{Z(z)}\frac{\partial^2Z(z)}{\partial z^2}}_{\text{only } z}$$

So, to be true for all x,y, both sides must equal the same constant. For this problem, we are asked to set that constant to zero

$$-\frac{1}{Z(z)}\frac{\partial^2Z(z)}{\partial z^2}=0\Rightarrow\frac{\partial^2Z(z)}{\partial z^2}=0$$

$$\Rightarrow Z(z) = Az + B$$

$$\frac{1}{rR(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) = 0 \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial R}{\partial r} = C \Rightarrow \int dR = \int \frac{C}{r}dr$$

So, $R(r) = C\ln r + D$, And

$$\Phi(r,z) = (Az+B)(C\ln r + D)$$

B

$$\Phi(r \le a, l) = V_0 \qquad \Phi(r, z=0) = 0$$

$$\Phi(r=b,z) = 0 \qquad \Phi(r=a_+,z) = \Phi(r=a_-,z)$$

$$\Phi(r=0,z) = \text{ finite}$$

C

$$0 \le r \le a : \Phi^{(1)}(r,z) = Az + Bz\ln r + C\ln r + D$$

$$\Phi^{(1)}(r=0,z) \text{ is finite} \Rightarrow B = C = 0$$

$$\Phi^{(1)}(r,l) = V_0, \Phi^{(1)}(r,0) = 0 \Rightarrow \Phi^{(1)} = \frac{V_0 z}{l}$$

$$a \le r \le b : \Phi^{(2)}(r,z) = Ez + Fz\ln r + G\ln r + H$$

$$\Phi^{(2)}(b,z) = 0 \Rightarrow E = F(-\ln b) \text{ and } H = G(-\ln b)$$

$$\text{so } \Phi^{(2)}(r,z) = Fz\ln\frac{r}{b} + G\ln\frac{r}{b}$$

$$\Phi^{(2)}(r,0) = 0 \Rightarrow \Phi^{(2)} = Fz\ln\frac{r}{b}$$

$$\Phi^{(1)}(r=a,z) = \Phi^{(2)}(r=a,z) \Rightarrow \Phi^{(2)} = \frac{V_0 z}{l\ln\frac{a}{b}}\ln\frac{r}{b}$$

D

$$\overrightarrow{E} = -\nabla\Phi = -\left(\hat{i}_r\frac{\partial\Phi}{\partial r} + \hat{i}_\phi\cancelto{0}{\frac{\partial\Phi}{\partial\phi}} + \hat{i}_z\frac{\partial\Phi}{\partial z}\right)$$

$$0 \le r < a : \overrightarrow{E}^{(1)} = -\nabla\left(\frac{V_0 z}{l}\right) \Rightarrow \overrightarrow{E}^{(1)} = -\hat{i}_z\frac{V_0}{l}$$

$$a \le r \le b : \overrightarrow{E}^{(2)} = -\nabla\left(\frac{V_0 z}{l\ln\left(\frac{a}{b}\right)}\ln\frac{r}{b}\right)$$

$$= -\hat{i}_r\frac{V_0 z}{l\ln\frac{a}{b}}\frac{1}{r} - \hat{i}_z\frac{V_0}{l\ln\frac{a}{b}}\ln\frac{r}{b}$$

$$\overrightarrow{E}^{(2)} = -\frac{V_0}{l\ln\frac{a}{b}}\left(\hat{i}_r\frac{z}{r} + \hat{i}_z\ln\frac{r}{b}\right)$$

E

$$\underbrace{\hat{n}}_{\hat{i}_r}\left(\varepsilon_0 \overrightarrow{E}^{(2)}(r=a^+) - \varepsilon \overrightarrow{E}^{(1)}(r=a^-)\right) = \sigma_{sf}$$

$$-\varepsilon_0 \frac{V_0}{l \ln \frac{a}{b}} \frac{z}{a} + 0 = \sigma_{sf}$$

$$\sigma_{sf} = -\frac{\varepsilon_0 V_0 z}{al \ln \frac{a}{b}}$$

F

We need $\Phi^{(2)}$ and $\overrightarrow{E}^{(2)}$

$$\Phi^{(2)}(r=r_0, z=z_0) = \frac{\cancel{V_0} z_0}{\cancel{l \ln \frac{a}{b}}} \ln \frac{r_0}{b} = \frac{\cancel{V_0} z}{\cancel{l \ln \frac{a}{b}}} \ln \frac{r}{b}$$

Equipot.: $z_0 \ln \frac{r_0}{b} = z \ln \frac{r}{b}$
Field Lines

$$\frac{dr}{dz} = \frac{E_r^{(2)}}{E_z^{(2)}} = \frac{\frac{z}{r}}{\ln \frac{r}{b}} = \frac{z}{r \ln \frac{r}{b}}$$

$$\int r \ln \frac{r}{b} dr = \int z dz$$

$$\frac{r^2}{2} \ln\left(\frac{r}{b}\right) - \frac{r^2}{4} = \frac{z^2}{2} + c$$

At $r = r_0$ and $z = z_0$

$$c = \frac{r_0^2}{2} \ln\left(\frac{r_0}{b}\right) - \frac{r_0^2}{4} - \frac{z_0^2}{2}$$

Are they perpendicular? For E-field:

$$\left.\frac{dr}{dz}\right|_{(r,z)=(r_0,z_0)}^{\text{E-field}} = \frac{z_0}{r_0 \ln \frac{r_0}{b}}$$

For Equipot:

$$\left.\frac{dr}{dz}\right|_{(r,z)=(r_0,z_0)}^{\text{equipot}} = \frac{-r_0 \ln \frac{r_0}{b}}{z_0}$$

$$\left(\left.\frac{dr}{dz}\right|^{\text{equipot}}\right)\left(\left.\frac{dr}{dz}\right|^{\text{E-field}}\right) = -1$$

Problem 6.3

Sphere, initial conditions: At $t = 0$, $p(t) = 0$. No charge anywhere. Dipole

$$p(t) = \begin{cases} 0 & t < 0 \\ p & t \geq 0 \end{cases}$$

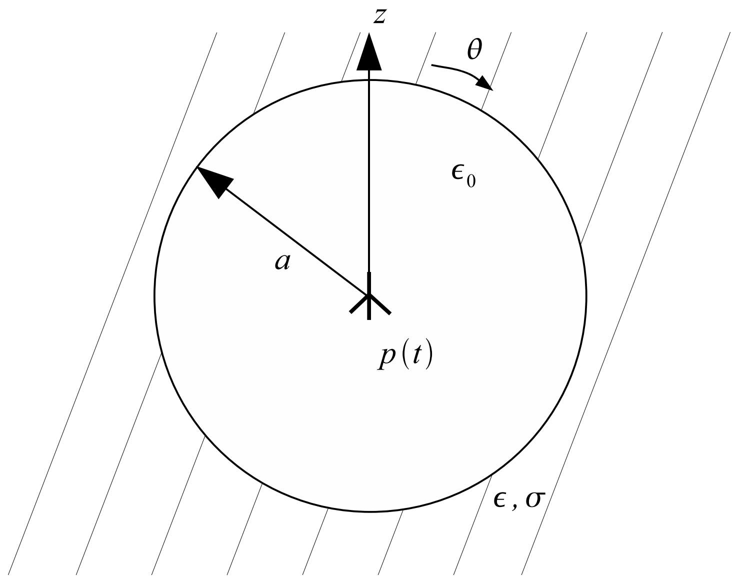

Figure 2: A z-directed electric dipole $p(t)$ at the center of a free space sphere of radius a surrounded by an infinite medium with permittivity ϵ and conductivity σ (Image by MIT OpenCourseWare.)

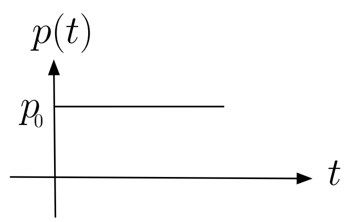

Figure 3: A dipole is turned on at $t = 0$ to a dipole moment of p_0 (Image by MIT OpenCourseWare.)

A

Since there is no volume charge anywhere in the system, we solve Laplace's equation

$$\nabla^2\Phi_{in} = 0 \qquad \Phi_{in} = A(t) r\cos\theta + \frac{B(t)}{r^2}\cos\theta$$

$$\nabla^2\Phi_{out} = 0 \qquad \Phi_{out} = \frac{C(t)}{r^2}\cos\theta$$

We know that since there is a dipole inside, $\Phi_{in}(r \to 0) = \frac{p_0 u(t)\cos\theta}{4\pi\varepsilon_0 r^2}$. Therefore, $B(t) = \frac{p_0(t)}{4\pi\varepsilon_0}$. At $t = 0^+$, the surface charge at the boundary $r = a$ is equal to 0. So

I.

$$(\varepsilon E_r(r = a^+) - \varepsilon_0 E_r(r = a^-)) = \sigma_s = 0$$

$$\begin{aligned}\bar{E}_{in} = -\nabla\Phi_{in} &= \left(\frac{p_0 u(t)\cos\theta}{2\pi\varepsilon_0 r^3} - A(t)\cos\theta\right)\hat{i}_r + \left(\frac{p_0 u(t)}{4\pi\varepsilon_0 r^3} + A(t)\right)\sin\theta\hat{i}_\theta \\ &= \left(\frac{p_0 u(t)}{2\pi\varepsilon_0 r^3} - A(t)\right)\cos\theta\hat{i}_r + \left(\frac{p_0 u(t)}{4\pi\varepsilon_0 r^3} + A(t)\right)\sin\theta\hat{i}_\theta\end{aligned}$$

$$\hat{E}_{out} = -\nabla\Phi_{out} = \left(\frac{2C(t)}{r^3}\cos\theta\right)\hat{i}_r + \left(\frac{C(t)}{r^3}\sin\theta\right)\hat{i}_\theta$$

$$\varepsilon_0\left(\frac{p_0 u(t)}{2\pi\varepsilon_0 a^3} - A(t)\right) = \varepsilon\frac{2C(t)}{a^3} \qquad (1)$$

II. Also, Φ is continuous at $r = a$, so

$$\frac{p_0 u(t)}{4\pi\varepsilon_0 a^3} + A(t) = \frac{C(t)}{a^3} \qquad (2)$$

Add 1 and 2 to get $\frac{3p_0 u(t)}{4\pi \cancel{a^3}} = \frac{(2\varepsilon+\varepsilon_0)C(t)}{\cancel{a^3}}$. Substitute $C(t) = \frac{3p_0 u(t)}{4\pi(2\varepsilon+\varepsilon_0)}$.

$$\frac{p_0 u(t)}{4\pi\varepsilon_0 a^3} + A(t) = \frac{3p_0 u(t)}{4\pi(2\varepsilon+\varepsilon_0)a^3}$$

Therefore $A(t) = \frac{p_0(\varepsilon_0-\varepsilon)u(t)}{2\pi\varepsilon_0(\varepsilon_0+2\varepsilon)a^3}$

So at $t = 0_+$

$$\Phi_{in} = \left(\frac{p_0(\varepsilon_0-\varepsilon)r}{2\pi\varepsilon_0(\varepsilon_0+2\varepsilon)a^3} + \frac{p_0}{4\pi\varepsilon_0 r^2}\right)u(t)\cos\theta \quad r \le a$$

$$\Phi_{out} = \frac{3p_0}{4\pi(\varepsilon_0+2\varepsilon)r^2}u(t)\cos\theta \quad r \ge a$$

B

For $t \to \infty$ at steady state, $\frac{\partial\sigma_s}{\partial t} = 0$

$$\left(\sigma_{out}E_{out,r} - \cancelto{0}{\sigma_{in}}E_{in,r}\right)\Big|_{r=a} = -\frac{\partial\sigma_s}{\partial t} = 0 \quad (\sigma = 0 \text{ inside sphere})$$

Therefore in steady state $C(t) = 0$. If $C(t) = 0$, then $\frac{p_0 u(t)}{4\pi\varepsilon_0 a^3} + A(t) = 0$, $A(t) = -\frac{p_0 u(t)}{4\pi\varepsilon_0 a^3}$. So for $t \to \infty$ in steady state

$$\Phi_{in} = \left(\frac{p_0}{4\pi\varepsilon_0 r^2} - \frac{p_0 r}{4\pi\varepsilon_0 a^3}\right)u(t)\cos\theta \quad r \le a$$

$$\Phi_{out} = 0 \quad r \ge a$$

C

Recall that $\Phi(t) = \underbrace{\Phi_{ss}}_{\text{steady state potential}} + \left(\underbrace{\Phi(t=0_+)}_{\text{initial potential}} - \underbrace{\Phi_{ss}}_{\text{steady state potential}}\right)e^{-\frac{t}{\tau}}$. We already have these terms. We have to find the time constant τ.

$$\sigma E_r(r=a^+) + \frac{\partial}{\partial t}\left(\varepsilon E_r(r=a^+) - \varepsilon_0 E_r(r=a^-)\right) = 0$$

$$\left(\frac{2\sigma C(t)}{a^3}\right) + \frac{\partial}{\partial t}\left(\varepsilon\frac{2C(t)}{a^3} - \frac{p_0 u(t)}{2\pi a^3} + \varepsilon_0 A(t)\right) = 0$$

$$\varepsilon_0 A(t) = \frac{\varepsilon_0 C(t)}{a^3} - \frac{p_0 u(t)}{4\pi a^3}$$

$$\left(\frac{2\varepsilon+\varepsilon_0}{a^3}\right)\frac{\partial}{\partial t}C(t) + \left(\frac{2\sigma}{a^3}\right)C(t) = \frac{\partial}{\partial t}\left(\frac{3p_0 u(t)}{4\pi\varepsilon_0 a^3}\right) = 0 \quad t > 0$$

so $\tau = \frac{2\varepsilon+\varepsilon_0}{2\sigma}$.

D

$$\sigma_s(t) = \sigma_s(t \to \infty) + \left(\cancelto{0}{\sigma_s(t = 0_+)} - \sigma_s(t \to \infty)\right) e^{-t/\tau}$$
$$= \sigma_s(t \to \infty)\left(1 - e^{-t/\tau}\right)$$
$$\sigma_s(t \to \infty) = \varepsilon \cancelto{0}{E_r}(r = a_+, t \to \infty) - \varepsilon_0 E_r(r = a_-, t \to \infty)$$
$$= \varepsilon_0 \left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=a_-, t\to\infty}$$
$$= -\frac{3p_0 \cos\theta}{4\pi\varepsilon_0 a^3}\left(1 - e^{-t/\tau}\right), \tau = \frac{2\varepsilon + \varepsilon_0}{2\sigma}$$

Problem 6.4

As no free current inside and outside the sphere, Maxwell's equations for MQS gives:

$$\nabla \cdot \bar{B} = 0, \nabla \times \bar{H} = 0$$

Here

$$\bar{H} = -\nabla \chi$$

So

$$\nabla^2 \chi = 0$$

The general solution for Laplace's equation in spherical coordinates is

$$\chi = Ar\cos\theta \quad r < R$$

$$\chi = (Dr + C/r^2)\cos\theta \quad r > R$$

The magnetic field is

$$\bar{H} = -\nabla\chi = -\left(\frac{\partial \chi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial \chi}{\partial \theta}\bar{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial \chi}{\partial \phi}\bar{i}_\phi\right)$$
$$= -A(\cos\theta\bar{i}_r - \sin\theta\bar{i}_\theta) = -A\bar{i}_z \quad r < R$$
$$= -(D - 2C/r^3)\cos\theta\bar{i}_r + (D + C/r^3)\sin\theta\bar{i}_\theta \quad r \geq R$$

The boundary conditions are

$$\bar{H}(r = \infty) = 0 \Rightarrow D = 0$$

$$H_\theta(r = R_+) = H_\theta(r = R_-) \Rightarrow A = D + C/R^3$$

$$B_r(r = R_+) = B_r(r = R_-) \Rightarrow H_r(r = R_+) = H_r(r = R_-) + M_0\cos\theta$$

with solutions

$$A = M_0/3$$

$$C = M_0 R^3/3$$

The scalar potential and the magnetic field are:

$$\chi = \frac{M_0 r}{3}\cos\theta \quad r < R$$

$$\chi = \frac{M_0 R^3}{3r^2}\cos\theta \quad r > R$$

$$\bar{H} = -\frac{M_0}{3}(\cos\theta\bar{i}_r - \sin\theta\bar{i}_\theta) = -\frac{M_0}{3}\bar{i}_z \quad r < R$$

$$= \frac{2M_0R^3}{3r^3}\cos\theta\bar{i}_r + \frac{M_0R^3}{3r^3}\sin\theta\bar{i}_\theta \quad r > R$$

$$= \left(\frac{M_0R^3}{3r^3}\right)(2\cos\theta\bar{i}_r + \sin\theta\bar{i}_\theta) \quad r > R$$

Problem 6.5

A

From Gauss' law $\nabla\cdot\bar{D} = \rho_f \Rightarrow \bar{n}\cdot(\bar{D}_1 - \bar{D}_2) = \sigma_{sf}$, the polarization surface charge is

$$-\nabla\cdot\bar{P} = \rho_p \Rightarrow -\bar{n}\cdot(\bar{P}_1 - \bar{P}_2) = \sigma_{sp}$$

so $\sigma_{sp} = P_r(r = a_-) = P_0\sin\phi$.

B

As no free charge inside and outside the cylinder, Maxwell's equations for EQS gives:

$$\nabla\cdot\bar{D} = 0, \nabla\times\bar{E} = 0$$

Here $\bar{E} = -\nabla\Phi$, So $\nabla^2\Phi = 0$. The general solution for Laplace's equation in cylindrical coordinates is

$$\Phi = Ar\sin\phi \quad r \le a$$

$$= (Br + C/r)\sin\phi \quad r \ge a$$

The electric field is

$$\bar{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\bar{i}_\phi + \frac{\partial\Phi}{\partial z}\bar{i}_z\right)$$

$$= -A(\sin\phi\bar{i}_r + \cos\phi\bar{i}_\phi) \quad r \le a$$

$$= -(B - C/r^2)\sin\phi\bar{i}_r - (B + C/r^2)\cos\phi\bar{i}_\phi \quad r \ge a$$

The boundary conditions are

$$\bar{E}(r = \infty) = 0 \Rightarrow B = 0$$

$$\Phi(r = a_+) = \Phi(r = a_-) \Rightarrow Aa = C/a$$

$$\varepsilon_0 E_r(r = a_+) = \varepsilon_0 E_r(r = a_-) + P_0\sin\phi$$

with solutions

$$A = \frac{P_0}{2\varepsilon_0}, \quad C = \frac{P_0a^2}{2\varepsilon_0}$$

The scalar potential and the electric field are

$$\Phi = \frac{P_0 r}{2\varepsilon_0}\sin\phi \quad r \le a$$

$$= \frac{P_0 a^2}{2\varepsilon_0 r}\sin\phi \quad r \ge a$$

$$\bar{E} = -\frac{P_0}{2\varepsilon_0}(\sin\phi\bar{i}_r + \cos\phi\bar{i}_\phi) \quad r < a$$

$$= \frac{P_0 a^2}{2\varepsilon_0 r^2}\sin\phi\bar{i}_r - \frac{P_0 a^2}{2\varepsilon_0 r^2}\cos\phi\bar{i}_\phi \quad r > a$$